MATH 512 – HOMEWORK 4
DUE WEDNESDAY OCTOBER 19

Problem 1. Show that if $V = L$, then there is a Δ^1_2 set that is not Lebesgue measurable. (Hint: You can assume Fubini's Theorem.)

Problem 2. Recall $H(\aleph_1)$ denotes the set of **hereditarily countable sets**, i.e., the collection of sets whose transitive closure is countable. Show a set $A \subseteq \omega^\omega$ is $\boldsymbol{\Sigma}^1_2$ iff it is Σ_1-definable (with parameters) over $(H(\aleph_1); \in)$.

Problem 3 (*). Assume there is a transitive model of ZFC. Show there is a Σ^1_2 statement that is not downwards absolute to all transitive models of ZFC.

Problem 4 (*). Show, in ZF, that the Axiom of Choice is equivalent to the statement that for all ordinals α, $\mathcal{P}(\alpha)$ can be wellordered.

Problem 5. Suppose $j : M \to N$ is a non-trivial elementary embedding of transitive ZFC-models. Show there is an ordinal $\alpha \in M$ so that $j(\alpha) \neq \alpha$.

Problem 6. Show an ultrafilter $\mathcal{U}$ is countably complete iff $\mathrm{Ult}(V,\mathcal{U})$ is wellfounded.

Problem 7. Suppose $\mathcal{U}$ is a normal ultrafilter. Show $\mathcal{U} \notin \mathrm{Ult}(V,\mathcal{U})$.

Problem 8. Suppose κ is the completeness of some countably complete ultrafilter $\mathcal{U}$ (i.e., κ is least so that there is a sequence $\{A_\xi \mid \xi < \kappa\} \subset \mathcal{U}$ with $\bigcap_{\xi<\kappa} A_\xi \notin \mathcal{U}$). Show κ is measurable.