MATH 512 – HOMEWORK 2
DUE MONDAY, SEPTEMBER 19

Problem 1. This problem outlines a proof of Gale-Stewart that doesn't use AC.

Let T be a tree on a set X. We say a quasistrategy S for Player I (Player II) is **winning in** $G(A;T)$ if $[S] \subseteq A$ ($[S] \cap A = \emptyset$).

Define an operator $F : \mathcal{P}(T) \to \mathcal{P}(T)$ by setting, for $R \subseteq T$,

$$F(R) = \{s \in T \mid |s| \text{ is even, and } (\forall x \in X)(\exists y \in X) s^\frown\langle x\rangle \in T \to s^\frown\langle x, y\rangle \in R\}.$$

That is, $F(R)$ is the set of positions where it is Player I's turn, so that Player II can always enter R on their next move. Inductively define $S_0 = \emptyset$, $S_\alpha = F(\bigcup_{\xi<\alpha} S_\xi)$.

(a) Argue that the S_α are $\subseteq$-increasing, and so for some α, $S_\alpha = S_{\alpha+1} =: S_\infty$.
(b) Show that if $\emptyset \in S_\infty$, then Player II has a winning quasistrategy in $G([T];T)$.
(c) Show that if $\emptyset \notin S_\infty$, then Player I has a winning quasistrategy in $G([T];T)$.
(d) Conclude that if X can be wellordered, then $G([T];T)$ is determined.

Problem 2 (*). Show the Axiom of Choice is equivalent to the determinacy of all games on trees with height 2 ($T \subseteq X^{\leq 2}$ some X); in particular, AC is equivalent to the Gale-Stewart Theorem (for arbitrary trees) over ZF.

Problem 3. Show, using the Axiom of Choice, that there is $A \subseteq \omega^\omega$ so that $G(A)$ is not determined. (You shouldn't appeal to a fact/exercise unless it was proved in class, or you show it here!)

Problem 4 (*). (Cantor-Bendixson.) Suppose $K \subseteq \omega^\omega$ is closed. Inductively define

- $K_0 = K$,
- $K_{\alpha+1} = K'_\alpha = \{x \in K_\alpha \mid x \text{ is a limit point of } K_\alpha\}$, and
- $K_\lambda = \bigcap_{\alpha<\lambda} K_\alpha$ for limit ordinals λ.

Show there exists $\alpha < \omega_1$ so that $K_{\alpha+1} = K_\alpha$, and that this K_α is perfect.

Conclude $K = P \cup C$ with P perfect and C countable. In particular, K has the perfect set property.

Problem 5. Fix $A \subseteq \omega^\omega$. The **perfect set game** $G_{\mathrm{PS}}(A)$ is played as follows:

$$\begin{array}{c|cccccccc} \mathrm{I} & s_0^0, s_0^1 & & s_1^0, s_1^1 & & \ldots & s_n^0, s_n^1 & & \ldots \\ \hline \mathrm{II} & & i_0 & & i_1 & \ldots & & i_n & \ldots \end{array}$$

Each $s_n^i \in \omega^{<\omega}$ and $i_n \in \omega$. Here are the rules: Player I plays s_0^0, s_0^1 with $s_0^0 \perp s_0^1$. Player II plays $i_n \in \{0, 1\}$. Having fixed $s_n^{i_n}$, Player I must choose incompatible extensions s_{n+1}^0, s_{n+1}^1 of $s_n^{i_n}$; that is, $s_{n+1}^0, s_{n+1}^1 \supsetneq s_n^{i_n}$, and $s_{n+1}^0 \perp s_{n+1}^1$.

A play of this game produces $x = \bigcup_{n\in\omega} s_n^{i_n}$. Player I wins if and only if $x \in A$.

(a) Show Player I has a winning strategy in $G_{\mathrm{PS}}(A)$ iff A has a non-empty perfect subset.
(b) Show Player II has a winning strategy in $G_{\mathrm{PS}}(A)$ iff A is countable.
(c) Conclude that under AD, all sets of reals have the perfect set property.