MATH 512 – HOMEWORK 5
DUE MONDAY NOVEMBER 7

Throughout, $M \models \mathsf{ZFC}$ is transitive, and $\mathbb{P}$ is a poset (and in M, where applicable).

Problem 1 (*). Show the following are equivalent for a filter G:

1. G is $\mathbb{P}$-generic over M.
2. $G \cap D \neq \emptyset$ for every *open* dense $D \in M$.
3. $G \cap A \neq \emptyset$ for every maximal antichain $A \in M$.
4. $G \cap E \neq \emptyset$ for every predense $E \in M$. (E is **predense** if $(\forall p \in \mathbb{P})(\exists q \in E) p \| q$.)

Problem 2 (Maximality of the Forcing Language). Show if $p \Vdash^M (\exists x)\phi(x, \tau_1, \ldots, \tau_n)$, there is a name $\sigma \in M^{\mathbb{P}}$ such that $p \Vdash^M \phi(\sigma, \tau_1, \ldots, \tau_n)$.

Problem 3. Let $i : \mathbb{P} \to \mathbb{P}$ be an automorphism, and let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Prove $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ iff $i(p) \Vdash \phi(i^*(\tau_1), \ldots, i^*(\tau_n))$.

Problem 4. A poset $\mathbb{P}$ is **weakly homogeneous** if for all $p, q \in \mathbb{P}$, there is an automorphism $i : \mathbb{P} \to \mathbb{P}$ such that $i(p) \| q$. Show

(a) For any cardinals δ, κ with κ infinite, $\mathrm{Add}(\delta, \kappa)$ is weakly homogeneous.
(b) If $\mathbb{P}$ is weakly homogeneous and $x_1, \ldots, x_n \in V$, then for any formula ϕ with n free variables, either $\mathbb{1}_{\mathbb{P}} \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$ or $\mathbb{1}_{\mathbb{P}} \Vdash \neg\phi(\check{x}_1, \ldots, \check{x}_n)$.

Problem 5. We say a real $x \in 2^\omega$ is **Cohen over** M if $x = \bigcup G$ for some $\mathrm{Add}(1, \omega)$-generic G over M.

(a) If $g = g_1 \oplus g_2 := \langle g_1(0), g_2(0), g_1(1), g_2(1), \ldots\rangle$ is Cohen over M, then g_1, g_2 are both Cohen over M.
(b) (*) There are reals g_1, g_2, each Cohen over M, so that $g_1 \oplus g_2$ is not; indeed, so that there is no transitive model N of ZFC with $\mathrm{ON} \cap M = \mathrm{ON} \cap N$ and $g_1, g_2 \in N$.

Problem 6 (*). A **chopped real** is a pair (x, Π) where $x \in 2^\omega$ and $\Pi = \{I_n \mid n \in \omega\}$ is partition of ω into finite intervals. Say a real **matches** a chopped real (x, Π) if $x \restriction I_n = y \restriction I_n$ for infinitely many $I_n \in \Pi$. Show a real g is Cohen over M iff it matches every chopped real in M.

Problem 7. Suppose G is $\mathbb{P}$-generic over M, where $\mathbb{P}$ is countable (in M). Show that if A is an uncountable set of ordinals in $M[G]$, then there is $B \subseteq A$, also uncountable (in $M[G]$), with $B \in M$.

Problem 8. Suppose $x \subset M$ satisfies $x \in M[G]$ for *all* M-generic filters $G \subseteq \mathbb{P}$. Show $x \in M$.

Problem 9 (*). Show there is an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ of size $2^{\aleph_0}$.

Problem 10. Recall we say a set $\mathcal{X}$ generates an ultrafilter on X if the smallest filter containing $\mathcal{X}$ is an ultrafilter. We define the **ultrafilter number** $\mathfrak{u}$ to be the smallest size of a generating set for a *nonprincipal* ultrafilter on ω.

(a) Show that $\omega < \mathfrak{u} \leq 2^\omega$.
(b) Show that MA implies $\mathfrak{u} = 2^\omega$.