MATH 512 – HOMEWORK 1
DUE WEDNESDAY, SEPT. 7

PROBLEM 1. Suppose $m : \mathcal{P}(S) \to [0,1]$ is a measure over S, and that $\{X_\alpha \mid \alpha < \omega_1\}$ is a family of subsets of S with $m(X_\alpha) > 0$ for all $\alpha < \omega_1$. Show there exist $\alpha \neq \beta$ with $m(X_\alpha \cap X_\beta) > 0$.

PROBLEM 2. Suppose κ is least so that there exists a measure over κ. Show every measure over κ is κ-additive. (Hint: The previous problem may help.)

PROBLEM 3. Show $\mathsf{AC}_\omega(\mathbb{R})$ is equivalent to the following seemingly weaker principle: For all countable collections $\{A_n\}_{n\in\omega}$ of non-empty sets of reals, there is some infinite $I \subseteq \omega$ and $f : I \to \bigcup_{n\in I} A_n$ with $f(n) \in A_n$ for every $n \in I$.

PROBLEM 4. Show that there are only countably many clopen subsets of 2^ω. Is the same true for Baire space, ω^ω?

PROBLEM 5. Show $\mathbb{R} = M \cup Z$ for some meager M and measure zero Z.

PROBLEM 6. Let $X \subseteq \omega^\omega$. Define X^* to be the collection

$$X^* = \{y \in \omega^\omega \mid (\exists x \in X)(\exists N \in \omega)(\forall n \geq N)x(n) = y(n)\}.$$

That is, X^* consists of all y that agree with some $x \in X$ on a tail end. Show the following.

1. If X is meager, then so is X^*.
2. If X is meager, then there is a nowhere dense set C such that $X \subseteq C^*$.

PROBLEM 7. Show:

1. If Player I has a winning strategy in $G(A)$, then $|A| = |2^\omega|$.
2. Suppose $A \subseteq \omega^\omega$, and there is no surjection $f : A \to \omega^\omega$. Show Player II has a winning strategy in $G(A)$.

PROBLEM 8 (*). ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, but not to 2^ω.

PROBLEM 9 (*). Consider the following game, played between two players, I and II.

$$\begin{array}{c|ccccc} \text{I} & A_0 & & A_2 & & \dots \\ \hline \text{II} & & A_1 & & A_3 & \dots \end{array}, \quad A_n \subset \mathbb{R}$$

The players must maintain, for all n: $A_{n+1} \subset A_n$; and each $A_n \subset \mathbb{R}$ is uncountable. After ω moves, Player I wins if $\bigcap_{n\in\omega} A_n = \emptyset$; otherwise Player II wins.

Is this game determined? If so, which player has a winning strategy? (Hint: The answer depends on the Axiom of Choice, which you will want to assume!)