Math 512: Set theory of the reals

Fall 2016

Instructor: Sherwood Hachtman
Time: MWF 10-10:50
Location: SEO 427

Course description: Vitali gave the first example of a non-Lebesgue measurable set. His proof made use of an appeal to the Axiom of Choice, and so this set is rather hard to descibe. Is choice necessary? Are there definable examples of such pathological sets? Perhaps surprisingly, these questions are not decided by ZFC. A general theme in set theory is that obtaining the well-behavedness of definable sets of real numbers often winds up requiring the consistent existence of large cardinals; what is more surprising is that often an equivalence holds.

We will explore this interplay, attempting to understand the structure of the reals via the methods of descriptive set theory, forcing, inner models, and large cardinals. These threads come together in a seminal theorem of Solovay: "ZF + All sets of reals are Lebesgue measurable" is equiconsistent with "ZFC + There is an inaccessible cardinal" (In particular, assuming large cardinals, choice is necessary for the existence of Vitali's nonmeasurable set). Pursuing this theme leads to a tight correspondence between large cardinals and determinacy axioms, either of which provides a fairly complete and attractive theory of definable sets of reals.

Topics:

- **Descriptive set theory**: Polish spaces; Borel and projective hierarchies; regularity properties of sets of reals; Gale-Stewart games, Axiom of Determinacy, and applications.
- **Large cardinals**: elementary embeddings of the universe; ultrapowers and iterations; the Martin-Solovay tree and absoluteness; proofs of determinacy.
- **Inner models**: constructible and ordinal definable sets; Shoenfield absoluteness; definable well-orders; mice and sharps; obtaining large cardinal strength from assumptions about the reals.
- **Forcing**: examples of generic reals; forcing absoluteness; Martin's Axiom, the Proper Forcing Axiom, and applications.

Prerequisites: Familiary with mathematical logic (models, definability) and set theory (ordinals, cardinals, transfinite induction) is essential; ideally a student will have seen Cohen's method of forcing and Gödel's constructible universe L, through the proof of the independence of the continuum hypothesis, but the basics of these methods will be reviewed.

Grading: Grades will be determined by performance on regular homework, assigned at least every other week, and course participation.

References:

- Sherwood Hachtman and Justin Palumbo, Notes on Determinacy, available at `http://math.uic.edu/~shac/determinacy/determinacy.html`.
- Tomek Bartoszyński and Haim Judah, *Set Theory: On the structure of the real line.*
- Akihiro Kanamori, *The Higher Infinite*, available through SpringerLink.
- Iannis Moschovakis, *Descriptive Set Theory*, 2nd Edition, available at `http://www.math.ucla.edu/~ynm/books.htm`.