MATH 512 – HOMEWORK 7
DUE DECEMBER 12, 2016

PROBLEM 1. Let $a \in \omega^\omega$. For x not random over $L[a]$, let $\nu(x)$ be the height in $\leq_{L[a]}$ of the $\leq_{L[a]}$-least Borel code $c \in L[a]$ such that $x \in A^c$ and $\mu(A^c) = 0$. Define a preorder $\preceq$ on $\mathbb{R}$ by $x \preceq y$ iff x, y are not random over $L[a]$ and $\nu(x) \leq \nu(y)$. Show:

(a) $\preceq$ is a $\Sigma^1_2(a)$ subset of $\mathbb{R} \times \mathbb{R}$.
(b) For all x not random over $L[a]$, $\mu(\{y \in \mathbb{R} \mid y \preceq x\}) = 0$.
(c) $\preceq$ is either nonmeasurable, or has measure zero. (Hint: Fubini.)
(d) If $\preceq$ is measurable, then $\{x \in \mathbb{R} \mid x \text{ is not random over } L[a]\}$ is Lebesgue null.

PROBLEM 2. Suppose $a \in \omega^\omega$ is such that $\{x \in \mathbb{R} \mid x \text{ is not random over } L[a]\}$ is Lebesgue null; show all $\Sigma^1_2(a)$ sets are Lebesgue measurable. (Hint: Imitate Solovay's proof of Lebesgue measurability in $L(\mathbb{R})$.)

PROBLEM 3. For $f, g : \omega \to [\omega]^{<\omega}$, write $f \subseteq^* g$ iff $(\forall^\infty n) f(n) \subseteq g(n)$. Define

$$\mathcal{C} := \left\{ f : \omega \to [\omega]^{<\omega} \;\middle|\; \sum_{n=1}^{\infty} \frac{|f(n)|}{n^2} < \infty \right\}$$

Show the partial order $(\mathcal{C}, \subseteq^*)$ is countably directed: whenever $\langle f_i \rangle_{i \in \omega}$ is an ω-sequence in $\mathcal{C}$, there is an upper bound $g \in \mathcal{C}$, i.e. $f_i \subseteq^* g$ for all i.

PROBLEM 4. Suppose $M \subseteq V$ are models of ZFC. Show the following are equivalent:

- $(\exists \phi : \omega \to [\omega]^{<\omega})(\forall n)|\phi(n)| \leq n$ and $(\forall x \in \omega^\omega \cap M)(\forall^\infty n) x(n) \in \phi(n)$.
- $(\exists \phi : \omega \to [\omega]^{<\omega})$ the map $n \mapsto |\phi(n)|$ is in M, and $(\forall x \in \omega^\omega \cap M)(\forall^\infty n) x(n) \in \phi(n)$.
- There is $\phi \in \mathcal{C}$ such that $f \subseteq^* \phi$ for all $f \in \mathcal{C} \cap M$ (see Problem 3).

PROBLEM 5. Suppose $M \models \mathsf{ZFC}^-$ and $U \subseteq \mathcal{P}(\kappa) \cap M$ satisfies the definition of M-normal ultrafilter with the possible exception of weak amenability. Show U is weakly amenable iff $\mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{\mathrm{Ult}(M,U)}$.

PROBLEM 6. Suppose U is an M-ultrafilter on κ and $i : M \to N = \mathrm{Ult}(M, U)$ is the ultrapower map. Let $W \subseteq \mathcal{P}(i(\kappa)) \cap N$ be defined by $[f]_U \in W$ iff $\{\xi < \kappa \mid f(\xi) \in U\} \in U$. Show $W = \bigcup\{i(x \cap U) \mid x \in M \cap [M]^\kappa\}$.

PROBLEM 7. Suppose M is a transitive model of ZFC and for some $U \in M$, $M \models$ "U is a normal measure". Suppose $\langle M_\alpha, U_\alpha, i_{\alpha,\beta} \rangle_{\alpha \leq \beta \leq \theta}$ is an iteration of $\langle M, U \rangle$ and θ is least such that M_θ is illfounded. Show θ is a limit ordinal and $\mathrm{cf}(\theta) = \omega$.

PROBLEM 8. Let $M = \langle L_\tau, U \rangle$ be a mouse and suppose $\langle \kappa_i \rangle_{i \in \omega_1}$ is the critical sequence of the length ω_1 iteration of M. Show the κ_i are indiscernibles for L, i.e., for all formulas φ in the language of set theory and sequences $\alpha_1 < \cdots < \alpha_k$, $\beta_1 < \cdots < \beta_k$ in ω_1,

$$L \models \varphi(\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_k}) \iff L \models \varphi(\kappa_{\beta_1}, \ldots, \kappa_{\beta_k}).$$

PROBLEM 9. Suppose $0^\#$ exists. Show for all uncountable cardinals κ that κ is inaccessible in L, and $\mathrm{cf}(\kappa^{+L}) = \omega$.