MATH 430: FORMAL LOGIC
SPRING 2018
SOLUTIONS TO HOMEWORK 6

1. A graph $G = (V; E)$ is called *bipartite* if there is a set $B \subseteq V$ so that if x, y are either both in B or both not in B, then $\langle x, y\rangle \notin E$.

A *cycle of length* n in G is a sequence $\langle x_0, x_1, \dots x_n\rangle$ so that $x_0 = x_n$, $x_i \neq x_j$ for $0 < i < j < n$, and $\langle x_i, x_{i+1}\rangle \in E$ for all $i < n$.

(a) Draw two graphs of size 6, one bipartite and one not.
(b) Show a graph is bipartite if and only if there are no cycles of *odd* length in G. (Hint: for the hard direction, define B inductively, starting with a point in each connected component of the graph and moving outwards.)
(c) Show the class of bipartite graphs is axiomatizable.
(d) Show the class of bipartite graphs is *not* finitely axiomatizable.

Solution. (b) If G contains an odd-length cycle, say $x_0 E x_1 E \dots E x_{2n} E x_0$. We claim G cannot be bipartite. For suppose otherwise and that there is a B witnessing bipartiteness. Then x_i and x_{i+1} cannot both be in B, or not in B, for each i, $0 \leq i \leq 2n$. Then if for instance $x_0 \in B$, then $x_1 \notin B$, then $x_2 \in B$, and so on; in particular $x_{2n} \in B$, and $x_0 \notin B$. This is a contradiction.

Suppose for simplicity that any two vertices of G are connected by a path. Let us first define a function on the vertex set V, with $f : V \to \omega$: fix some arbitrary vertex x_0. Let $f(x_0) = 0$, and for all other vertices y, $f(y)$ is the length of the shortest path from x_0 to y. Note $1 \leq f(y) < \omega$ for all $y \in V$.

We now set

$$B = \{x \in V \mid f(x) \text{ is even}\}.$$

We claim B witnesses that G is bipartite. Suppose not: then there are x and y so that xEy, and $f(x) \leq f(y)$ are both even or both odd.

Now if $f(x) < f(y)$, we must have $f(y) = f(x) + 1$, since the path from x_0 to x can be extended by one to a path from x_0 to y using the fact that xEy. But this contradicts that $f(x)$ are both even or both odd.

So we must have $f(x) = f(y)$. Let us also assume they take the minimum possible value; say this value is n. In particular there are paths $x_0, x_1, \dots, x_n = x$ and $x_0 = y_0, y_1, \dots y_n = y$ from x_0 to x and y, respectively.

Let i be largest so that $x_i = y_j$ for some j. Note that for this to happen, then by the minimality of the length of paths witnessing the values $f(x) = f(y)$, we have $i = j$. But then $x_i, x_{i+1}, \dots, x_n, y_n, y_{n-1}, \dots, y_i = x_i$ is a cycle of length $2(n - i) + 1$, contradicting our assumption.

(c) Using part (b), we just need to find sentences forbidding the existence of cycles of odd length. Letting σ_n be the sentence

$$\neg \exists v_0 \exists v_1 \dots \exists v_n \left(\bigwedge_{1 \leq i < j \leq n} v_i \neq v_j \right) \wedge \left(\bigwedge_{0 \leq i < n} v_i E v_{i+1} \right) \wedge v_0 = v_n,$$

the desired theory is $\Sigma = \{\sigma_{2k+1} \mid k \geq 1\}$.

(d) Suppose not. Then there is some single sentence τ so that a graph $G = (V; E)$ is bipartite iff $G \models \tau$. In particular, if Σ is our theory from part (c), then $\Sigma \models \tau$. **By Compactness**, there is a finite subset Σ_0 of Σ with $\Sigma_0 \models \tau$. We can assume $\Sigma_0 = \{\sigma_3, \sigma_5, \ldots, \sigma_{2k+1}$ for some $k \geq 1$.

Now we have that if $G \models \Sigma_0$, then $G \models \tau$, hence G is bipartite. But if G is just the graph consisting of a single cycle of length $2k + 3$, then $G \models \Sigma_0$ but is not bipartite. This is a contradiction.

2. Let E be the set of even integers. Show the structures $(\mathbb{Z}; E)$ and $(\mathbb{R}; E)$, in the language with one unary relation symbol, are elementarily equivalent: $(\mathbb{Z}; E) \equiv (\mathbb{R}; E)$.

Solution. By Downward Löwenheim-Skolem, there is a countable elementary substructure $(A; E)$ of $(\mathbb{R}; E)$ with $E \subseteq A$. For all n, $(A; E) \models$"there are at least n distinct elements x so that $\neg E(x)$". In particular, $A \setminus E$ is countably infinite. So we can fix a bijection $f : \mathbb{N} \to A \setminus E$. Then $g : \mathbb{Z} \to A$ given by $g(2n) = 2n$ and $g(2n + 1) = f(n)$ for all n, is an isomorphism between $(\mathbb{Z}; E)$ and $(A; E)$.

Since isomorphic, these structures are elementarily equivalent, and $(A; E)$ and $(\mathbb{R}; E)$ were elementarily equivalent by Löwenheim-Skolem. So $(\mathbb{Z}; E) \equiv (\mathbb{R}; E)$ as needed.