Math 430: Formal Logic **Name (Print):** ______________________________

Midterm #2

April 13, 2018

- This exam contains 6 pages and 7 problems worth a total of 100 points. Please check to make sure your exam contains all pages and problems!
- No notes, books, calculators or other electronic devices should be out at any point during the exam. Phones should be off and put away.
- Write all proofs in complete sentences. Be sure to define any notation you introduce if it wasn't used in class. If proving a statement by induction, carefully state what you are proving, and what your inductive hypothesis is.
- Good luck!

1. (12 points) Give the full definition or statement of each of the following.
   (a) ordinal

   (b) transitive set

   (c) Completeness Theorem

   (d) complete theory

2. (9 points) For each of the following sentences in the language with one binary relation symbol $<$, give a translation into English that makes the meaning (in the context of strict linear orders) clear.
   (a) $\forall x(\exists y\, y < x)$

   (b) $\exists x \exists y(x < y \wedge \forall z \neg(x < z \wedge z < y))$

   (c) $\forall x[(\exists y\, y < x) \rightarrow \exists y(y < x \wedge \forall z(z < x \rightarrow (z < y \vee z = y)))]$

3. (12 points) Fill in the table below with "TRUE" or "FALSE" to indicate whether the corresponding sentence from Problem 2 is satisfied by that row's structure.

	(a)	(b)	(c)
$(\mathbb{Z}; <)$			
$([0,1); <)$			
$(\mathbb{N}^2; <_{\text{Lex}})$			
$(\omega + 1; \in)$			

4. (20 points) In the following table, the first row gives three structures with universe $\mathbb{R}$; the symbols $<, +, \cdot$ are interpreted as usual. The first column gives three subsets of the reals.

	$(\mathbb{R}; <)$	$(\mathbb{R}; +)$	$(\mathbb{R}; \cdot)$
$[0, 1]$	①	②	③
$(-\infty, 0]$	④	⑤	⑥
$\mathbb{R} \setminus \mathbb{Q}$	⑦	⑧	⑨

(a) For each of the 9 cells, write "yes" if the subset of $\mathbb{R}$ in its row is the universe of a substructure of the structure heading its column. In the other cells, write an equation or expression that witnesses that it isn't a substructure.

(b) Of those cells in which you wrote "yes," exactly one should be an elementary substructure. Which is it? How do you know?

(c) For the remaining (non-elementary) substructures, write down a formula witnessing that the substructure is not elementary.

5. (15 points) Let $\mathcal{E}$ be the class of structures $(A; \approx)$ so that $\approx$ is an equivalence relation on A with infinitely many classes.

   (a) Show $\mathcal{E}$ is axiomatizable.

   We must show that there is some collection Σ of sentences so that $(A; \approx) \in \mathcal{E}$ if and only if $(A; \approx) \models \Sigma$.

   Σ must include the axioms for an equivalence relation: . . .

   Now $(A; \approx)$ is an equivalence relation, then it has infinitely many equivalence classes if and only if for each n, there are at least n elements of A, $x_1, \ldots, x_n \in A$, so that $x_i \not\approx x_j$ whenever $i \neq j$. So let σ_n be the sentence. . .

   (b) Show $\mathcal{E}$ is not finitely axiomatizable.

   Suppose towards a contradiction that $\mathcal{E}$ is finitely axiomatizable. Then there is a single sentence σ so that $(A; \approx) \models \sigma$ if and only if $(A; \approx) \in \mathcal{E}$. In particular, if Σ is as in part (a), then $\Sigma \models \sigma$. By compactness . . .

   (c) Show $\mathcal{E}$ is not ω-categorical.

   We must find two structures $\mathfrak{A}, \mathfrak{B}$, both countable and in $\mathcal{E}$, that are not isomorphic. Note that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic equivalence relations, then every equivalence class of $\mathfrak{A}$ is in one-to-one correspondence with some class of $\mathfrak{B}$, and vice versa. . .

6. (18 points) If $G = (V; E)$ is a graph, we say vertices $x, y \in V$ are *joined by a path* (of length $n \geq 1$) if there are $z_1, z_2, \ldots, z_n \in V$ so that xEz_1, $y = z_n$, and z_iEz_{i+1} for all $1 \leq i < n$.

   A graph is *connected* if any two distinct vertices are joined by a path.

   (a) Consider the language of graphs expanded by two constant symbols, $\mathcal{L} = \{E, c, d\}$. Give an $\mathcal{L}$-sentence σ_n expressing "c and d are not joined by a path of length n."

   (Your answer should use only the logical symbols $\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \forall, \exists, x, y, v_i, x_i$, etc., and the non-logical symbols E, c, d of the language $\mathcal{L}$; you may use abbreviations for finite conjunctions, $\bigwedge_{1 \leq i < j \leq n}$, also.)

   (b) Suppose T is a theory in the language of graphs that is satisfied by every connected graph. Show $T \cup \{\sigma_n \mid n \in \mathbb{N}, n \geq 1\}$ is satisfiable.

   By the Compactness Theorem, it is enough to show $T \cup \{\sigma_n \mid n \in \mathbb{N}, n \geq 1\}$ is finitely satisfiable. So let Σ_0 be a finite subset of this theory. Let n be largest natural number with $\sigma_n \in \Sigma_0$. . .

   (c) Show the class of connected graphs is not axiomatizable.

   Suppose towards a contradiction that T is a theory so that $(V; E) \models T$ if and only if V is a connected graph. By the previous part, there is a model $\mathcal{G}^* = (V; E, c^{\mathcal{G}^*}, d^{\mathcal{G}^*})$ of . . .

7. (14 points) Suppose $\mathfrak{A} = (A; <)$ is a (strict) linear order, and that every countable elementary substructure of $\mathfrak{A}$ is a wellorder. Show $\mathfrak{A}$ is a wellorder.

   Suppose $\mathfrak{A}$ is not a wellorder. Then there is a subset $B \subseteq A$ with no $<$-least element. By downward Lowenheim-Skolem, there is . . .