MATH 512 – HOMEWORK 6
DUE FRIDAY NOVEMBER 18

Problem 1. Show the random real forcing $\mathbb{B}$ has the countable chain condition.

Problem 2. Let M be a transitive ZFC model. Show $x \in 2^\omega$ is Cohen over M iff $x \in A^c$ for all Borel codes $c \in M$ such that A^c is comeager.

Problem 3 (*). A map $e : \mathbb{P} \to \mathbb{Q}$ between posets is an **embedding** if $p \leq p'$ implies $e(p) \leq e(p')$, and $p \perp p'$ implies $e(p) \perp e(p')$. It is **complete** if whenever A is a maximal antichain in $\mathbb{P}$, the pointwise image $e[A]$ is a maximal antichain in $\mathbb{Q}$. It is **dense** if its range $e[\mathbb{P}]$ is dense in $\mathbb{Q}$. Let $e : \mathbb{P} \to \mathbb{Q}$ be an embedding in a transitive ZFC model M.

a) If e is complete, then for all M-generic $H \subseteq \mathbb{Q}$, $i^{-1}[H] \subseteq \mathbb{P}$ is M-generic.
b) If e is dense, then whenever $G \subseteq \mathbb{P}$ is an M-generic filter, then $i^{\uparrow}[G] \subseteq \mathbb{Q}$ is an M-generic filter, where here $i^{\uparrow}[X] := \{q \in \mathbb{Q} \mid (\exists p \in X) e(p) \leq q\}$.

Problem 4 (*). A poset $\mathbb{P}$ is **separative** if whenever $p \not\leq q$, then there is some $r \leq p$ such that $r \perp q$. Suppose generic filters containing p exist for all $p \in \mathbb{P}$. Show:

a) $\mathbb{P}$ is separative iff for all $p, q \in \mathbb{P}$, $p \leq q \iff p \Vdash \check{q} \in \dot{G}$.
b) For all $\mathbb{P}$, there is a separative $\mathbb{Q}$ and a surjective (hence dense and complete) embedding $\pi : \mathbb{P} \to \mathbb{Q}$.

Problem 5 (Feferman-Levy). Let G be $\mathrm{Col}(\omega, {<}\aleph_\omega)$-generic over V. Show there is a model N of ZF, $V \subseteq N \subseteq V[G]$, such that $\mathrm{cf}(\aleph_1) = \omega$ in N.

The next two problems outline an argument due to A. Miller of the relative consistency of $\mathsf{ZF} + V{=}L(\mathbb{R}) + \neg\mathsf{AC}$. See Kunen Classic Ch. VII Exercise (E3-4) for more hints.

Problem 6. Let $\kappa \leq \lambda$ be uncountable. Show, for all formulas ϕ and ordinals α,

$$\mathbb{1} \Vdash_{\mathrm{Add}(\kappa,\omega)} \phi(\check{\alpha})^{L(\mathbb{R})} \quad \text{iff} \quad \mathbb{1} \Vdash_{\mathrm{Add}(\lambda,\omega)} \phi(\check{\alpha})^{L(\mathbb{R})}.$$

(Hint: This is clear if $|\kappa| = |\lambda|$. So note that if G is V-generic for $\mathrm{Col}(\kappa, \lambda)$ and H is $V[G]$-generic for $\mathrm{Add}(\kappa, \omega)$, then H is V-generic also, and $\mathbb{R}^{V[G][H]} = \mathbb{R}^{V[H]}$. Use weak homogeneity to conclude $\mathbb{1} \Vdash^V_{\mathrm{Add}(\kappa,\omega)} \phi(\check{\alpha})^{L(\mathbb{R})}$ iff $\mathbb{1} \Vdash^{V[G]}_{\mathrm{Add}(\kappa,\omega)} \phi(\check{\alpha})^{L(\mathbb{R})}$; similarly for λ.)

Problem 7. Show that if G is $\mathrm{Add}(\aleph_1, \omega)$-generic, then there is no wellorder of $\mathbb{R}$ in $L(\mathbb{R})^{V[G]}$. (Hint: Otherwise there is a bijection $f : \mathbb{R} \to \alpha$ in $L(\mathbb{R})^{V[G]}$ for some ordinal α. Use the previous problem with sufficiently large λ.)

Problem 8 (*). Recall $A \subseteq^* B$ iff $A \setminus B$ is finite. A sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \subseteq [\omega]^\omega$ is a **tower** if $A_\beta \subseteq^* A_\alpha$ whenever $\alpha < \beta < \kappa$, and there is no **pseudointersection** for $\langle A_\alpha \rangle_{\alpha<\kappa}$, that is, for no infinite $B \subseteq \omega$ do we have $B \subseteq^* A_\alpha$ for all α. The **tower number** $\mathfrak{t}$ is the least length of a tower. Show:

a) $\mathfrak{t}$ is a regular cardinal and $\omega < \mathfrak{t} \leq 2^\omega$.
b) If $\omega \leq \kappa < \mathfrak{t}$, then $2^\kappa = 2^{\aleph_0}$. (Hint: Embed a tall binary tree into $([\omega]^\omega, \supsetneq^*)$.)
c) $\mathfrak{t} \leq \mathrm{cf}(2^{\aleph_0})$.
d) Under MA, $\mathfrak{t} = 2^{\aleph_0}$.