MATH 430: FORMAL LOGIC
SPRING 2018
HOMEWORK 8

Due Friday, May 4.

1. A 6-state Turing machine is described by the following program w:
$(1,0,2,3),(1,1,1,3),(2,0,3,2),(2,1,2,3),(3,0,1,3),(3,1,4,1),$
$(4,0,5,2),(4,1,4,2),(5,0,6,3),(5,1,5,2),(6,0,1,4),(6,1,3,1).$

(a) What is M_w's behavior on input corresponding to the pair $(2,3)$? $(3,1)$?
(b) What is the (possibly partial) binary function $\varphi_w^{(2)}$?

2. Give a program (as in the previous problem) for a Turing machine that computes the function $x \mapsto 2x$.

3. Show that if A, B are recursive subsets of $\mathbb{N}$, then so are $A \cap B$, $A \cup B$, and $\mathbb{N} \setminus A$.
Which parts of this problem are true if "recursive" is replaced by "recursively enumerable"?

4. Show the set of (Gödel numbers of) true statements of arithmetic is not recursively enumerable.

5 (Extra credit)**.** Suppose for this problem that Peano Arithmetic is consistent and sound (i.e., all statements provable in PA are true in the structure $(\mathbb{N}; 0, 1, +, \cdot, <)$). By the fixed point lemma, there are sentences in the language of PA that are equivalent to the following self-referential statements. (Below, "provable" means provable in PA.)

(a) "This statement is provable."
(b) "This statement is not provable."
(c) "The negation of this statement is provable."
(d) "The negation of this statement is not provable."
(e) "Either this statement or its negation is provable."
(f) "Neither this statement, nor its negation, is provable."

Which of these are necessarily true, or necessarily false? How does the truth or falsity of each relate to its provability in PA?