NOTES FOR MATH 512: SET THEORY OF THE REALS

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§1. The measure problem.

QUESTION 1.1 (Lebesgue). Does there exist a function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ satisfying

(a) $m(\mathbb{R}) = \infty$ and $m([0, 1]) < \infty$,
(b) (translation invariance) $m(X) = m(X + r)$ for all $r \in \mathbb{R}$ and $X \subseteq \mathbb{R}$, and
(c) (countable additivity) If $\{X_n \mid n \in \omega\}$ is a pairwise disjoint collection of subsets of $\mathbb{R}$, then $m(\bigcup_{n\in\omega} X_n) = \Sigma_{n\in\omega} m(X_n)$?

THEOREM 1.2 (Vitali). *No.*

PROOF. Define an equivalence relation on $\mathbb{R}$ by $x \sim_{\mathbb{Q}} y$ if and only if $x - y \in \mathbb{Q}$. Note each $\sim_{\mathbb{Q}}$-equivalence class is countable. Let $\{x_\alpha \mid \alpha < 2^{\aleph_0}\}$ be a system of representatives for the $\sim_{\mathbb{Q}}$-classes, i.e. for each x there is a unique α so that $x \sim_{\mathbb{Q}} x_\alpha$; translating, we can further assume $x_\alpha \in [0, 1)$ for each α.

Now suppose towards a contradiction we have m witnessing a positive answer to Lebesgue's question. Notice $\{A + q \mid q \in \mathbb{Q}\}$ is a partition of $\mathbb{R}$ into countably many pieces, so by countable additivity (and since $m(\mathbb{R}) = \infty$), we have $0 < m(A)$. But then by countable additivity again, $\bigcup_{n\in\omega} A + \frac{1}{n+1}$ is a subset of $[0, 2)$ with infinite measure, contradicting (a). ⊣

Note that Vitali's example of a *nonmeasurable set* A in the proof above relied heavily on choice. Is there a more explicit description of a nonmeasurable set? We return to this question later.

Banach proposed a generalization of Lebesgue's measure problem relaxing the condition (b).

QUESTION 1.3 (Banach). Does there exist $S \neq \varnothing$ and a function $m : \mathcal{P}(S) \to [0, 1]$ satisfying

(i) $m(S) = 1$,
(ii) (nontriviality) $m(\{x\}) = 0$, for all $x \in S$, and
(iii) (countable additivity) If $\{X_n \mid n \in \omega\}$ is pairwise disjoint, then $m(\bigcup_{n\in\omega} X_n) = \Sigma_{n\in\omega} m(X_n)$?

Call such a function m a **measure over** S. Note that the conditions together imply that if there is a measure over S, then S is uncountably infinite. By the following exercise, we automatically get a bit more than countable additivity. Say a measure is **κ-additive** if whenever $\{A_\xi \mid \xi < \gamma\} \subseteq \mathcal{P}(S)$ is pairwise disjoint with $\gamma < \kappa$, we have

$$m\left(\bigcup_{\xi<\gamma} A_\xi\right) = \sum_{\xi<\gamma} m(A_\xi) := \sup_{F\subset\gamma \text{ finite}} \sum_{\xi\in F} m(A_\xi).$$

EXERCISE 1.4. If κ is least such that there is a measure over κ, then every measure over κ is κ-additive.

We say $\kappa > \omega$ is **real-valued measurable** if there is a κ-additive measure over κ.

REMARK 1.5. Such a κ must be **regular**, i.e., whenever $\alpha < \kappa$ and $f : \alpha \to \kappa$ is a function, we have that f is bounded in κ: $\sup f[\alpha] < \kappa$. To see this, note that otherwise we could write $\kappa = \bigcup_{\xi<\alpha} \lambda_\xi$ for some $\alpha < \kappa$ and with each $\lambda_\xi < \kappa$. But then κ-additivity implies each λ_ξ, and then also κ, has measure 0, a contradiction.

We now see that a positive answer to Banach's measure problem has some large cardinal strength.

THEOREM 1.6 (Ulam). *If κ is real-valued measurable, then κ is weakly inaccessible.*

LEMMA 1.7. *For each cardinal λ, there is an array $\langle A^\xi_\alpha \mid \xi < \lambda, \alpha < \lambda^+\rangle$ of subsets of λ^+ such that*

(a) $A^\xi_\alpha \cap A^\xi_\beta = \varnothing$, *for all* $\alpha < \beta < \lambda^+$, $\xi < \lambda$;
(b) $\lambda^+ \setminus \bigcup_{\xi<\lambda} A^\xi_\alpha| \leq \lambda$ *for all* $\alpha < \lambda^+$.

*Such an array is called an **Ulam matrix**.*

PROOF. For $\eta < \lambda^+$, fix $f_\eta : \lambda \to \eta$ onto. Then set, for $\xi < \lambda$ and $\alpha < \lambda^+$,

$$A^\xi_\alpha = \{\eta < \lambda^+ \mid f_\eta(\xi) = \alpha\}.$$

(a) is immediate. And for (b), we have if $\eta \notin \bigcup_{\xi<\lambda} A^\xi_\alpha$ for some $\alpha < \lambda^+$, then for all $\xi < \lambda$, $f_\eta(\xi) \neq \alpha$. Since f_η is a surjection onto η, this means $\eta \leq \alpha$. So $\lambda^+ \setminus \bigcup_{\xi<\lambda} A^\xi_\alpha \subset \alpha + 1$, which proves (b). ⊣

PROOF OF THEOREM 1.6. Let m be a measure over κ. We have remarked above that real-valued measurable cardinals are regular, so we just need to show that κ is a limit cardinal. Otherwise $\lambda^+ = \kappa$ for some $\lambda < \kappa$, so we may fix an Ulam matrix $\langle A^\xi_\alpha \mid \xi < \lambda, \alpha < \lambda^+\rangle$. Now since $\lambda < \kappa$, κ-additivity and property (b) implies that the union of each column, $\bigcup_{\xi<\lambda} A^\xi_\alpha$ for each α, has m-measure 1. So again by κ-additivity there must be a ξ_α, for each $\alpha < \lambda^+$, with positive m-measure. Now by the pigeonhole principle, there is some fixed $\xi < \lambda$ so that

$$S = \{\alpha < \lambda^+ \mid \xi_\alpha = \xi\}$$

has size λ^+. So $\{A^\xi_\alpha \mid \alpha \in S\}$ is a λ^+-sized collection with $m(A^\xi_\alpha) > 0$ for all $\alpha \in S$. But then the following exercise contradicts (b) in the definition of Ulam matrix:

EXERCISE 1.8. Whenever m is a measure over a set S and $\{B_\alpha \mid \alpha < \omega_1\}$ is an uncountable sequence of sets with $m(B_\alpha) > 0$, we must have $m(B_\alpha \cap B_\beta) > 0$ for some $\alpha < \beta < \omega_1$.

⊣

§2. The Axiom of Choice. We have seen that choice implies the existence of pathological sets; but we would like to develop the basics of analysis, and we need at least some choice to even show, e.g., that the countable union of countable sets is countable. We here recall the statement of AC, and introduce some weakenings.

DEFINITION 2.1. The Axiom of Choice (AC) states that whenever $\{A_i\}_{i\in I}$ is a collection of non-empty sets, there is a function $f : I \to \bigcup_{i\in I} A_i$ such that $f(i) \in A_i$ for each $i \in I$. Such a function is called a **choice function** for the family $\{A_i\}_{i\in I}$.

We can restrict the Axiom of Choice by requiring the sets A_i to be a subset of some fixed set A, or by fixing the index set I. The following definition allows for both of these restrictions.

DEFINITION 2.2. Let $\mathsf{AC}_I(A)$ be the axiom which states: whenever $\{A_i\}_{i\in I}$ is a collection of non-empty subsets of A, there is a function $f : I \to A$ such that $f(i) \in A_i$ for all $i \in I$.

The **Axiom of Countable Choice**, abbreviated AC_ω, states: for all sets A, $\mathsf{AC}_\omega(A)$ holds.

Let's see AC_ω in action. For the following theorem, recall that a set B is infinite if there is no surjection $f : n \to B$ with $n \in \omega$.

THEOREM 2.3. *Assume AC_ω. If B is infinite, there is an injection $f : \omega \to B$.*

PROOF. For each $n \in \omega$, let $\mathcal{B}_n$ be the collection of subsets of B containing exactly 2^n elements; note that $\mathcal{B}_n$ is non-empty for each n since B is infinite. By AC_ω, there is a choice function $g : \omega \to \bigcup_{n\in\omega} \mathcal{B}_n$ so that for all n, $g(n) \in \mathcal{B}_n$.

Now define A_n by

$$A_n = g(n) \setminus \bigcup_{i<n} g(i).$$

Note that $|\bigcup_{i<n} g(i)| \le \Sigma_{i<n}|g(i)| = \Sigma_{i<n}2^i = 2^n - 1$. Since $|g(n)| = 2^n$, it follows that A_n is non-empty, and the A_n are pairwise disjoint. Then again by AC_ω, we obtain a choice function $f : \omega \to \bigcup_{n\in\omega} A_n$, which is the desired injection into B. ⊣

Some choice is necessary to prove this! Ditto the next theorem:

THEOREM 2.4. *Assume AC_ω. Let $\{A_n\}_{n\in\omega}$ be a collection of countable sets. Then $\bigcup_{n\in\omega} A_n$ is countable.*

PROOF. We may assume some A_n is non-empty; then we need to find a surjection $f : \omega \to \bigcup_{n\in\omega} A_n$. We know for each n there is some surjection $g : \omega \to A_n$; the problem is picking a collection of such g_n for all n simultaneously.

Let $\mathcal{F}_n = \{h \in {A_n}^\omega \mid h \text{ is surjective}\}$. By assumption, each $\mathcal{F}_n$ is non-empty, so by AC_ω, we obtain a choice function $g : \omega \to \bigcup_{n\in\omega} \mathcal{F}_n$. Fix some $a \in \bigcup_{n\in\omega} A_n$. Let f be defined by setting

$$f(n) = \begin{cases} g(i)(j) & \text{if } n = 2^i 3^j \text{ for some } i, j \in \omega; \\ a & \text{otherwise.} \end{cases}$$

Then it is easy to see that $f : \omega \to \bigcup_{n\in\omega} A_n$ is onto. ⊣

From now on, we take the Axiom of Countable Choice for granted, and won't draw attention to its use.

DEFINITION 2.5. The **Principle of Dependent Choices**, abbreviated DC, states: Suppose R is a binary relation on a non-empty set X so that for all $x \in X$ there is an element $y \in X$ with $x\,R\,y$. Then there is a sequence $\langle x_n\rangle_{n\in\omega}$ of elements of X so that $x_n\,R\,x_{n+1}$ for all $n \in \omega$.

First observe this follows from AC: if R is such a relation, then for each $x \in X$, let $A_x = \{y \in X \mid x\,R\,y\}$. Each A_x is non-empty, and so we obtain a choice function $f : X \to X$ with $x\,R\,f(x)$ for all $x \in X$. The desired sequence is obtained by letting $x_0 \in X$ be arbitrary, and inductively letting $x_{n+1} = f(x_n)$.

THEOREM 2.6. *Assume* DC*. Then* AC_ω *holds.*

PROOF. Let $\{A_n\}_{n\in\omega}$ be a collection of non-empty sets. Let X be the set of functions f so that each $f \in X$ has $\mathrm{dom}(f) \in \omega$, and $f(n) \in A_n$ for each n.

Now define R on X by letting $f\,R\,g$ if $\mathrm{dom}(g) = \mathrm{dom}(f) + 1$ and $f \subseteq g$. It's easy to see that given $f \in X$, we may fix $a \in A_{\mathrm{dom}(f)}$ and obtain $f\,R\,f^\frown\langle a\rangle$. By DC, we obtain a sequence $\langle f_n\rangle_{n\in\omega}$ with $f_n\,R\,f_{n+1}$ for all n. Then $f = \bigcup_{n\in\omega}$ is the desired choice function. ⊣

Heuristically, DC is like a "dynamic" version of AC_ω: it says that we can make countably many choices without knowing in advance where the choices have to come from. We have an alternate characterization of DC in terms of trees.

DEFINITION 2.7. Given a non-empty set X, we let $X^{<\omega}$ denote the set of finite sequences of elements of X. A non-empty set $T \subseteq X^{<\omega}$ is a **tree** (on X) if it is closed under initial segment: if $t \in T$ and $s \subseteq t$, then $s \in T$.

The **body** of a tree T is

$$[T] = \{f \in X^\omega \mid (\forall n) f \restriction n = \langle f(0), \ldots, f(n-1)\rangle \in T,$$

the set of **infinite branches** through T.

For $s \in T$, T_s is the **subtree of T with stem** s, $T_s = \{t \in T \mid s \subseteq t \vee t \subseteq s\}$.

A tree T on a set X is **finitely branching** if for all $s \in T$, the set $\{x \in X | s^\frown\langle x\rangle \in T\}$ of immediate successors of s in T is finite.

LEMMA 2.8 (König). *Let T be an infinite finitely branching tree. Then T has an infinite branch.*

PROOF. Let $S \subseteq T$ be the set of elements s of T so that T_s is infinite. Define R on S by setting $s\,R\,t$ if $s \subseteq t$.

Since $T_s = \bigcup\{T_{s^\frown\langle x\rangle} \mid s^\frown\langle x\rangle \in T\}$ and T is finitely branching, we have for each $s \in S$ some x so that $T_{s^\frown\langle x\rangle}$ is infinite. Then $s^\frown\langle x\rangle \in S$, and $s\,R\,s^\frown\langle x\rangle$.

By DC, there is an infinite sequence $\langle s_n\rangle_{n\in\omega}$ so that $s_n \in S$, and $s_n \subseteq s_{n+1}$ for all n. Then $f = \bigcup_{n\in\omega} s_n$ is an infinite branch through T. ⊣

Recall that a binary relation R on X is **wellfounded** if every non-empty subset of X has an R-minimal element. It is not too hard to see that DC is equivalent to the usual characterization of wellfoundedness: R is wellfounded if there is no infinite R-descending sequence $\langle x_n \mid n \in \omega\rangle$.

We say a tree is **wellfounded** if the relation $\supsetneq$ is a wellfounded relation on T. A tree is **ranked** if there is a function $rho : T \to \mathrm{ON}$ so that $s \subsetneq t \in T$ implies $\rho(s) > \rho(t)$.

THEOREM 2.9. *A tree T is wellfounded if and only if it is ranked.*

PROOF. If $\rho : T \to \mathrm{ON}$ is a ranking function, then clearly T is wellfounded, since if $A \subseteq T$ is nonempty then by the definition of rank function, any $s \in A$ with $\rho(s)$ minimal in $\rho[A]$ is $\supsetneq$-minimal in A.

We prove the converse using transfinite induction. Let $T_0 = T$; for limit λ, let $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$. For successors,

$$T_{\alpha+1} = \{s \in T_\alpha \mid (\forall t \supsetneq s) t \notin T_\alpha\}.$$

That is, we obtain $T_{\alpha+1}$ by deleting all terminal nodes from T_α. This process stabilizes; i.e. $T_{\alpha+1} = T_\alpha$ for some α (since otherwise we get an injection . And since T is wellfounded, this T_α must be empty.

Define ρ on T by setting $\rho(s) = \min\{\alpha \mid s \notin T_{\alpha+1}\}$. It's easy to check this is a rank function. ⊣

It's not too hard to see that the ρ we defined takes the minimum possible values on T, and that its range is onto some ordinal $\alpha < |T|^+$.

Exercise 2.10. (DC) A tree T is wellfounded iff $[T] \neq \varnothing$.

We will see that trees are a fundamental object in our study of sets of reals, and the prime ingredient for obtaining nice representations for such sets, which yield many structural consequences.

§3. Baire Space and Cantor Space. We regard the set ω^ω of functions $x : \omega \to \omega$ as a topological space by taking as a basis the collection of all sets of the form

$$N_s = \{x \in \omega^\omega \mid s \subseteq x\} = [(\omega^{<\omega})_s],$$

where $s \in \omega^{<\omega}$. So endowed, ω^ω is called **Baire space**. Note that this is just the usualy product topology on ω^ω induced by the discrete topology on ω. It is left as an exercise to show Baire space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, which to some extent justifies our preoccupation with these "set theorists' reals".

Every open set U in Baire space is then of the form $U = \bigcup_{s\in B} N_s$ with $B \subseteq \omega^{<\omega}$. We regard open subsets of ω^ω as the simplest subsets of ω^ω, because they come with a finite certificate for membership: $x \in U$ if and only if $x \restriction n \in B$ for some $n \in \omega$.

As an example, the set $U_0 = \{x \in \omega^\omega \mid (\exists n)x(n) = 3\}$ is open. Its complement, $U_1 = \{x \in \omega^\omega \mid (\forall n)x(n) \neq 3\}$, is not open, since every $s \in \omega^{<\omega}$ extends to some $x \in U_0$. Similarly, the set $U_2 = \{x \in \omega^\omega \mid \Sigma_{n<\omega} x(n) > 9{,}000\}$ is open, but its complement is not.

We may similarly put a topology on 2^ω by using as a basis the sets N_s, for $s \in 2^{<\omega}$. Note that 2^ω is the set of branches through the infinite binary tree. We call 2^ω with this topology **Cantor space**.

A few remarks are in order regarding our basic open sets N_s. If $s \subseteq t$, then clearly $N_s \supseteq N_t$. If it is not the case that $s \subseteq t$ or $t \subseteq s$, then there must be some $i < \min\{|s|, |t|\}$ so that $s(i) \neq t(i)$. In this situation, $N_s \cap N_t = \varnothing$. We write $s \perp t$, and say s, t are **incompatible**.

We now isolate a number of important topological properties of the spaces $\omega^\omega, 2^\omega$.

Definition 3.1. A subset C of a topological space is **clopen** if it is both closed and open. A topological space is **totally disconnected** if it has a basis of clopen sets.

Proposition 3.2. *ω^ω and 2^ω are totally disconnected.*

Proof. Note that $\omega^\omega \setminus N_s = \bigcup_{t\perp s} N_t$. So N_s is clopen for all $s \in \omega^{<\omega}$. So ω^ω is totally disconnected; similarly for 2^ω. ⊣

We have the following simple characterization of convergence.

Proposition 3.3. *Let $\langle x_n \rangle_{n\in\omega}$ be a sequence in Baire space (or Cantor space). Then $\lim_{n\to\infty} x_n = x$ if and only if for all $m \in \omega$, there is some N so that $x_n \restriction m = x \restriction m$ for all $n \geq N$.*

Let X be a topological space. A set $D \subseteq X$ is **dense in** X if $U \cap D \neq \varnothing$ whenever $U \subseteq X$ is open. X is called **separable** if it has a countable dense subset. A set $D \subseteq \omega^\omega$ is dense if and only if $N_s \cap D \neq \varnothing$ for all $s \in \omega^{<\omega}$; that is, if for all s there is some $x \in D$ with $s \subseteq x$. It follows that if D_0 is the set of eventually zero sequences in ω^ω, then D_0 is dense, and clearly countable. So ω^ω (and 2^ω) is separable.

Let us also mention that ω^ω and 2^ω can be regarded as metric spaces. Define, for $x, y \in \omega^\omega$,

$$d(x,y) = \begin{cases} 0 & \text{if } x = y; \\ 2^{-n}, \text{where } n \text{ is least so that } x(n) \neq x(y) & \text{if } x \neq y. \end{cases}$$

The reader should verify that this is a metric on ω^ω (2^ω), and that it generates the topology of Baire space (Cantor space).

The following characterization of closed sets in Baire space is fundamental. We say that a tree T is **pruned** if it has no terminal nodes.

Theorem 3.4. *A set $C \neq \varnothing$ is closed in Baire space (or Cantor space) if and only if $C = [T]$ for some pruned tree $T \subseteq \omega^{<\omega}$ ($2^{<\omega}$).*

Proof. If C is closed, set $T = \{\varnothing\} \cup \{x \restriction n \mid x \in C, n \in \omega\}$. It is immediate that T is a pruned tree, and $C \subseteq [T]$. Conversely, suppose $x \in [T]$. For each n, there is some $x_n \in C$ so that $x_n \restriction n = x \restriction n$, by definition of T. Then $x = \lim_{n\to\infty} x_n \in C$, since C is closed.

For the reverse, suppose T is a tree on ω; we need to show $[T]$ is closed as a subset of Baire space. Suppose $\lim_{n\to\infty} x_n = x$, where each $x_n \in [T]$. For each $m \in \omega$, we have some n so that $x_n \restriction m = x \restriction m$; in particular, $x \restriction m \in T$ for all m. This implies $x \in [T]$, as needed. $\dashv$

Recall that a set K in a topological space is **compact** if every open cover of K admits a finite subcover: that is, if $K \subseteq \bigcup_{i\in I} U_i$ for some collection $\{U_i\}_{i\in I}$ of open sets, then there is some finite $F \subseteq I$ with $K \subseteq \bigcup_{i\in F} U_i$. We mention two facts about compactness: First, if $C_0 \subseteq K \subseteq X$ with K compact and C_0 closed, then C_0 is compact. Second, if X is a metric space, then whenever K is compact, K is automatically closed (in particular, this holds for ω^ω and 2^ω).

The Heine-Borel Theorem states that a set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded. The following is the analogue for compact subsets of ω^ω, and has a similar proof.

Theorem 3.5. *A non-empty set $K \subseteq \omega^\omega$ is compact if and only $K = [T]$ for some finitely branching pruned tree T.*

Proof. Suppose first that K is compact. Then K is closed, and by Theorem 3.4, there is a pruned tree T on ω with $[T] = K$. We claim T is finitely branching. If not, there is some $s \in T$ so that $s^\frown\langle a\rangle \in T$ for infinitely many $a \in \omega$; let $\langle a_n\rangle_{n\in\omega}$ enumerate these a in increasing order. Note that $[T] \cap N_{s^\frown\langle a_n\rangle} \neq \varnothing$ for all n, by the assumption that T is pruned, and the sets $N_{s^\frown\langle a_n\rangle}$ are pairwise disjoint. It follows that $\{N_{s^\frown\langle a_n\rangle}\}_{n\in\omega}$ is an infinite cover of K with no finite subcover. This contradicts compactness of K, so T must be finitely branching.

Conversely, suppose T is a finitely branching pruned tree on ω. We claim $[T]$ is compact. Suppose otherwise. Then there is some collection $\{U_i\}_{i\in I}$ of open sets covering $[T]$, but so that no finite subcover covers T.

We now inductively construct $x \in [T]$ so that for all n, $[T_{x\restriction n}]$ cannot be covered by finitely many of the U_i. This gives the desired contradiction, since $x \in [T]$ implies $x \in U_i$ for some i; then by openness of U_i, there must be some n so that $[T_{x\restriction n}] \subseteq N_{x\restriction n} \subseteq U_i$.

For $n = 0$, we have by assumption that $[T_\varnothing] = [T]$ cannot be covered by finitely many of the U_i. Suppose inductively that we have defined $x \restriction n = \langle x(0), x(1), \dots x(n-1)\rangle$ so that $[T_{x\restriction n}]$ cannot be covered by finitely many of the U_i. Since T is finitely branching, we have $T_{x\restriction n} = \bigcup_{k<m} T_{(x\restriction n)^\frown\langle a_k\rangle}$ for some finite list $a_0, a_1, \dots, a_{m-1}$ of elements of ω. Suppose towards a contradiction that each $[T_{(x\restriction n)^\frown\langle a_k\rangle}]$ can be covered by $\{U_i\}_{i\in F_k}$ for some finite set $F_k \subseteq I$. Then we have $F = \bigcup_{k<m} F_k$ a finite set so that $[T_{x\restriction n}]$ is covered by $\{U_i\}_{i\in F}$, contradicting our inductive hypothesis.

So there must be some $k < m$ so that $[T_{(x\restriction n)^\frown\langle a_k\rangle}]$ cannot be covered by finitely many of the U_i. Set $x(n) = a_k$; by induction, we obtain the desired x. $\dashv$

§4. Regularity properties of sets of reals. In this section we introduce three regularity properties of sets of reals of central importance; we will see later that all of these are implied by a powerful axiom involving infinite games.

4.1. Meager sets and the Baire property.

Definition 4.1. Let X be a topological space. We say that $A \subseteq X$ is **nowhere dense** if for every non-empty open $U \subseteq X$, there is a non-empty open V with $V \subseteq U$ and $V \cap A = \varnothing$.

So a set $A \subseteq \omega^\omega$ is nowhere dense if for every $s \in \omega^{<\omega}$, there is some extension $t \supseteq s$ with $N_t \cap A = \varnothing$. For example, the set U_1 defined earlier is nowhere dense. Also, 2^ω is nowhere dense as a subset of ω^ω (but not as a subset of itself, of course!).

Any nowhere dense set can be enlarged a bit and still be nowhere dense.

Proposition 4.2. *If $A \subseteq X$ is nowhere dense, then so is its closure $\bar{A} = A \cup \{x \in X \mid x$ is a limit point of $A\}$.*

Proof. This follows immediately from the fact that if V is an open set with $V \cap A = \varnothing$, then $V \cap \bar{A} = \varnothing$. $\dashv$

The reader should verify that a finite union of nowhere dense sets is nowhere dense. Of course, it is not the case that a countable union of nowhere dense sets is nowhere dense. For example, a countable dense set is a countable union of singletons, each of which is nowhere dense. The following notion of smallness is more robust.

Definition 4.3. A set $A \subseteq X$ is **meager** if it is contained in the union of countably many nowhere dense sets.

It is immediate that if $M_0 \subseteq M$ with M meager, M_0 is meager as well. Nowhere dense sets are obviously meager. And since countable unions of countable sets are countable, we have that the countable union of meager sets is meager. In particular, countable sets are meager. This is a good thing: we think of countable sets as intuitively small.

Of course, one set that shouldn't be small is ω^ω itself. This is the content of the following theorem.

THEOREM 4.4 (The Baire Category Theorem). *The space ω^ω is not meager as a subset of itself. (Similarly for 2^ω.)*

PROOF. We have to show that no countable union of nowhere dense subsets of ω^ω is equal to all of ω^ω. Let $\{C_n\}_{n\in\omega}$ be a countable collection of nowhere dense sets. Replacing the sets C_n with their closures $\bar{C_n}$ if necessary, we may assume each C_n is closed. Then $U_n = \omega^\omega \setminus C_n$ is an open dense set.

To show $\bigcup_{n\in\omega} C_n$ is not all of ω^ω, it is sufficient to show $\bigcap_{n\in\omega} U_n$ is non-empty. In fact we can do even better.

CLAIM. *The countable intersection of dense open sets is dense.*

Fix $s \in \omega^{<\omega}$. We need to show $N_s \cap \bigcap_{n\in\omega} U_n$ is non-empty. Set $s_0 = s$. Suppose inductively that s_n has been defined. Since U_n is dense open, the set $U_n \cap N_{s_n}$ is open and non-empty, so there is some proper extension $s_{n+1} \supseteq s_n$ so that $N_{s_{n+1}} \subseteq U_n$.

Put $x = \bigcup_{n\in\omega} s_n$. Clearly $x \in N_s = N_{s_0}$. And for each n, we have $x \in N_{s_n} \subseteq U_n$. It follows that $x \in N_s \cap \bigcap_{n\in\omega} U_n$. This proves the claim, and the theorem. ⊣

A set A in ω^ω is called **comeager** if its complement $\omega^\omega \setminus A$ is meager. A countable intersection of open sets is called a G_δ set. By the proof just given, the countable intersection of open dense sets is always a dense G_δ set. Thus a set is comeager if and only if it contains a dense G_δ.

We think of the meager sets as small, or thin. The next class of sets we define are those which are just a meager set away from being open.

DEFINITION 4.5. Let X be a Polish space. A set $B \subseteq X$ has the **Baire property** if there is some open set $U \subseteq X$ such that $B \triangle U$ is meager. (Here $B \triangle U$ is the symmetric difference $(B \setminus U) \cup (U \setminus B)$.)

This is our first regularity property. Let us see that the Baire property is preserved under the basic set operations of complement and countable union. The next definition is fundamental.

DEFINITION 4.6. A collection $\mathcal{F} \subseteq \mathcal{P}(S)$ is called a **σ-algebra** on S if

- For all $A \in \mathcal{F}$, we have $S \setminus A \in \mathcal{F}$, and
- Whenever $A_n \in \mathcal{F}$ for $n \in \omega$, we have $\bigcup_{n\in\omega} A_n \in \mathcal{F}$.

The reader should check that a σ-algebra is also closed under countable intersections.

Given any $\mathcal{C} \subseteq \mathcal{P}(S)$, there is a smallest σ-algebra $\mathcal{F}$ with $\mathcal{C} \subseteq \mathcal{F}$, the **$\sigma$-algebra generated by** $\mathcal{C}$. Of particular importance is the **Borel σ-algebra** on a topological space X; this is the σ-algebra generated by the collection of open sets. The subsets of X in the Borel σ-algebra are the **Borel** subsets of X.

PROPOSITION 4.7. *The collection of subsets of ω^ω with the Baire property forms a σ-algebra; in particular, every Borel set has the Baire property.*

PROOF. It is clear that open sets have the Baire property, so we will be done if we show the first claim.

For countable unions, suppose $B_0, B_1, \ldots$ all have the Baire property. Thus for each B_n there is an open set U_n so that $B_n \triangle U_n$ is meager. It follows that the union $\bigcup_{n<\omega} B_n \triangle U_n$ is meager. Let $B = \bigcup_{n<\omega} B_n$ and let $U = \bigcup_{n<\omega} U_n$. Since $B \triangle U \subseteq \bigcup_{n\in\omega} B_n \triangle U_n$ we see that B has the Baire property.

Now for complements. Suppose B has the Baire property. We want to show $\omega^\omega \setminus B$ has the Baire property. Let U be open with $B \triangle U$ meager. Let C be the closure of U; so $\omega^\omega \setminus C$ is an open set. Notice that $(\omega^\omega \setminus B) \triangle (\omega^\omega \setminus U)$ is equal to $B \triangle U$. Also notice that $(\omega^\omega \setminus B) \triangle (\omega^\omega \setminus C) \subseteq (\omega^\omega \setminus B) \triangle (\omega^\omega \setminus U) \cup (C \setminus U)$. If we can show that $C \setminus U$ is nowhere dense we will be done.

Let $V \subseteq \omega^\omega$ be an open set; we want to find an open $W \subseteq V$ disjoint from $C \setminus U$. If $V \cap (C \setminus U)$ is empty there is nothing for us to do. Otherwise, let $x \in V \cap (C \setminus U)$. Then x is a limit point of U; hence by definition of a limit point $V \cap U$ is non-empty. Take $W = V \cap U$; then $W \cap (C \setminus U)$ is empty. $\dashv$

4.2. Null sets and Lebesgue measurability. At the outset we saw that Lebesgue's measure problem had a negative answer in the presence of choice. Lebesgue's way around this was to isolate a collection of sets well-behaved enough that measure values could be coherently defined, avoiding pathologies, but rich enough to accommodate most applications.

For each $A \subseteq \mathbb{R}$ we define the **outer measure of** A,

$$\mu^*(A) = \inf\left\{ \sum_{n<\omega} (b_n - a_n) \;\middle|\; A \subseteq \bigcup_{n<\omega} (a_n, b_n),\ \text{each } a_n, b_n \in \mathbb{R} \right\}.$$

Say a set A is **null** or has **Lebesgue measure zero** if $\mu^*(A) = 0$.

Proposition 4.8. *μ^* has the following properties:*

1. *$\mu^*(\mathbb{R}) = \infty$; $\mu^*([0,1]) = 1$;*
2. *$\mu^*(A + r) = \mu^*(A)$, for all $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$;*
3. *(countable subadditivity) For all $\{A_n \mid n < \omega\}$, $\mu^*(\bigcup_{n<\omega} A_n) \leq \sum_{n<\omega} \mu^*(A_n)$.*

Note that Vitali's example provides an instance where a strict inequality holds in (3).

Definition 4.9. A set $A \subseteq \mathbb{R}$ is **Lebesgue measurable** if there is a Borel set B such that $\mu^*(A \triangle B) = 0$. In this case the **Lebesgue measure of** A is $\mu(A) = \mu^*(A)$. Let $\mathcal{L}$ denote the collection of Lebesgue measurable subsets of $\mathbb{R}$.

We remark here that a similar notion of measurability is defined for 2^ω or ω^ω by starting with some coherently defined outer measure (e.g. coin-flipping measure for 2^ω; for ω^ω, $\mu^*(N_s) := \prod_{i<|s|} 2^{-(s(i)+1)}$), and we will have such a development in mind when we talk about Lebesgue measurable subsets of Cantor or Baire space. We catalog some properties of Lebesgue measure.

Proposition 4.10. *The union of countably many measure zero sets has measure zero.*

Proposition 4.11. *$\mathcal{L}$ is a σ-algebra, and is in fact the smallest σ-algebra containing the Borel sets and the sets of outer measure zero.*

Proposition 4.12. *$\mathcal{B} \neq \mathcal{L}$.*

Theorem 4.13. *μ satisfies countable additivity on $\mathcal{L}$.*

Theorem 4.14. *For all $A \in \mathcal{L}$ and $\varepsilon > 0$, there exist a closed F and open G with $F \subseteq A \subseteq G$, such that $\mu(G \setminus F) < \varepsilon$.*

Consequently, we have that for all $A \in \mathcal{L}$ there is a G_δ set $H \supseteq A$ with $\mu(H \setminus A) = 0$.

Definition 4.15. A set $P \subseteq \omega^\omega$ is **perfect** if it is closed and has no isolated points: $P' = P$.

Claim. *If P is non-empty and perfect, then $|P| = 2^{\aleph_0}$; in fact there is a continuous injection $i : 2^\omega \to P$.*

Proof. Let $P = [T]$ for some pruned tree $T \subseteq \omega^{<\omega}$. Note that perfection of P implies for all $s \in T$, there exist incompatible extensions $s_0, s_1 \supset s$ in T. This allows us to inductively define a map $i_0 : 2^{<\omega} \to T$ such that

- $s \subsetneq t$ implies $i_0(s) \subsetneq i_0(t)$,
- $s \perp t$ implies $i_0(s) \perp i_0(t)$.

We thus obtain a copy of the complete binary tree $2^{<\omega}$ inside T. For $x \in 2^\omega$ set $i(x) = \bigcup_{n\in\omega} i_0(x \upharpoonright n)$; it is easy to check that this is continuous and injective. ⊣

Theorem 4.16 (Cantor-Bendixon). *If $F \subseteq \omega^\omega$ is closed, then F may be written as a union $F = P \cup C$ with P perfect and C countable.*

In particular, every uncountable closed set has size continuum: There are no closed counterexamples to the continuum hypothesis.

Proof. Exercise. The proof closely resembles that of 2.9. ⊣

Definition 4.17. $A \subseteq \omega^\omega$ has the **perfect set property** if it is countable or contains a perfect set.

Theorem 4.18. (AC) *There is a set $A \subseteq \omega^\omega$ without the perfect set property.*

Proof. Note that every closed set is a countable intersection of basic open neighborhoods; so there are precisely $2^{\aleph_0}$ closed subsets of ω^ω. Let $\langle F_\alpha \rangle_{\alpha<\omega_1}$ be an enumeration of all closed $F \subseteq \omega^\omega$ such that $|F| = |\omega^\omega \setminus F| = 2^{\aleph_0}$.

We define by transfinite induction disjoint sequences $\langle a_\alpha \rangle_{\alpha<2^{\aleph_0}}$, $\langle b_\alpha \rangle_{\alpha<2^{\aleph_0}}$, satisfying, for all α,

- $a_\alpha \notin F_\alpha$,
- $b_\alpha \in F_\alpha$,
- $a_\alpha, b_\alpha \notin \bigcup_{\xi<\alpha}\{a_\xi, b_\xi\}$.

Note that such a_α and b_α may always be found by our assumption on the F_α's. We claim $A = \{a_\alpha \mid \alpha < 2^{\aleph_0}\}$ is the desired set. Clearly A is uncountable, and since it contains no b_α, so is its complement. If $P \subseteq A$ was a perfect set, then we would have $P \cap \{b_\alpha \mid \alpha < 2^{\aleph_0}\} = \varnothing$, so that $P = F_\alpha$ for some $\alpha < 2^{\aleph_0}$. But $b_\alpha \in C_\alpha \setminus A$, contradiction. ⊣

§5. Games, strategies, and the Axiom of Determinacy. We will see that the regularity properties of the previous section may be obtained from winning strategies in certain infinite games. Our primary interest will be games played with natural number moves, but we give our definitions in more generality.

Definition 5.1. Let T be a tree on a set X, and let $A \subseteq [T]$. The **game on T with payoff A**, denoted $G(A; T)$, is played as follows: two players, Player I and Player II, alternate choosing elements of X,

$$\begin{array}{c|ccccccc} \mathrm{I} & x_0 & & x_2 & \dots & x_{2n} & & \dots \\ \hline \mathrm{II} & & x_1 & & \dots & & x_{2n+1} & \dots \end{array}$$

so that for all n, $\langle x_0, \dots, x_{n-1} \rangle$ is an element of T. The game ends if either a terminal node of T is reached, or if an infinite branch $\langle x_0, x_1, \dots \rangle \in [T]$ is produced. A sequence

s is a **play in** T if s is terminal in T or $s \in [T]$ is an infinite branch. Player I **wins the play** s if either

- $s \in T$ is a terminal node, and $|s|$ is odd;
- $s \in [T]$ is an infinite branch, and $s \in A$.

Otherwise, Player II wins the play s.

When $T = \omega^{<\omega}$, we write $G(A)$ for the game $G(A;T) = G(A;\omega^{<\omega})$.

In the game $G(A;T)$, Player I is trying to produce a branch f through T with $f \in A$; Player II is trying to ensure $f \notin A$. If a terminal node is reached, then the last player who made a move is the winner.

Intuitively, a strategy for Player I in the game $G(A;T)$ should be a function that takes positions s of even length as input, and tells Player I what move to make next. There are a number of equivalent ways to formalize this. We elect to regard strategies as *trees.*

DEFINITION 5.2. Let T be a tree. A **strategy for Player I in** T is a set $\sigma \subseteq T$ so that

(1) σ is a tree.
(2) If $s \in \sigma$, $|s|$ is odd, and $x \in X$ is such that $s^\frown\langle x\rangle \in T$, then $s^\frown\langle x\rangle \in \sigma$.
(3) If $s \in \sigma$ and $|s|$ is even, then there is a unique $x \in X$ so that $s^\frown\langle x\rangle \in \sigma$.

We say an infinite play f is **compatible with** σ if $f \in [\sigma]$; a strategy σ is **winning for Player I in** $G(A;T)$ if $[\sigma] \subseteq A$ (that is, every play compatible with σ is winning for Player I).

Strategies τ for Player II are defined similarly (exchanging "even" with "odd"); τ is winning for Player II in $G(A;T)$ if $[\tau] \cap A = \varnothing$.

So a strategy for Player I is a subtree σ of T that picks out moves for Player I, but puts no restrictions on moves for Player II. We will often abuse notation and regard σ as a function, writing $\sigma(s) = x$ for the unique element guaranteed by (3).

Note that (3) implies that no finite play in a strategy is won by the opponent. It is then not obvious at this stage that given a tree T, a strategy in T (winning or not) exists for *either* player!

DEFINITION 5.3. Let T be a tree on X with $A \subseteq [T]$. If one of the players has a winning strategy in $G(A;T)$, then we say the game is **determined**.

When $T = \omega^{<\omega}$, we often say simply that $A \subseteq \omega^\omega$ **is determined**.

Let AD_X denote the statement that for every set $A \subseteq X^{<\omega}$, the game $G(A;X^{<\omega})$ is determined. The **Axiom of Determinacy**, denoted AD, is AD_ω: *Every set* $A \subseteq \omega^\omega$ *is determined.*

Note that every strategy in $\omega^{<\omega}$ is a subset of $\omega^{<\omega}$, so that the collection of strategies in $\omega^{<\omega}$ has size at most $2^{\aleph_0}$; furthermore, for each strategy σ in $\omega^{<\omega}$, the set $[\sigma]$ of plays compatible with σ has size $2^{\aleph_0}$.

Our first observation is that not all sets are determined.

THEOREM 5.4. *If there is a wellorder of* ω^ω*, then there is a set* $B \subseteq \omega^\omega$ *so that* $G(B)$ *is not determined; in particular, if* AC *holds, then* AD *fails.*

PROOF. Exercise. The argument closely resembles that of Theorem 4.18. ⊣

THEOREM 5.5 (Gale-Stewart). *(Using the Axiom of Choice.) Let* T *be a tree. Then* $G([T];T)$ *is determined.*

To help us prove this theorem, we introduce a more general notion of strategy.

Definition 5.6. Let T be a tree. A **quasistrategy for Player I in** T is a tree $S \subseteq T$ satisfying (1) and (2) in Definition 5.2, but instead of (3), satisfying

(3′) If $s \in S$ has odd length, then there is some $x \in X$ so that $s^\frown\langle x\rangle \in S$.

A quasistrategy can be thought of as a "multi-valued strategy". Quasistrategies are typically obtained from the following lemma:

Lemma 5.7. *(Using the Axiom of Choice.) Let T be a tree on X, $A \subseteq [T]$, and suppose Player II does not have a winning strategy in $G(A;T)$. Define*

$$S = \{s \in T \mid (\forall i \leq |s|) \text{ Player II has no winning strategy in } G(A;T_{s\restriction i})\}.$$

Then S is a quasistrategy for Player I in T, the ***non-losing quasistrategy for I in*** *$G(A;T)$.*

Proof. Clause (1) of the definition of quasistrategy is immediate; closure under initial segment follows from the definition of S, and the assumption that Player II has no winning strategy ensures S is non-empty, a requirement for S to be a tree.

For clause (2), suppose $s \in S$ has odd length. If $s^\frown\langle x\rangle \in T \setminus S$ for some $x \in X$, then by definition of S, there is some strategy $\tau \subseteq T_{s^\frown\langle x\rangle}$ that is winning for Player II in the game $G(A;T_{s^\frown\langle x\rangle})$. But clearly $\tau \subseteq T_s$ is also winning for Player II in $G(A;T_s)$, contradicting our assumption that $s \in S$.

The key part of the proof is clause (3′). So suppose $s \in S$ has even length. We claim $s^\frown\langle x\rangle \in S$ for some $x \in X$. Suppose instead for a contradiction, that for each $x \in X$, there is some strategy τ_x in $T_{s^\frown\langle x\rangle}$ that is winning for Player II in $G(A;T_{s^\frown\langle x\rangle})$. Define a strategy $\tau \subseteq T_s$ for Player II by setting

$$t \in \tau \iff t \subseteq s \text{ or } (\exists x)(s^\frown\langle x\rangle \subseteq t \text{ and } t \in \tau_x).$$

Note τ does not restrict Player I's move at s, so τ is a strategy for Player II. If f is a play compatible with τ, then we have $s^\frown\langle x\rangle \subseteq f$ for some $x \in X$, so that f is compatible with τ_x. It follows that f is a win for II in $G(A;T_s)$, and τ is a winning strategy for Player II in $G(A;T_s)$, contradicting our assumption that $s \in S$. ⊣

Proof of Theorem 5.5. Let T be a tree, and suppose Player II does not have a winning strategy in $G([T];T)$. By Lemma 5.7, we obtain a quasistrategy S for Player II. This can be refined to a strategy $\sigma \subseteq S$ for I, by choosing a single successor node at each $s \in S$ of even length. It is clear that σ is winning for Player I. ⊣

We remark that when $X = \omega$, it was enough in the proof to assume $\mathsf{AC}_\omega(\mathbb{R})$. However, we can get away with even less choice; in particular, it is sufficient to assume X can be well-ordered, so that the determinacy of $G([T];T)$ follows without choice if e.g. $T \subseteq \alpha^{<\omega}$ for some ordinal α.

For the record we state closed determinacy for Baire space separately; it follows immediately from the more general Gale-Stewart theorem as we have stated it.

Theorem 5.8. *Let $C \subseteq \omega^\omega$ be closed. Then $G(C)$ is determined.*

Proof. Fix a tree T on ω so that $C = [T]$. Let T' be the tree defined by

$$T' = \{s \in \omega \mid s \in T \text{ or } (\exists n) s = t^\frown\langle n\rangle \text{ for some } t \in T \text{ with } |t| \text{ odd.}\}$$

Note that $[T'] = [T] = C$, and all terminal nodes in T' have even length. By the Gale-Stewart Theorem 5.5, the game $G([T'];T')$ is determined. The reader can check that if Player I has a winning strategy σ, then this is a winning strategy for I in $G(C) = G(C;\omega^{<\omega})$; and a winning strategy for II in T' can be extended to one for II in $\omega^{<\omega}$. $\dashv$

§6. The Baire Property and the Banach-Mazur Game. Recall that a set $A \subseteq X$ is meager in X if it is contained in some countable union of nowhere dense sets; A is comeager in X if its complement is meager. Note that meagerness is a relative notion, in the sense that a set meager in X may not be meager in a subset of X; for example, 2^ω is non-meager in 2^ω by the Baire category theorem, but is meager as a subset of ω^ω. The following proposition shows that meagerness persists between a space and its subsets, provided those subsets are *open*.

PROPOSITION 6.1. *Suppose $X \subseteq Y$ and X is an open set in Y. Then $A \subseteq X$ is meager in X exactly when it is meager in Y.*

PROOF. First suppose A is meager as a subset of X. Then $A \subseteq \bigcup_{n\in\omega} C_n$ where each $C_n \subseteq X$ is nowhere dense as a subset of X. We claim each C_n is also nowhere dense as a subset of Y, from which it follows that A is meager in Y. For if $U \subseteq Y$ is a non-empty open set, then $U \cap X$ is open in X. If it is empty, then we already have $U \cap C_n = \varnothing$; otherwise, since C_n is nowhere dense there is $V_0 \subseteq U \cap X$ which is open in X and disjoint from C_n. Then we have $V_0 = V \cap X$ for some open $V \subseteq Y$, and we easily see that V is disjoint from C_n as needed.

Going the other way, suppose A is meager as a subset of Y. Then $A \subseteq \bigcup_{n\in\omega} C_n$, where each C_n is nowhere dense as a subset of Y. The reader may check that $C_n \cap X$ is nowhere dense as a subset of X, which is enough. $\dashv$

PROPOSITION 6.2. *Suppose $B \subseteq \omega^\omega$ has the Baire property. Then either B is meager or there is some s such that $B \cap N_s$ is comeager in the topology on N_s.*

PROOF. Since B has the Baire property there is some open set U such that $(B \setminus U) \cup (U \setminus B)$ is meager. If U is empty then B is meager. So assume U is non-empty. Let $N_s \subseteq U$. Since $U \setminus B$ is meager so is $N_s \setminus B$.

We have $N_s = (B \cap N_s) \cup (N_s \setminus B)$. Since $N_s \setminus B$ is meager in N_s by the last proposition, the former set is comeager in N_s. $\dashv$

DEFINITION 6.3. We say that $A \subseteq \omega^\omega$ is a **tail set** if for every $x, y \in \omega^\omega$ if there exists some $k \in \omega$ such that $x(j) = y(j)$ for all $j > k$, then x belongs to A exactly when y belongs to A.

In other words, if we consider the equivalence relation E_0 defined by $x \, E_0 \, y$ if and only if there exists some $k \in \omega$ such that $x(j) = y(j)$ for all $j > k$, then a tail set is one which is a union of E_0 equivalence classes.

THEOREM 6.4. *If A is a tail set with the Baire property, then A is either meager or comeager.*

PROOF. Let us suppose towards a contradiction that A is neither comeager nor meager. Since A has the Baire property, there is by Proposition 6.2 some s so that $A \cap N_s$ is comeager in N_s. By the same reasoning applied to the complement of A, there is some t so that $(\omega^\omega \setminus A) \cap N_t$ is comeager in N_t; that is, $A \cap N_t$ is meager in N_t.

Extending one of s, t if necessary, we may by Proposition 6.1 assume $|s| = |t| = k$. Consider the map $\varphi : N_s \to N_t$ defined by

$$\varphi(x)(i) = \begin{cases} t(i) \text{ if } i < k, \\ x(i) \text{ otherwise.} \end{cases}$$

This is clearly a homeomorphism. Thus the image of $A \cap N_s$ under φ should be comeager in N_t. But in fact $\varphi(x) \in A$ if and only if $x \in A$ because A is a tail set. Then $\varphi[A \cap N_s] = A \cap N_t$ with the latter meager in N_t, a contradiction! ⊣

The same argument of course works for subsets of 2^ω.

This theorem is handy in immediately identifying that certain sets are meager or comeager. For example, the set $\{x \in \omega^\omega : \lim_{n\to\infty} x(n) = \infty\}$ is a tail set and is Borel, so has the Baire property. So without even thinking about it we know it must be either meager or comeager.

Under the Axiom of Choice there are sets which do not have the Baire property (exercise). On the other hand, under AD there are no such examples. This we aim to show next.

Definition 6.5. Let $A \subseteq \omega^\omega$. We define the **Banach-Mazur Game** $G_{\mathrm{BM}}(A)$ to be the game with moves in $\omega^{<\omega}$, played as follows: Player I plays s_0, Player II plays $s_1 \supsetneq s_0$, Player I plays $s_2 \supsetneq s_1$, and so on.

I	s_0		s_2		$\ldots$	s_{2n}		$\ldots$
II		s_1		s_3	$\ldots$		s_{2n+1}	$\ldots$

Figure 1. The Banach-Mazur game $G_{\mathrm{BM}}(A)$.

A play of the game is an increasing sequence $\langle s_n \rangle_{n\in\omega}$ of elements of s_n. Set $x = \bigcup_{n\in\omega} s_n$. Then Player I wins if $x \in A$; otherwise, Player II wins.

Claim. *A is meager if and only if Player II has a winning strategy in $G_{\mathrm{BM}}(A)$.*

Proof. First suppose that A is meager. Write $A \subseteq \bigcup_{n\in\omega} C_n$ where each C_n is nowhere dense. Player II's strategy essentially consists in proving the Baire category theorem with the sets C_n. Namely, given s_{2n}, since C_n is nowhere dense, there is some $s_{n+1} \supsetneq s_n$ with $N_{s_{n+1}} \cap C_n = \varnothing$. Playing in this fashion clearly produces a real $x \notin A$, so this strategy is winning for Player II.

Conversely, suppose that Player II has some winning strategy τ. For each position p in the Banach-Mazur game, let $s_p = \bigcup_{i<|p|} p(i)$ denote the node in $\omega^{<\omega}$ reached by p (so if $p = \langle s_0, \ldots, s_n \rangle$ then $s_p = s_n$, and $s_\varnothing = \varnothing$). For each even-length position $p \in \tau$, we define a set $D_p \subseteq \omega^\omega$ by

$$D_p = \bigcup \{N_t \mid t \perp s_p \text{ or } (\exists s \supsetneq s_p)\tau(p^\frown\langle s \rangle) = t\}.$$

So x belongs to D_p if and only if $x \perp p$, or there is some move by Player I at p which prompts τ to respond with an initial segment of x.

Clearly D_p is open; we claim it is dense. Fix $s \in \omega^{<\omega}$ with $|s| > |s_p|$. If $s \perp s_n$, then by definition of D_p we have $N_s \subseteq D_p$. Otherwise $s \supsetneq s_p$, so s is a legal move for Player I at p. Set $t = \tau(p^\frown\langle s \rangle)$. Then $t \supsetneq s$ and $N_t \subseteq D_p$ as needed.

Now $\bigcap_p D_p$ is a dense G_δ, so is comeager. Suppose $x \in \bigcap_p D_p$. Then we can inductively construct a play $\langle s_0, s_1, s_2, \ldots\rangle$ of $G_{\mathrm{BM}}(A)$ compatible with τ and so that $x = \bigcup_{n\in\omega} s_n$: let s_0 be a move by Player I witnessing $x \in D_\varnothing$, and s_1 τ's response. And inductively, set s_{2n} a witness to membership of x in $D_{\langle s_0,\ldots,s_{2n}\rangle}$, and $s_{2n+1} = \tau(\langle s_0, \ldots, s_{2n}\rangle) \subseteq x$.

Since τ is winning for Player II, $x \notin A$. So A is disjoint from a comeager set, hence meager. ⊣

Claim. *Player I has a winning strategy in $G_{\mathrm{BM}}(A)$ if and only if there is some $s \in \omega^{<\omega}$ with A comeager in N_s.*

Proof. The same arguments in the proof of the previous claim show this. Just note that after Player I plays a first move s_0, the game is essentially $G_{\mathrm{BM}}(N_{s_0} \setminus A)$ with the roles of the players reversed. ⊣

Claim. *Given $A \subseteq \omega^\omega$ there is an open set U_A such that if $G_{\mathrm{BM}}(A\setminus U_A)$ is determined, then A has the Baire property.*

Proof. Let $U_A = \bigcup\{N_s \mid A \text{ is comeager in } N_s\}$. We claim that I cannot have a winning strategy in $G_{\mathrm{BM}}(A \setminus U_A)$; supposing otherwise, we have by the last claim that $(A \setminus U_A)$ is comeager in N_s for some s. But then clearly A is comeager in N_s, so that $N_s \subseteq U_A$, so that $(A \setminus U_A) \cap N_s$ is empty, a contradiction.

Then if the game is determined it must be Player II who has the winning strategy. By the first claim $A \setminus U_A$ is meager. And $U_A \setminus A$ is also meager, being contained in the union of all the $N_s \setminus A$ for which this set is meager. So $A \triangle U_A = (A \setminus U_A) \cup (U_A \setminus A)$ is meager. Thus A has the Baire property. ⊣

Theorem 6.6. *Assume* AD. *Then every set $A \subseteq \omega^\omega$ has the Baire property.*

Proof. By the previous claims it is enough to see that $G_{\mathrm{BM}}(A)$ is determined for all $A \subseteq \omega^\omega$. But $G_{\mathrm{BM}}(A)$ can clearly be coded by a game on ω of the form $G(A^*)$, for example by fixing an enumeration $\langle t_i\rangle_{\in\omega}$ of $\omega^{<\omega} \setminus \{\varnothing\}$ and setting $x \in A^*$ if and only if $\varphi(x) = t_{x(0)}{}^\frown t_{x(1)}{}^\frown t_{x(2)}{}^\frown \cdots \in A$. ⊣

§7. Polish Spaces and Pointclasses. In the last section, we introduced the spaces ω^ω and 2^ω and isolated some useful topological properties of these. In this section, we abstract these properties into a definition of a class of structures that includes the spaces ω^ω, 2^ω, $\mathbb{R}$, as well as their products, and many others.

Recall that a metric space (X, d) is **complete** if every Cauchy sequence in X converges: that is, whenever $\langle x_n\rangle_{n\in\omega}$ is a sequence of elements of X so that for all ε, there is some N so that $m, n \geq N$ implies $d(x_m, x_n) < \varepsilon$, then $\lim_{n\to\infty} x_n$ exists in X. We leave it as an exercise to verify that Baire space and Cantor space are both complete when endowed with the metric defined in the last section.

Definition 7.1. A topological space $(X, \mathcal{T})$ is a **Polish space** if it is separable, and there exists a metric $d : X \times X \to \mathbb{R}$ that generates the topology $\mathcal{T}$ of X, and so that (X, d) is a complete metric space.

Notice that the definition of a Polish space asserts the existence of some complete metric, but there needn't be a unique such. The point is that if $\mathcal{T}$ is generated by a complete metric it will have nice properties, but we are more interested just in the topology than the particular metric generating it. Nonetheless, we typically suppress mention of the topology $\mathcal{T}$, using the domain set X to denote the Polish space $(X,\mathcal{T})$ when $\mathcal{T}$ is clear.

We have seen that $\mathbb{R},\omega^\omega$ and 2^ω are Polish spaces. So is ω with the discrete topology, as witnessed by the metric d on ω with $d(m,n)=1$ for all $m\neq n$. Furthermore, if X,Y are Polish spaces, then so is their product $X\times Y$. This can be seen by setting

$$d_{X\times Y}(\langle x_1,y_1\rangle,\langle x_2,y_2\rangle)=\max\{d_X(x_1,x_2),d_Y(y_1,y_2)\}$$

where d_X,d_Y are the complete metrics witnessing Polishness of X,Y, respectively.

We remark that the proof of the Baire category theorem in the last section was sufficiently general to go through for arbitrary Polish spaces. We obtain: *If X is Polish, then X is not meager in itself.*

The following theorem gives an indication of the important status of ω^ω among all Polish spaces.

Theorem 7.2. *Let Y be Polish. Then there is a continuous surjection $f:\omega^\omega\to Y$.*

Proof. Let d be a complete metric generating the topology on Y, and fix a countable dense subset $D=\{c_0,c_1,\dots\}$ of Y. We define recursively, for each non-empty $s\in\omega^{<\omega}$, an element y_s of D. We ensure for each $x\in\omega^\omega$ that the sequence $\langle y_{x\restriction n}\rangle_{n\in\omega}$ is Cauchy in Y.

For each $a\in\omega$, set $y_{\langle a\rangle}=c_a$. Now suppose inductively that we have defined c_s for some $s\in\omega^{<\omega}$ with $|s|=n\geq 1$. For each $a\in\omega$, define $y_{s^\frown a}$ as follows: if $d(y_s,c_a)<2^{-n}$, let $y_{s^\frown\langle a\rangle}=c_a$; otherwise, set $y_{s^\frown\langle a\rangle}=y_s$.

We claim for each x, $\langle y_{x\restriction n}\rangle_{n\in\omega}$ is Cauchy. For we have, for all $1\leq m\leq n$,

$$\begin{aligned}d(y_{x\restriction m},y_{x\restriction n})&\leq d(y_{x\restriction m},y_{x\restriction m+1})+d(y_{x\restriction m+1},y_{x\restriction m+2})+\cdots+d(y_{x\restriction n-1},y_{x\restriction n})\\&<2^{-m}+2^{-(m+1)}+\cdots+2^{-(n-1)}<2^{-(m-1)}.\end{aligned}$$

Given $\varepsilon>0$, take N to be large enough that $2^{-(N-1)}<\varepsilon$; this witnesses Cauchyness of $\langle y_{x\restriction n}\rangle_{n\in\omega}$.

Now by completeness of Y, we may set $f(x)=\lim_{n\to\infty}y_{x\restriction n}$ for each $x\in\omega^\omega$. We claim $f:\omega^\omega\to Y$ is continuous. For suppose $x_0,x_1\in\omega^\omega$ and $d(x_0,x_1)<2^{-n}$, $n\geq 0$. This implies $x_0\restriction n=x_1\restriction n$, so that in particular, $y_{x_0\restriction n}=y_{x_1\restriction n}$. We then have $d(f(x_0),f(x_1))\leq d(f(x_0),y_{x_0\restriction n})+d(y_{x_1\restriction n},f(x_1))\leq 2^{-n}+2^{-n}=2^{-(n-1)}$. By the ε-δ characterization of continuity, we are done.

Finally, we need to show f is onto. Fix $y\in Y$. Define a sequence of elements of D tending quickly towards y: for all n, let $x(n)\in\omega$ be least so that $d(y,c_{x(n)})<2^{-(n+2)}$. We obtain $x\in\omega^\omega$; using the triangle inequality, it's easy to check that $d(c_{x(n)},c_{x(n+1)})<2^{-n+1}$ for all n. Then by induction, we always have $y_{x\restriction n}=a_{x(n)}$, so that $f(x)=\lim_{n\to\infty}a_{x(n)}=y$. ⊣

As remarked above, we regard open sets as the simplest subsets of a Polish space. Shortly we will define larger classes of sets that may contain more complicated sets. We would like to have a way of working with the class of all sets of a particular complexity in arbitrary Polish spaces (not just ω^ω). For this reason, we introduce the following new concept.

Definition 7.3. We call $\mathbf{\Gamma}$ a **pointclass** if it consists of pairs (A,X), where A is a subset of the Polish space X. We say $\mathbf{\Gamma}$ is **closed under continuous substitution** if, whenever X, Y are Polish spaces, $f : X \to Y$ is continuous, and (A, Y) belongs to $\mathbf{\Gamma}$, then also $(f^{-1}[A], X) \in \mathbf{\Gamma}$.

Given a pointclass $\mathbf{\Gamma}$ and a Polish space X, the **restriction of $\mathbf{\Gamma}$ to** X is the collection of sets A so that $(A, X) \in \mathbf{\Gamma}$; that is, $\mathbf{\Gamma}(X) = \mathbf{\Gamma} \cap \mathcal{P}(X)$.

The **dual pointclass of $\mathbf{\Gamma}$**, denoted $\neg\mathbf{\Gamma}$, is the class of complements of elements of $\mathbf{\Gamma}$: that is, $(A, X) \in \neg\mathbf{\Gamma}$ if and only if $(\neg A, X) \in \mathbf{\Gamma}$; here $\neg A = X \setminus A$. A pointclass is **self-dual** if $\mathbf{\Gamma} = \neg\mathbf{\Gamma}$.

Typically, the ambient space X will be understood and we simply write $A \in \mathbf{\Gamma}$ or say "A is $\mathbf{\Gamma}$" to mean that $(A, X) \in \mathbf{\Gamma}$.

Notice that $\mathbf{\Gamma}$ is closed under continuous substitution if and only if $\neg\mathbf{\Gamma}$ is. The class of open sets in Polish spaces is an example of a pointclass closed under continuous substitution; the class of closed sets (and that of clopen sets) is also closed under continuous substitution.

Let us connect the notions we are developing to determinacy.

Definition 7.4. Let $\mathbf{\Gamma}$ be a pointclass. We say **$\mathbf{\Gamma}$ determinacy** holds (and write $\mathbf{\Gamma}$-DET) if whenever $A \in \mathbf{\Gamma}(\omega^\omega)$, the game $G(A)$ is determined.

We saw that $\mathbf{\Gamma}$-DET holds when $\mathbf{\Gamma}$ is the class of closed sets. By the following theorem, we also have $\neg\mathbf{\Gamma}$ (open) determinacy.

Theorem 7.5. *Suppose $\mathbf{\Gamma}$ is a pointclass closed under continuous substitution. Then $\mathbf{\Gamma}$ determinacy is equivalent to $\neg\mathbf{\Gamma}$ determinacy.*

For contrast, recall that (under the Axiom of Choice) $G(A)$ may be determined while $G(\omega^\omega \setminus A)$ is not.

Proof. Let $A \in \neg\mathbf{\Gamma}(\omega^\omega)$ and suppose $\mathbf{\Gamma}$ determinacy holds. We wish to show $G(A)$ is determined. The proof illustrates a common technique in proofs of determinacy: the simulation of play in $G(A)$ by that in an auxiliary game.

Define $f : \omega^\omega \to \omega^\omega$ by $f(x)(n) = x(n+1)$ for all $n \in \omega$. Clearly f is continuous. Since $\neg\mathbf{\Gamma}$ is closed under continuous substitution, we have $f^{-1}[A] \in \neg\mathbf{\Gamma}$. Let $B = \neg f^{-1}[A]$. Then $B \in \mathbf{\Gamma}$ is determined by hypothesis.

Suppose Player II wins $G(B)$ with strategy τ. We obtain a strategy σ for Player I to win $G(A)$ by pretending we are Player II in $G(B)$, and that Player I played first move 0. That is, let $\sigma(s) = \tau(\langle 0\rangle^\frown s)$ for all $s \in \omega^{<\omega}$ for which the latter is defined.

Then σ is a strategy for Player II in $\omega^{<\omega}$. Suppose x is a play compatible with σ; then $\langle 0\rangle^\frown x$ is a play compatible with τ. Since τ is winning for Player II, $\langle 0\rangle^\frown x \notin B$, so that $f(\langle 0\rangle^\frown x) = x \in A$. Thus σ is winning for Player I in $G(A)$.

The argument when Player I wins $G(B)$ is similar: If σ is the winning strategy, then use it to play as Player II to win $G(A)$ (now ignoring the first move made by σ).

We have shown $\mathbf{\Gamma}$ determinacy implies $\neg\mathbf{\Gamma}$ determinacy; the converse holds by symmetry. $\dashv$

We have obtained that open sets and closed sets are determined. In order to investigate determinacy for more complicated sets, we first explore a way of producing sets that are more complicated.

The complement of an open set is not, in general, open; and the intersection of countably many open sets may be neither open nor closed. Iterating the operations of complement and countable union gives us a hierarchy of increasingly complicated sets. The next definition is central to our study of sets of reals.

Definition 7.6. Let X be a Polish space. We define a hierarchy of pointclasses $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, $\mathbf{\Delta}^0_\alpha(X)$ for $1 \leq \alpha < \omega_1$ by transfinite recursion.

1. $U \in \mathbf{\Sigma}^0_1(X)$ iff U is an open set in X.
2. Assuming $\mathbf{\Sigma}^0_\alpha(X)$ is defined, $\mathbf{\Pi}^0_\alpha(X) = \{A \subseteq X \mid X \setminus A \in \mathbf{\Sigma}^0_\alpha(X)\}$.
3. Assuming $\mathbf{\Pi}^0_\beta(X)$ is defined for all $1 \leq \beta < \alpha$, we let $\mathbf{\Sigma}^0_\alpha(X)$ be the set of countable unions of sets in $\bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$. That is, $A \in \mathbf{\Sigma}^0_\alpha(X)$ if and only if $A = \bigcup_{n\in\omega} A_n$ for some sequence $\langle A_n\rangle_{n\in\omega}$ with each $A_n \in \mathbf{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$.

We furthermore define the **ambiguous pointclasses** $\mathbf{\Delta}^0_\alpha(X)$ to consist of those sets that are in both $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$. That is, $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.

We define $\mathbf{\Sigma}^0_\alpha$ to be the pointclass consisting of (X, A) with A in $\mathbf{\Sigma}^0_\alpha(X)$ as X ranges over all Polish spaces. We define $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ similarly.

The classes $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha$ are the **Borel pointclasses**. A set A is a **Borel set in** X if $A \in \mathbf{\Sigma}^0_\alpha(X)$ or $A \in \mathbf{\Pi}^0_\alpha(X)$ for some $\alpha < \omega_1$. We set $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$.

Let's look at the first few levels of this hierarchy. Of course, $\mathbf{\Pi}^0_1$ is exactly the collection of closed sets. The collection $\mathbf{\Sigma}^0_2$ consists of all the countable unions of closed sets; sometimes these are also called F_σ sets. Since singletons are closed, any countable set is $\mathbf{\Sigma}^0_2$.

$\mathbf{\Pi}^0_2$ consists of all countable intersections of open sets; these are also called the G_δ sets. In the proof of the Baire category theorem we showed the countable intersection of dense open sets is dense; such sets are called dense G_δ sets. A set is meager precisely when it is disjoint from a dense G_δ set.

Example 7.7. Let $a < b \in \mathbb{R}$. Then the half-open interval $[a, b)$ is $\mathbf{\Delta}^0_2$: It can be written both as the countable union of closed sets and as the countable intersection of open sets.

Example 7.8. The set $\mathbb{Q}$ is $\mathbf{\Sigma}^0_2$ as a subset of $\mathbb{R}$. Since $\mathbb{Q}$ is meager, it cannot be the intersection of countably many (necessarily dense) open sets, since this would imply $\mathbb{R}$ is the union of two meager sets, contradicting the Baire category theorem. So Q is not $\mathbf{\Pi}^0_2$ (and so not $\mathbf{\Delta}^0_2$).

We pursue a systematic study of this hierarchy in the next section.

§8. The Borel Hierarchy.

Theorem 8.1. *Each pointclass $\mathbf{\Sigma}^0_\alpha, \mathbf{\Pi}^0_\alpha, \mathbf{\Delta}^0_\alpha$ is closed under continuous substitution.*

Proof. We proceed by induction, just as the Borel pointclasses were defined. For $\mathbf{\Sigma}^0_1$, this is immediate from the definition of continuity. Having shown $\mathbf{\Sigma}^0_\alpha$ is closed under continuous substitution, suppose $A \subseteq Y$ is in $\mathbf{\Pi}^0_\alpha$ and $f : X \to Y$ is continuous. Then since $Y \setminus A \in \mathbf{\Sigma}^0_\alpha$, we have that $f^{-1}[Y \setminus A] \in \mathbf{\Sigma}^0_\alpha$. It follows that $f^{-1}[A] = X \setminus f^{-1}[Y \setminus A]$ belongs to $\mathbf{\Pi}^0_\alpha$, as needed.

Finally suppose $\mathbf{\Pi}^0_\beta$ is closed under continuous substitution for all $\beta < \alpha$. Let $A \in \mathbf{\Sigma}^0_\alpha(Y)$; then $A = \bigcup_{n\in\omega} A_n$ where each A_n is in $\mathbf{\Pi}^0_{\beta_n}(Y)$ for some $\beta_n < \alpha$. By inductive hypothesis, $f^{-1}[A_n] \in \mathbf{\Pi}^0_{\beta_n}(X)$ for each $n < \omega$, and then $f^{-1}[A] = \bigcup_{n\in\omega} f^{-1}[A_n] \in \mathbf{\Sigma}^0_\alpha(X)$, as needed.

The claim for $\mathbf{\Delta}^0_\alpha$ follows immediately. ⊣

Let us analyze the hierarchy of Borel sets a little further. First, we note that it really is a hierarchy.

PROPOSITION 8.2. *If* $1 \leq \beta < \alpha < \omega_1$*, we have* $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$*,* $\mathbf{\Pi}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$*, while* $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$ *and* $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Pi}^0_\alpha$*.*

PROOF. If we can show the former claim, then $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Delta}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$ follows from the definition. We prove $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$; that $\mathbf{\Pi}^0_\beta \subseteq \mathbf{\Delta}^0_\alpha$ is immediate by taking complements. For this it is clearly enough to only deal with successor ordinals and show $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Delta}^0_{\beta+1}$ for all $1 \leq \beta < \omega_1$.

Showing $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Pi}^0_{\beta+1}$ is easy: say $A \in \mathbf{\Sigma}^0_\beta$. Then $B = X \setminus A \in \mathbf{\Pi}^0_\beta$. Setting $B_n = B$ for all $n \in \omega$ we have $B = \bigcup_{n\in\omega} B_n$ belongs to $\mathbf{\Sigma}^0_{\beta+1}$ and thus the complement A belongs to $\mathbf{\Sigma}^0_{\beta+1}$.

It remains to show $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Sigma}^0_{\beta+1}$. This is a little tougher; we do it by induction on $\beta \geq 1$. If $\beta = 1$, notice that any open set in a Polish space is a countable union of closed sets. Thus $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$. Now assume inductively that we have $\mathbf{\Sigma}^0_\gamma \subseteq \mathbf{\Sigma}^0_{\gamma+1}$ for $\gamma < \beta$; by taking complements we also have $\mathbf{\Pi}^0_\gamma \subseteq \mathbf{\Pi}^0_{\gamma+1}$. Let $A \in \mathbf{\Sigma}^0_\beta$. Then $A = \bigcup_{n\in\omega} A_n$ where each A_n is $\mathbf{\Pi}^0_{\gamma_n}$ for some $\gamma_n < \beta$; thus each A_n is also $\mathbf{\Pi}^0_{\gamma_n+1}$ where $\gamma_n + 1 < \beta + 1$. Thus indeed A is $\mathbf{\Sigma}^0_{\beta+1}$. ⊣

Next we will be interested in the closure properties that the Borel pointclasses enjoy.

PROPOSITION 8.3. *Let X be a Polish space. Then for all* $1 \leq \alpha < \omega_1$*,*

1. $\mathbf{\Sigma}^0_\alpha(X)$ *is closed under countable unions, and* $\mathbf{\Pi}^0_\alpha(X)$ *is closed under countable intersections.*
2. $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ *are each closed under finite unions and intersections.*
3. $\mathbf{\Delta}^0_\alpha(X)$ *is closed under complements; in particular, each* $\mathbf{\Delta}^0_\alpha$ *is self-dual.*
4. $\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ *is closed under the operations of countable union, countable intersection, and complementation; that is,* $\mathcal{B}(X)$ *is a* ***σ-algebra****, and it is the smallest σ-algebra containing the open sets of* X*.*

PROOF. The first and third items are immediate by definition; we leave the second as an exercise. For the last, closure under countable unions follows from the fact that ω_1 is regular: if $A_n \in \mathbf{\Sigma}^0_{\alpha_n}$ for each n, then $\alpha = \sup_{n\in\omega} \alpha_n < \omega_1$, and $\bigcup_{n\in\omega} A_n \in \mathbf{\Sigma}^0_\alpha$.

For the final claim, suppose $\mathcal{F}$ is a σ-algebra containing the open sets of X. Then $\mathbf{\Sigma}^0_1(X) \subseteq \mathcal{F}$, and whenever $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{F}$ we must have $\mathbf{\Pi}^0_\alpha(X) \subseteq \mathcal{F}$ by closure of $\mathcal{F}$ under complements; similarly, if $\mathbf{\Pi}^0_\beta(X) \subseteq \mathcal{F}$ for all $1 \leq \beta < \alpha$, then $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{F}$ by closure of $\mathcal{F}$ under countable unions. Thus by transfinite induction we obtain $\mathcal{B}(X) \subseteq \mathcal{F}$. ⊣

We next define an operation on sets in product Polish spaces of the form $\omega \times X$.

DEFINITION 8.4. Let X be Polish, and $A \subseteq \omega \times X$. We define

$$\exists^\omega A = \{x \in X \mid (\exists n \in \omega)\langle n, x\rangle \in A\}$$

and

$$\forall^\omega A = \{x \in X \mid (\forall n \in \omega)\langle n, x\rangle \in A\}.$$

These operations determine corresponding operations on pointclasses:

$$\exists^\omega \mathbf{\Gamma} = \{(\exists^\omega A, X) \mid (A, \omega \times X) \in \mathbf{\Gamma}\}$$

and similarly for $\forall^\omega \mathbf{\Gamma}$.

Before proceeding, we make a general comment about taking *slices*. If X, Y are Polish and $A \subseteq X \times Y$, the **slice of A at x** is defined to be

$$A_x = \{y \in Y \mid \langle x, y\rangle \in A\}.$$

Note that if $\mathbf{\Gamma}$ is closed under continuous substitution, then $A \in \mathbf{\Gamma}$ implies $A_x \in \mathbf{\Gamma}$ for all $x \in X$. This gives us the following:

PROPOSITION 8.5. *Each $\mathbf{\Sigma}^0_\alpha$ is closed under $\exists^\omega$, and each $\mathbf{\Pi}^0_\alpha$ is closed under $\forall^\omega$. In symbols: $\exists^\omega \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\alpha$ and $\forall^\omega \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_\alpha$.*

PROOF. It suffices to show $\exists^\omega \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\alpha$, since $\forall^\omega A = \neg(\exists^\omega \neg A)$. By the previous remarks, each A_n is in $\mathbf{\Sigma}^0_\alpha$. But $\exists^\omega A = \bigcup_{n\in\omega} A_n$, and so we're done. $\dashv$

There is one more important fact about the Borel hierarchy we would like to show. Namely, we want to show new sets are obtained at each level, so that in particular $\mathbf{\Sigma}^0_\alpha \neq \mathbf{\Sigma}^0_{\alpha+1}$ for all $1 \leq \alpha < \omega_1$. This is accomplished by the next theorem.

THEOREM 8.6 (The Hierarchy Theorem). *Let X be an uncountable Polish space. Then for each α there is some $A \subseteq X$ with $A \in \mathbf{\Sigma}^0_\alpha \setminus \mathbf{\Pi}^0_\alpha$.*

Notice that by taking complements we get the existence of a set in $\mathbf{\Pi}^0_\alpha \setminus \mathbf{\Sigma}^0_\alpha$. To prove this theorem the main technical tool we will make use of is the notion of a *universal set.*

DEFINITION 8.7. Let X and Y be Polish spaces, and let $\mathbf{\Gamma}$ be a pointclass. A set $W \subseteq X \times Y$ is $\mathbf{\Gamma}$-universal for Y if $W \in \mathbf{\Gamma}$ and for every $A \subseteq Y$ with $A \in \mathbf{\Gamma}$ there is some $x \in X$ such that $A = W_x$.

THEOREM 8.8. *Let X be a Polish space. For each α with $\mathbf{\Gamma}$ equal to either $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$, there is a $\mathbf{\Gamma}$-universal set $W \subseteq 2^\omega \times X$ for X.*

We start with the open sets.

PROPOSITION 8.9. *For each Polish space X there is a universal open ($\mathbf{\Sigma}^0_1$) set $W \subseteq 2^\omega \times X$.*

PROOF. The idea is simple: Since the space X is Polish it has a countable basis $\{U_i\}_{i\in\omega}$. Thus the open subsets of X are exactly the countable unions of sets of the form U_i, and since there are only $2^{\aleph_0} = |2^\omega|$ of these we can use each $x \in 2^\omega$ to encode the possible unions.

Thus we define $W \subseteq 2^\omega \times X$ by

$$\langle f, x\rangle \in W \text{ if and only if } (\exists n \in \omega) x \in U_n \text{ and } f(n) = 1.$$

We need to see W is open. But this is clear, since

$$W = \bigcup_{n\in\omega} \{f \in 2^\omega \mid f(n) = 1\} \times U_n$$

and each set $\{f \in 2^\omega \mid f(n) = 1\} \times U_n$ is open.

Next note W is universal. For let $A \subseteq X$ be open. Then A can be written as a countable union of the U_n; we let $f \in 2^\omega$ indicate which, so that $A = \bigcup\{U_n \mid f(n) = 1\}$. It follows that $W_f = A$ straight from the definition. $\dashv$

The next proposition is clear.

PROPOSITION 8.10. *If $W \subseteq 2^\omega \times X$ is $\mathbf{\Gamma}$-universal, then $(2^\omega \times X) \setminus W$ is $\neg\mathbf{\Gamma}$-universal.*

The last step in the proof of Theorem 8.8 is the following proposition.

Proposition 8.11. *Let $1 \leq \alpha < \omega_1$, and assume that for each $\beta < \alpha$ there is a $\mathbf{\Pi}^0_\beta$-universal set $W^\beta \subseteq 2^\omega \times X$. Then there is a $\mathbf{\Sigma}^0_\alpha$-universal set $W \subseteq 2^\omega \times X$.*

In order to prove this last proposition we first need to bring up coding. It will be useful for us to have a way of encoding an infinite sequence of elements of 2^ω by a single $f \in 2^\omega$. One way to do this is by defining, for $f \in 2^\omega$, $(f)_n$ by $(f)_n(m) = f(2^m 3^n)$. There are two important things to notice. One: the map sending f to $(f)_n$ is continuous, and two: any countable sequence $\langle g_n \rangle_{n\in\omega}$ of members of 2^ω is coded by some f so that $(f)_n = g_n$ for all n.

Proof of Proposition 8.11. Let $\{\gamma_k\}_{k\in\omega}$ be an enumeration of all the ordinals below α, enumerated in such a way that each one repeats infinitely often. Notice that then $A \subseteq X$ belongs to $\mathbf{\Sigma}^0_\alpha$ exactly when there are sets A_k in $\mathbf{\Pi}^0_{\gamma_k}$ with $A = \bigcup_{k\in\omega} A_k$.

Now define W by

$$\langle f, x\rangle \in W \text{ if and only if } (\exists k)\langle (f)_k, x\rangle \in W^{\gamma_k}.$$

We claim W belongs to $\mathbf{\Sigma}^0_\alpha$. To see this, for each $k \in \omega$ let B_k be equal to the collection of $\langle f, x\rangle$ such that $x \in W^{\gamma_k}_{(f)_k}$. Then clearly $W = \bigcup_{k\in\omega} B_k$. Now define $\varphi_k : 2^\omega \times X \to 2^\omega \times X$ by $\varphi_k(f, x) = \langle (f)_k, x\rangle$. This map is continuous, and $\varphi_k^{-1}[W^{\gamma_k}]$ is exactly B_k since $\langle f, x\rangle \in B_k$ if and only if $\langle x, (f)_k\rangle \in W^{\gamma_k}$.

We finish by showing W is $\mathbf{\Sigma}^0_\alpha$-universal. Let $A \subseteq X$ be $\mathbf{\Sigma}^0_\alpha$. Then $A = \bigcup_{k\in\omega} A_k$ where each $A_k \subseteq X$ is in $\mathbf{\Pi}^0_{\gamma_k}$. For each k using the universality of W^{γ_k}, we fix g_k so that $A_k = W^{\gamma_k}_{g_k}$. Let f be such that $(f)_k = g_k$. Then we have that $x \in W_f$ exactly when $\langle f, x\rangle$ belongs to W. This holds exactly when for some k we have $\langle (f)_k, x\rangle \in W^{\gamma_k}$, that is $\langle g_k, x\rangle \in W^{\gamma_k}$ which itself is equivalent to $x \in A_k$. So $W_f = A$ as needed. ⊣

Proof of Theorem 8.6. We just do the case $X = 2^\omega$, leaving the general perfect Polish case as an exercise. Let $W \subseteq 2^\omega \times 2^\omega$ be a $\mathbf{\Sigma}^0_\alpha$-universal set. Set

$$A = \{x \in 2^\omega \mid x \in W_x\}.$$

Then $A \in \mathbf{\Sigma}^0_\alpha$: for if $f : 2^\omega \to 2^\omega \times 2^\omega$ is given by $f(x) = \langle x, x\rangle$, then f is continuous, and $A = f^{-1}W$.

We claim the set A does not belong to $\mathbf{\Pi}^0_\alpha$. For, supposing for contradiction that A did belong to $\mathbf{\Pi}^0_\alpha$, its complement $\neg A = 2^\omega \setminus A$ would belong to $\mathbf{\Sigma}^0_\alpha$. By universality of W that means there is a $y \in 2^\omega$ such that $\neg A = W_y$. But then by definition of A, we have $y \in A$ if and only if $y \in W_y$ if and only if $y \notin A$, a contradiction. ⊣

§9. Borel codes and the lightface Borel pointclasses. In this section we describe a canonical way of associating a Borel set with a real number.

Definition 9.1. The **Borel codes of rank** α are defined by induction on α, as follows:

$$\mathrm{BC}_1 = \{x \in \omega^\omega \mid x(0) = 0\},$$

and for $\alpha > 1$,

$$\mathrm{BC}_\alpha = \{x \in \omega^\omega \mid x(0) = 0, \text{ or } x(0) > 0 \text{ and } (\forall n \in \omega)(x)_n \in \mathrm{BC}_\beta \text{ for some } \beta < \alpha\}.$$

We set $\mathrm{BC} = \mathrm{BC}_{\omega_1}$.

It is easy to argue inductively that the sets BC_α are $\subseteq$-increasing, and that $\mathrm{BC} = B_{\omega_1} = B_\alpha$ all $\alpha > \omega_1$. Let X be a Polish space with open basis $\{U_i\}_{i\in\omega}$. Given $x \in \mathrm{BC}$, we can uniquely decode a Borel set in X

$$A^x \begin{cases} \bigcup\{U_i \mid x(i+1) > 0\} & \text{if } x(0) = 0; \\ \bigcup X \setminus A^{(x)_n} & \text{otherwise.} \end{cases}$$

This is well-defined (by induction on the rank of the Borel code x).

We next describe an effective refinement of the Borel hierarchy.

Definition 9.2. Let $X = \omega^k \times (\omega^\omega)^\ell$, where $k, \ell \in \omega$ are not both 0. Regard the Borel codes defined above as using a *recursive* open basis (via some effective coding by naturals of tuples designating open neighborhoods in X). The **lightface Borel pointclasses** are obtained by setting

$$\Sigma^0_\alpha = \{A^x \subset X \mid x \in \mathrm{BC}_\alpha \text{ and } x \text{ is recursive}\}.$$

The dual pointclasses Π^0_α and ambiguous Δ^0_α are then defined as before. We furthermore define the **relativization** to reals $z \in \omega^\omega$ by

$$\Sigma^0_\alpha(z) = \{A^x \subset X \mid x \in \mathrm{BC}_\alpha \text{ and } x \leq_T z\},$$

where here $\leq_T$ is Turing reducibility.

It is immediate that each Σ^0_α is countable, and that $\boldsymbol{\Sigma}^0_\alpha = \bigcup_{z\in\omega^\omega} \Sigma^0_\alpha(z)$.

We remark that this definition can be made more generally for X any *effectively presented* perfect Polish space, but we refrain from defining this notion. Most (if not all) of the theorems we prove below may be sharpened to obtain "lightface" versions, often (but not always) by simply keeping track of the complexity of the definitions used in defining witnesses to those theorems. But our main purpose in introducing the lightface pointclasses is to have a handy notational device for keeping track of real parameters used in definitions of sets of reals.

It is worth noting some facts (proofs left to the reader):

1. A set $A \subseteq \omega^\omega$ is Π^0_1 iff $A = [T]$ for some tree $T \subseteq \omega^{<\omega}$ such that T is recursive.
2. For all α, $\Sigma^0_{\alpha+1} = \exists^\omega \Sigma^0_\alpha$.
3. A set $a \subseteq \omega$ is Σ^0_n iff it is Σ_n-definable over the structure $(\omega; 0, 1, +, \cdot, <)$.
4. Let ω_1^{CK} ("Church-Kleene ω_1") be the least non-recursive ordinal, i.e. the supremum of the order-types $\mathrm{otp}(\omega; R)$ over all recursive well-orders $R \subseteq \omega \times \omega$ of ω. Then
$$\Sigma^0_{\omega_1^{\mathrm{CK}}} = \Sigma^0_{\omega_1^{\mathrm{CK}}+1} = \Sigma^0_\gamma, \text{ for all} \gamma > \omega_1^{\mathrm{CK}}.$$
5. For all $\alpha < \omega_1^{\mathrm{CK}}$, there is a universal Σ^0_α set $W \subseteq \omega \times X$. In particular (and in contrast to the boldface hierarchy), the Hierarchy Theorem 8.6 holds for the lightface pointclasses even for the space $X = \omega$.

Each of these facts relativizes to real parameters z.

§10. The Projective Hierarchy. Our basic operations for generating the Borel sets were negation and countable union, and we saw how the latter could be realized as quantification $\exists^\omega$ over the set of natural numbers. What if we allow ourselves to quantify over a bigger set?

DEFINITION 10.1. Let X be a Polish space, and $A \subseteq \omega^\omega \times X$. We define

$$\exists^{\omega^\omega} A = \{y \in X \mid (\exists x \in \omega^\omega)\langle x, y\rangle \in A\}.$$

For a pointclass $\mathbf{\Gamma}$, we let

$$\exists^{\omega^\omega} \mathbf{\Gamma} = \{(A, X) \mid A = \exists^{\omega^\omega} B \text{ for some } (B, \omega^\omega \times X) \in \mathbf{\Gamma}\}.$$

Similarly define operations $\forall^{\omega^\omega} A$ and $\forall^{\omega^\omega}\mathbf{\Gamma}$ for sets and pointclasses.

DEFINITION 10.2. For $n \in \omega$ we define the **projective pointclasses** $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$ by induction, as follows:

1. $\mathbf{\Pi}^1_0 = \mathbf{\Pi}^0_1$, and $\mathbf{\Sigma}^1_0 = \mathbf{\Sigma}^0_1$.
2. Given $\mathbf{\Sigma}^1_n$, let $\mathbf{\Pi}^1_n$ be the dual pointclass, $\mathbf{\Pi}^1_n = \neg\mathbf{\Sigma}^1_n$.
3. Given $\mathbf{\Pi}^1_n$, put $\mathbf{\Sigma}^1_{n+1} = \exists^{\omega^\omega}\mathbf{\Pi}^1_n$.

The **ambiguous projective pointclasses** are defined to be $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

The sets obtained in the projective hierarchy are the **projective sets**. We call the members of $\mathbf{\Sigma}^1_1$ the **analytic sets**; those of $\mathbf{\Pi}^1_1$ are the **coanalytic sets**.

PROPOSITION 10.3. *Each projective pointclass is closed under continuous substitution.*

PROOF. We show something a bit more general: if $\mathbf{\Gamma}$ is closed under continuous substitution, then so is $\exists^{\omega^\omega}\mathbf{\Gamma}$. Let $f : X \to Y$ be continuous, and suppose $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$ with $A \subseteq Y$; so $A = \exists^{\omega^\omega} B$ with $B \subseteq \omega^\omega \times Y$. Define $g : \omega^\omega \times X \to \omega^\omega \times Y$ by $g(\langle z, x\rangle) = \langle z, f(x)\rangle$. Then

$$\begin{aligned} f^{-1}[A] &= \{x \in X \mid f(x) \in A\} \\ &= \{x \in X \mid (\exists z \in \omega^\omega)\langle z, f(x)\rangle \in B\} \\ &= \exists^{\omega^\omega} g^{-1}[B], \end{aligned}$$

And this last set belongs to $\exists^{\omega^\omega}\mathbf{\Gamma}$ by closure of $\mathbf{\Gamma}$ under continuous substitution.

The proposition now follows by induction on the levels of the projective hierarchy (using the fact that $\mathbf{\Gamma}$ is closed under continuous substitution if and only if $\neg\mathbf{\Gamma}$ is). $\dashv$

PROPOSITION 10.4. *Each* $\mathbf{\Sigma}^1_n$ *is closed under* $\exists^{\omega^\omega}$*; each* $\mathbf{\Pi}^1_n$ *is closed under* $\forall^{\omega^\omega}$.

PROOF. Notice that the second claim follows from the first and De Morgan's law $\forall^{\omega^\omega} A = \neg\exists^{\omega^\omega}\neg A$. Then we have it for $n = 0$, since the projection of an open set is open: If $B \subseteq \omega^\omega \times X$ is open and $x \in \exists^{\omega^\omega} B$, we have $\langle z, x\rangle \in B$; taking $N_s \times U \subseteq B$ with $\langle z, x\rangle \in N_s \times U$, and $U \subseteq \exists^{\omega^\omega} B$.

Suppose now that $A \subseteq \omega^\omega \times X$ is in $\mathbf{\Sigma}^1_{n+1}$ for some $n \in \omega$. By definition $A = \exists^{\omega^\omega} B$ with $B \subseteq \omega^\omega \times \omega^\omega \times X$, $B \in \mathbf{\Pi}^1_n$. We need to show $\exists^{\omega^\omega} A \in \mathbf{\Sigma}^1_{n+1}$. Let $\phi : \omega^\omega \to \omega^\omega \times \omega^\omega$ be a homeomorphism, with $\phi(w) = \langle \phi_0(w), \phi_1(w)\rangle$ for all $w \in \omega^\omega$. Now the set

$$C = \{\langle w, x\rangle \in \omega^\omega \mid \langle \phi_0(w), \phi_1(w), x\rangle \in B\}$$

belongs to $\mathbf{\Pi}^1_n$ by closure under continuous substitution, and $\exists^{\omega^\omega} C = \exists^{\omega^\omega} A$ is in $\mathbf{\Sigma}^1_{n+1}$ as needed. $\dashv$

Just as with the Borel hierarchy, we have universal sets at each level of the projective hierarchy.

Theorem 10.5. *Suppose $W \subseteq 2^\omega \times \omega^\omega \times X$ is $\mathbf{\Gamma}$-universal for $\omega^\omega \times X$ where $\mathbf{\Gamma}$ is closed under continuous substitution. Then there is $W^* \subseteq 2^\omega \times X$ which is $\exists^{\omega^\omega}\mathbf{\Gamma}$-universal for X.*

Proof. Fix a $\mathbf{\Gamma}$-universal set W for $\omega^\omega \times X$. The $\exists^{\omega^\omega}\mathbf{\Gamma}$-universal set will be the obvious one, obtained by projecting along the ω^ω coordinate; that is

$$W^* = \{\langle u, x\rangle \in 2^\omega \times X \mid (\exists w \in \omega^\omega)\langle u, w, x\rangle \in W\} = \exists^{\omega^\omega}\{\langle w, u, x\rangle \mid \langle u, w, x\rangle \in W\}.$$

Closure of $\mathbf{\Gamma}$ under continuous substitution implies the set on the inside of the $\exists^{\omega^\omega}$ is in $\mathbf{\Gamma}$; so W^* is in $\exists^{\omega^\omega}\mathbf{\Gamma}$.

Now if $A \in \exists^{\omega^\omega}\mathbf{\Gamma}$, we have $A = \exists^{\omega^\omega}B$ with $B \subseteq \omega^\omega \times X$ in $\mathbf{\Gamma}$. Say $B = W_u$ with $u \in 2^\omega$. It is now easy to check that $A = W^*_u$. ⊣

Corollary 10.6. *For every Polish space X and $n \in \omega$, there exist $\mathbf{\Sigma}^1_n$-universal and $\mathbf{\Pi}^1_n$-universal sets for X.*

With the same diagonalization argument we used on the Borel hierarchy, we have

Corollary 10.7. *For ever $n \in \omega$ there is a set $A \subseteq 2^\omega$ in $\mathbf{\Sigma}^1_n \setminus \mathbf{\Pi}^1_n$ (and so $\neg A \in \mathbf{\Pi}^1_n \setminus \mathbf{\Sigma}^1_n$).*

Let us mention some more closure properties. First we note that our existential quantifier can range over any Polish space.

Definition 10.8. Let X, Y be Polish spaces, with $B \subseteq X \times Y$. Then $\exists^X B$ is the set $\{y \in Y \mid \langle x, y\rangle \in B\}$.

Proposition 10.9. *Each $\mathbf{\Sigma}^1_n$ is closed under $\exists^X$, for all Polish spaces X.*

Proof. Exercise. ⊣

Proposition 10.10. *Each $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n, \mathbf{\Delta}^1_n$ for $n > 0$ is closed under $\exists^\omega, \forall^\omega$, countable unions, and countable intersections.*

Proof. We first show $\mathbf{\Sigma}^1_n$ is closed under $\forall^\omega$ and $\exists^\omega$; then closure for $\mathbf{\Pi}^1_n$ follows from De Morgan's laws, and for the $\mathbf{\Delta}^1_n$ by definition.

Closure under $\exists^\omega$ follows from the last proposition, since ω is a Polish space. So suppose $B \subseteq \omega \times X$ with $B \in \mathbf{\Sigma}^1_n$. We need to show $\forall^\omega B \in \mathbf{\Sigma}^1_n$. We have by definition of $\mathbf{\Sigma}^1_n$ that there is a set $C \in \mathbf{\Pi}^1_{n-1}$ so that $B = \exists^{\omega^\omega} C$. Now

$$x \in \forall^\omega B \iff (\forall n \in \omega)\langle n, x\rangle \in B \iff (\forall n \in \omega)(\exists w \in \omega^\omega)\langle w, n, x\rangle \in C.$$

We require a way of reversing the order of quantifiers $\forall^n, \exists^{\omega^\omega}$. Let $w \mapsto \langle (w)_n\rangle_{n\in\omega}$ be a homeomorphism of ω^ω with $(\omega^\omega)^\omega$, so that each ω-sequence of elements of ω^ω is coded by a single w. We have, for each $x \in X$,

$$(\forall n \in \omega)(\exists w \in \omega^\omega)\langle w, n, x\rangle \in C \iff (\exists w \in \omega^\omega)(\forall n)\langle (w)_n, n, x\rangle \in C.$$

The right to left direction is clear; for the reverse, suppose for each n there is some $u_n \in \omega^\omega$ so that $\langle u_n, n, x\rangle \in C$, and (by countable choice) let w be a real with $(w)_n = u_n$ for all n.

Now the set $D = \{\langle w, x\rangle \in \omega^\omega \times X \mid (\forall n)\langle (w)_n, n, x\rangle \in C\}$ belongs to $\mathbf{\Pi}^1_{n-1}$, since $C \in \mathbf{\Pi}^1_{n-1}$ and this pointclass is closed under continuous substitution and $\forall^\omega$. Since $\forall^\omega B = \exists^{\omega^\omega} D$, we have $\forall^\omega B \in \mathbf{\Sigma}^1_n$ as needed.

Finally, we need to show $\mathbf{\Sigma}^1_n$ is closed under countable unions and intersections. Suppose $\langle A_n \rangle_{n\in\omega}$ is a sequence of members of $\mathbf{\Sigma}^1_n(X)$. Let $W \subseteq 2^\omega \times X$ be $\mathbf{\Sigma}^1_n$-universal for X. For each $n \in \omega$, pick some $y_n \in 2^\omega$ with $A_n = W_{y_n}$. By closure under continuous substitution, the set $C = \{\langle n, x\rangle \mid \langle y_n, x\rangle \in W\} = \{\langle n, x\rangle \mid x \in A_n\}$ is in $\mathbf{\Sigma}^1_n$ (since the map $n \mapsto y_n$ is automatically continuous). But $\exists^\omega C = \bigcup_{n\in\omega} A_n$ and $\forall^\omega C = \bigcap_{n\in\omega} A_n$ then both belong to $\mathbf{\Sigma}^1_n$. ⊣

Corollary 10.11. *Every Borel set belongs to* $\mathbf{\Delta}^1_1$.

Proof. Every closed $F \subseteq X$ belongs to $\mathbf{\Sigma}^1_1$, since $F \times F$ is closed in $X \times X$, and so $F = \exists^X F \times F \in \mathbf{\Sigma}^1_1$. Every open set is the countable union of closed sets, and so is in $\mathbf{\Sigma}^1_1$ by the previous proposition. Then the closed and open sets are in $\mathbf{\Pi}^1_1$ as well.

Now $\mathbf{\Delta}^1_1$ contains the open and closed sets, and is closed under complement, countable union, and countable intersection. It follows that $\mathbf{\Delta}^1_1$ contains all the Borel sets. ⊣

We have seen that the analytic sets contain the Borel sets, and that there is a set that is in $\mathbf{\Sigma}^1_1$ but not in $\mathbf{\Delta}^1_1$—in particular, this set is not Borel. That the projection of a Borel set in the plane is Borel was incorrectly asserted by Lebesgue; the existence of a counterexample was discovered by Suslin, a graduate student at the time. And so descriptive set theory was born.

In a surprising and useful turn of events, the converse of the previous corollary holds: $\mathbf{\Delta}^1_1$ consists of exactly the Borel sets! In order to show this, we need some tools to help us analyze $\mathbf{\Sigma}^1_1$. Recall that closed sets in Baire space were precisely the sets of branches through trees $T \subseteq \omega^{<\omega}$. Since sets in $\mathbf{\Sigma}^1_1$ are projections of closed sets in $\omega^\omega \times \omega^\omega$, it will be useful to introduce a system of notation to study trees

Definition 10.12. We say a non-empty set $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ is a **tree** if

1. For all $\langle s, t\rangle \in T$, we have $|s| = |t|$.
2. If $s \subseteq s'$, $t \subseteq t'$, $|s| = |t|$ and $\langle s', t'\rangle \in T$, then $\langle s, t\rangle \in T$.

We say that $\langle x, y\rangle \in \omega^\omega \times \omega^\omega$ is a **branch** through the tree T if for all n, $\langle x \restriction n, y \restriction n\rangle \in T$, and write $[T] \subseteq \omega^\omega \times \omega^\omega$ for the set of branches.

Similar definitions are made for the higher products $\omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ and so forth.

Of course, there is an obvious correspondence between trees T on $\omega \times \omega$ and trees in $\omega^{<\omega} \times \omega^{<\omega}$ as defined here. This new definition essentially introduces a systematic abuse of notation, identifying the sequence of pairs $\langle\langle s(0), t(0)\rangle, \ldots, \langle s(n-1), t(n-1)\rangle\rangle \in T$ with the pair of sequences $\langle s, t\rangle$.

Proposition 10.13. *A set* $C \subseteq \omega^\omega \times \omega^\omega$ *is closed if and only if* $C = [T]$ *for a tree* $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$.

Proof. Set $T = \{\langle x \restriction n, y \restriction n\rangle \mid \langle x, y\rangle \in C\}$; the proof that $[T]$ is closed when T is a tree is the same as before. ⊣

Corollary 10.14. *A set* $A = \omega^\omega \times \omega^\omega$ *is* $\mathbf{\Sigma}^1_1$ *if and only if* $A = \exists^{\omega^\omega}[T]$ *for some tree* $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$.

As expected, for $\langle s, t\rangle \in T$ we denote

$$T_{s,t} = \{\langle s', t'\rangle \in T \mid s \subseteq s' \text{ and } t \subseteq t', \text{ or } s' \subseteq s \text{ and } t' \subseteq t\}.$$

Observe we have the equality

$$T_{s,t} = \bigcup_{m,n\in\omega} T_{s^\frown\langle m\rangle, t^\frown\langle n\rangle}.$$

We are just about ready to prove that all $\mathbf{\Delta}^1_1$ sets are Borel. First, one more definition.

Definition 10.15. Suppose A, B are disjoint sets. We say C **separates** A from B if $A \subseteq C$ and $B \cap C = \varnothing$.

The key fact is the following theorem.

Theorem 10.16 (Lusin). *Suppose $A, B \in \mathbf{\Sigma}^1_1(\omega^\omega)$ are disjoint. Then there is a Borel set $C \subseteq \omega^\omega$ that separates A from B.*

Proof. We take advantage of the following simple fact.

Claim. *Suppose $A = \bigcup_{i\in I}$ and $B = \bigcup_{j\in J} B_j$, and suppose for each $i \in I$ and $j \in J$ there is a set $C_{i,j}$ which separates A_i from B_j. Then the set $C = \bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}$ separates A and B.*

Proof of claim. Suppose $x \in A$; then $x \in A_i$ for some i. Since $A \subseteq C_{i,j}$ for all $j \in J$, we have $x \in C$. So $A \subseteq C$.

Now suppose $x \in B$. Then $x \in B_j$ for some $j \in J$. For every i, we have $C_{i,j} \cap B_j = \varnothing$. In particular, $x \notin \bigcap_{j\in J} C_{i,j}$ for each i; so $x \notin C$, and $B \cap C = \varnothing$. ⊣

Now let A, B be disjoint in $\mathbf{\Sigma}^1_1$. Let $S, T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be trees with $A = \exists^{\omega^\omega}[S]$ and $B = \exists^{\omega^\omega}[T]$. We proceed by contradiction: Suppose A, B cannot be separated by a Borel set.

Now we have

$$A = \exists^{\omega^\omega}[S] = \bigcup_{k,l\in\omega} \exists^{\omega^\omega}[S_{\langle k\rangle,\langle l\rangle}], \qquad B = \exists^{\omega^\omega}[T] = \bigcup_{m,n\in\omega} \exists^{\omega^\omega}[T_{\langle m\rangle,\langle n\rangle}].$$

By (the contrapositive of) the claim, there must exist some $k_0, l_0, m_0, n_0 \in \omega$ so that $\exists^{\omega^\omega}[S_{\langle k_0\rangle,\langle l_0\rangle}], \exists^{\omega^\omega}[T_{\langle m_0,n_0\rangle}]$ cannot be separated by a Borel set. Clearly then $\langle\langle k_0\rangle, \langle l_0\rangle\rangle \in S$ and $\langle\langle m_0\rangle, \langle n_0\rangle\rangle \in T$, since otherwise one of these sets would be empty and so easily separated by a Borel set. Notice also that we must have $l_0 = n_0$, for clearly $\exists^{\omega^\omega}[S_{\langle k_0\rangle,\langle l_0\rangle}] \subseteq N_{\langle l_0\rangle}$ and $\exists^{\omega^\omega}[T_{\langle m_0,n_0\rangle}] \subseteq N_{\langle n_0\rangle}$; if these were distinct, then $N_{\langle l_0\rangle}$ would separate A from B.

Now suppose inductively that we have sequences $s = \langle k_0, \dots, k_{i-1}\rangle$, $t = \langle m_0, \dots, m_{i-1}\rangle$, and $u = \langle n_0, \dots, n_{i-1}\rangle$, so that the sets $\exists^{\omega^\omega}[S_{s,u}]$ and $\exists^{\omega^\omega}[T_{t,u}]$ cannot be separated by a Borel set. By the same argument, we have some k_i, m_i, n_i so that $\langle s^\frown\langle k_i\rangle, u^\frown\langle n_i\rangle\rangle \in S$, $\langle t^\frown\langle m_i\rangle, u^\frown\langle n_i\rangle\rangle \in T$, and the sets $\exists^{\omega^\omega}[S_{s^\frown\langle k_i\rangle, u^\frown\langle n_i\rangle}]$ and $\exists^{\omega^\omega}[T_{\langle t^\frown\langle m_i\rangle, u^\frown\langle n_i\rangle}]$ cannot be separated by a Borel set.

By induction we obtain $x = \langle k_i\rangle_{i\in\omega}$, $y = \langle m_i\rangle_{i\in\omega}$ and $z = \langle n_i\rangle_{i\in\omega}$. By construction we have $\langle x, z\rangle \in S$ and $\langle y, z\rangle \in T$. But then $z \in \exists^{\omega^\omega}[S] \cap \exists^{\omega^\omega}[T] = A \cap B$, contradicting our assumption that A, B were disjoint. ⊣

Corollary 10.17 (Suslin). *The Borel subsets of ω^ω are exactly those in the class $\mathbf{\Delta}^1_1$.*

Proof. We already saw that every Borel set is $\mathbf{\Delta}^1_1$. Suppose that $A \subseteq \omega^\omega$ is in $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. Then both A and $\neg A$ are in $\mathbf{\Sigma}^1_1$; by the theorem, we have a Borel set C in ω^ω with $A \subseteq C$ and $\neg A \cap C = \varnothing$. But the only possibility for such a C is $C = A$! ⊣

§11. Analyzing Co-analytic Sets. We now restrict our attention to the first level of the projective hierarchy, that of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$. Our analysis will hinge on the fact that the sets in $\mathbf{\Sigma}^1_1$ are the projections of trees. We start off by defining a notion that lets us convert trees on ω into linear orders.

Definition 11.1. The **Kleene-Brouwer order** is the order $<_{\mathrm{KB}}$ defined on $\omega^{<\omega}$ as follows. We say $s <_{\mathrm{KB}} t$ if and only if

1. $s \supsetneq t$, or
2. $s(n) < t(n)$, where n is least such that $s(n) \neq t(n)$.

Proposition 11.2. *$<_{\mathrm{KB}}$ is a linear order.*

Any linear order restricted to a subset of its domain is again a linear order; in particular, $<_{\mathrm{KB}}$ is a linear order on any tree $T \subseteq \omega^{<\omega}$. The following proposition is our reason for introducing $<_{\mathrm{KB}}$.

Proposition 11.3. *Suppose T is a tree on ω, and that $\langle s_n \rangle_{n\in\omega}$ is an infinite sequence of nodes in T with $s_{n+1} <_{\mathrm{KB}} s_n$ for all n. Then there is an infinite branch through T.*

Proof. By our definition of $<_{\mathrm{KB}}$, we have $s_{n+1}(0) \leq s_n(0)$ for all n. In particular, the sequence $\langle s_n(0) \rangle_{n\in\omega}$ is eventually constant, so there must be some k_0 so that $s_n(0) = k_0$ for all but finitely many n.

Now suppose inductively that we have found $t = \langle k_0, \ldots, k_{i-1} \rangle$ so that $t \subseteq s_n$ for all but finitely many n. For each such n, we again have $s_{n+1}(i) \leq s_n(i)$, and so there exists k_i so that $s_n(i) = k_i$ for all but finitely many n; thus eventually $t^\frown\langle k_i \rangle \subseteq s_n$.

By construction each finite string $\langle k_0, \ldots, k_i \rangle$ is an initial segment of some s_n, and so the sequence $\langle k_n \rangle_{n\in\omega}$ is a branch through T. ⊣

Corollary 11.4. *Let T be a tree on ω. Then $[T] = \varnothing$ if and only if $<_{\mathrm{KB}}$ restricted to T is a well-order.*

Proof. We have just shown that if $<_{\mathrm{KB}}$ is not a well-order on T, then it $[T] \neq \varnothing$. Conversely, suppose $x \in [T]$; then $x \restriction n+1 <_{\mathrm{KB}} x \restriction n$ for all n, so $<_{\mathrm{KB}}$ is ill-founded on T. ⊣

This justifies the following terminology: A tree T is **well-founded** if it has no infinite branches.

We can now give a useful characterization of $\mathbf{\Pi}^1_1$ sets. We need one more piece of notation.

Let $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be a tree. For $y \in \omega^\omega$, we let $T(y) \subseteq \omega^{<\omega}$ be the set

$$T(y) = \{s \in \omega^{<\omega} \mid \langle s, y \restriction |s| \rangle \in T\}.$$

Then $T(y)$ is a tree.

Proposition 11.5. *For all $x, y \in \omega^\omega$ and trees $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$, we have $\langle x, y \rangle \in [T]$ if and only if $x \in [T(y)]$.*

Proof. This falls right out of the definitions: $x \in [T(y)]$ iff $(\forall n \in \omega) x \restriction n \in T(y)$ iff $(\forall n \in \omega)\langle x \restriction n, y \restriction n \rangle \in T$ iff $\langle x, y \rangle \in [T]$. ⊣

Corollary 11.6. *A set B is $\mathbf{\Pi}^1_1$ if and only if there is some tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that*

$$B = \{y \in \omega^\omega \mid T(y) \text{ is well-founded}\}.$$

Proof. If B is $\mathbf{\Pi}^1_1$, then $\neg B$ is $\mathbf{\Sigma}^1_1$ and so $\neg B = \exists^{\omega^\omega}[T]$ for some tree T. Then $y \in B$ iff $y \notin \exists^{\omega^\omega}[T]$ iff $(\forall x)\langle x, y\rangle \notin [T]$ iff $(\forall x) x \notin [T(y)]$ iff $T(y)$ is well-founded. $\dashv$

With this characterization down, we can isolate a particularly interesting $\mathbf{\Pi}^1_1$ set. But first, let's talk about coding. We want to regard elements of 2^ω as coding binary relations on ω, that is, members of $\mathcal{P}(\omega\times\omega)$. For this, we set down a canonical way of identifying ω and $\omega\times\omega$: Set $\ulcorner i, j\urcorner = 2^i(2j+1)$ for $i, j \in \omega$. As the reader can check, this is a bijection from $\omega\times\omega$ to ω.

Now, given $x \in 2^\omega$, define R_x to be the relation on ω obtained by setting

$$i \, R_x \, j \iff x(\ulcorner i, j\urcorner) = 1.$$

We can now encode classes of countable mathematical structures as sets of reals, and talk about the complexity of these in terms of descriptive set theory. For example, let LO be the set of all x encoding a linear order:

$$\mathrm{LO} = \{x \in 2^\omega \mid R_x \text{ is a linear order of some subset of } \omega\}.$$

Proposition 11.7. LO *is Borel.*

Proof. This amounts to writing down the definition of a linear order and observing that the only quantifiers we use are first-order—that is, we only quantify over elements of the linear order (as opposed to its subsets). We have

$$\begin{aligned} x \in \mathrm{LO} \iff & (\forall i \in \omega)(i\, R_x\, i) \\ & \wedge (\forall i, j, k \in \omega)(i\, R_x\, j \wedge j\, R_x\, k \to i\, R_x\, k) \\ & \wedge (\forall i, j \in \omega)(i\, R_x\, j \wedge j\, R_x\, i \to i = j) \\ & \wedge (\forall i, j \in \omega)(i\, R_x\, i \wedge j\, R_x\, j \to i\, R_x\, j \vee j\, R_x\, i). \end{aligned}$$

Since for any fixed $i, j \in \omega$, the set of $\langle x, i, j\rangle$ satisfying $i\, R_x\, j$ (equivalently, $x(\ulcorner i, j\urcorner) = 1$) is clopen, we have a Borel definition of LO. $\dashv$

We obtain a more complicated class of structures by restricting to *well-orders*:

$$\mathrm{WO} = \{x \in 2^\omega \mid R_x \text{ is a well-order}\}.$$

Proposition 11.8. WO *is* $\mathbf{\Pi}^1_1$.

Proof. Notice that $x \in \mathrm{WO}$ if and only if $x \in \mathrm{LO}$ and R_x has no infinite descending chains. This last condition is the same as saying there is no infinite sequence $i_0, i_1, \ldots$ such that $i_{n+1}\, R_x\, i_n$ and $i_n \neq i_{n+1}$ for all $n \in \omega$. Thus $x \in \mathrm{WO}$ if and only if

$$x \in \mathrm{LO} \wedge \neg(\exists y \in \omega^\omega)(\forall n \in \omega)\neg(y(n+1) \neq y(n) \wedge x(\ulcorner y(n+1), y(n)\urcorner) = 1).$$

Now the set of $\langle x, y, n\rangle$ such that $y(n+1) \neq y(n) \wedge x(\ulcorner y(n+1), y(n)\urcorner) = 1$ is clearly open. So by the closure properties of $\mathbf{\Sigma}^1_1$, we have a $\mathbf{\Pi}^1_1$ definition of WO. $\dashv$

Now if $x \in \mathrm{WO}$ then its associated well-ordering of ω is isomorphic to some countable ordinal, the **order-type of** x. Let us write $\mathrm{ot}(x) = \gamma$ if and only if (ω, R_x) is isomorphic to $(\gamma, \in)$.

Proposition 11.9. *For each* γ, *let* $\mathrm{WO}_\gamma = \{x \in 2^\omega \mid \mathrm{ot}(x) = \gamma\}$. *Then* WO_γ *is* $\mathbf{\Sigma}^1_1$.

Proof. Exercise. $\dashv$

Notice that this gives us (without using choice) an equivalence relation on 2^ω with precisely ω_1 equivalence classes, each of which is $\mathbf{\Sigma}^1_1$. Under AD, there is no selector for this relation.

Let's now see that WO is as complicated as a $\mathbf{\Pi}^1_1$ set can get.

THEOREM 11.10. *Let $A \subseteq \omega^\omega$. Then A is $\mathbf{\Pi}^1_1$ if and only if there is a continuous function $f : \omega^\omega \to 2^\omega$ such that for all $x \in \omega^\omega$, $f(x) \in \mathrm{LO}$, and f satisfies the equivalence*

$$x \in A \iff f(x) \in \mathrm{WO}\,.$$

PROOF. If we have such a function f, then $A = f^{-1}[\mathrm{WO}]$. That $A \in \mathbf{\Pi}^1_1$ follows from closure of this pointclass under continuous preimages.

Now, suppose $A \in \mathbf{\Pi}^1_1$. We have a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ so that $x \in A$ precisely when $x \notin \exists^{\omega^\omega}[T]$; equivalently, $x \in A$ if and only if $T(x)$ is well-founded if and only if $<_{\mathrm{KB}}$ restricted to $T(x)$ is a well-order.

The trick, then, is to try to define f so that $f(x)$ will encode the Kleene-Brouwer order on $T(x)$. Let's fix an enumeration $\langle s_i \rangle_{i \in \omega}$ of $\omega^{<\omega}$; let's also require that our enumeration has the property that $i \leq j$ whenever $s_i \subseteq s_j$ (that is, we list all proper initial segments of s_i before we list s_i). We will define $f(x)$ so that $i \, R_{f(x)} \, j$ when $s_i, s_j \in T(x)$ and $s_i <_{\mathrm{KB}} s_j$. For those i for which $s_i \notin T(x)$, we simply put i on the top of $R_{f(x)}$ in the usual order; this ensures that $R_{f(x)}$ has domain all of ω.

Formally, we define

$$f(x)(\ulcorner i, j \urcorner) = \begin{cases} 1 & \text{if } s_i, s_j \in T(x), \text{ and } s_i <_{\mathrm{KB}} s_j, \\ 1 & \text{if } i < j \text{ and } s_i, s_j \notin T(x), \\ 1 & \text{if } s_i \in T(x) \text{ and } s_j \notin T_x, \\ 0 & \text{otherwise.} \end{cases}$$

Now for any x, $f(x)$ is a linear order because $<_{\mathrm{KB}}$ is; and $f(x)$ is well-founded exactly when $<_{\mathrm{KB}}$ is, that is, when $x \in A$.

Finally, notice that f is continuous: given $x \restriction n$, we know exactly which elements of $\omega^{<\omega}$ of length at most n are in $T(x)$, and so know the values of $f(x)(\ulcorner i, j \urcorner)$ whenever $\ulcorner i, j \urcorner = 2^i(2j+1) \leq n$—in fact, because of how we enumerated the s_i, this guarantees f is Lipschitz. $\dashv$

COROLLARY 11.11. *The set* WO *is $\mathbf{\Pi}^1_1$ and not $\mathbf{\Sigma}^1_1$.*

PROOF. Because by closure under continuous substitution $\mathrm{WO} \in \mathbf{\Sigma}^1_1$ would imply $\mathbf{\Pi}^1_1 \subseteq \mathbf{\Sigma}^1_1$! $\dashv$

The following theorem is usually invoked as "$\mathbf{\Sigma}^1_1$ Boundedness".

THEOREM 11.12. *Suppose $B \in \mathbf{\Sigma}^1_1$ and $B \subseteq \mathrm{WO}$. Then there is some $\gamma < \omega_1$ so that for all $x \in B$, we have $\mathrm{ot}(x) < \gamma$.*

PROOF. Suppose otherwise towards a contradiction, so members of B achieve arbitrarily high countable order-type. We'll show that WO would then be a member of $\mathbf{\Sigma}^1_1$.

For each $x \in \mathrm{WO}$, we have by assumption some $y \in B$ with $\mathrm{ot}(x) \leq \mathrm{ot}(y)$. In particular, we have an injective map $f : \omega \to \omega$ which embeds the linear order coded by x into that coded by y; that is to say, $x(\ulcorner i, j \urcorner) = 1$ if and only if $y(\ulcorner f(i), f(j) \urcorner) = 1$. Conversely, given $x \in \mathrm{LO}$ and such a map f and $y \in B$, we have $x \in \mathrm{WO}$. That is,

$$x \in \mathrm{WO} \iff (x \in \mathrm{LO}) \wedge (\exists y \in B)(\exists f : \omega \to \omega) f \text{ embeds } R_x \text{ into } R_y.$$

We claim this definition is $\mathbf{\Sigma}^1_1$. Since LO is Borel and B is $\mathbf{\Sigma}^1_1$, this is guaranteed by the closure properties of $\mathbf{\Sigma}^1_1$, provided the set $\{\langle x, y\rangle \mid (\exists f : \omega \to \omega) f \text{ embeds } R_x \text{ into } R_y\}$ is shown to be $\mathbf{\Sigma}^1_1$. This sacred task we entrust to the reader. $\dashv$

§12. Models of set theory and absoluteness. The central objects of study in group theory are groups. But it would be a mistake to say that the central objects of study in set theory are sets. Rather, the objects we care most about are set-theoretic universes, or models of set theory.

Definition 12.1. A **model of set theory** is a (set or class) structure $\mathcal{M} = \langle M, \varepsilon, ...\rangle$ in some language including a binary relation ε, satisfying some subset (often, all) of the axioms of ZFC.

The best models of set theory are those whose membership relation is the real $\in$; these, we can trust.

Definition 12.2. A set or class M is transitive if $x \in y \in M$ implies $x \in M$. We regard transitive M as a model of set theory by interpreting ε by $\in$.

The proof of the next theorem is essentially the same as that of Theorem 2.9. It tells us that it is often possible to restrict our attention to transitive models.

Theorem 12.3 (Mostowski). *Suppose $\mathcal{M}$ is a (set or class) model that is*

1. *Extensional: For all $x, y \in M$, $x = y$ iff for all $z \in M$, $z \,\varepsilon\, x \leftrightarrow z \,\varepsilon\, y$.*
2. *Wellfounded: Every $A \subseteq M$ has a ε-minimal element in M.*
3. *Set-like: For all $x \in M$, $\{y \in M \mid y \,\varepsilon\, x\}$ is a set.*

Then $\mathcal{M}$ is isomorphic to a unique transitive model.

Definition 12.4. A formula φ with n free variables (possibly with parameters) is **absolute** between transitive models $M \subseteq N$ if whenever $\vec{x} \in M^n$, we have $M \models \varphi(\vec{x}) \iff N \models \varphi(\vec{x})$. We say φ is **upwards absolute** if the left-to-right implication always holds, and **downwards absolute** if the right-to-left implication holds.

We will often simply say "φ is absolute" if it is absolute between any two *transitive models* that contain the parameters appearing in φ. Sufficiently simple formulas are absolute. Recall a quantifier is **bounded** if it has the form $(\forall x \in a)$ or $(\exists x \in b)$ (where these are really abbreviations: $(\forall x \in a)\phi$ means $(\forall x) x \in a \to \phi$, and $(\exists x \in a)\phi$ means $(\exists x) x \in a \wedge \phi$).

Definition 12.5. A formula ϕ in the language of set theory is Δ_0 if all of its quantifiers are bounded. It is Σ_1 if it has the form $(\exists x)\psi$ where ψ is Δ_0; and Π_1 if it has the form $(\forall x)\psi$ with ψ Δ_0. A formula ϕ is Δ_1 in T if it is provably equivalent in T both to a Σ_1 and to a Π_1 formula.

Note Δ_1-ness is not a syntactic condition on ϕ, and depends on the ambient theory T. However, when T is a subtheory of ZFC, we often simply say that a formula is Δ_1. The next fact is immediate, but very important.

Fact. *Δ_0 formulas are absolute. Σ_1 formulas are upwards absolute; Π_1 formulas are downwards absolute. Δ_1 (in T) formulas are absolute between any two models (of T).*

Examples:

1. "u is an ordinal" is Δ_0 (by the Axiom of Foundation), so is absolute.
2. "u is a real number" is Δ_0 in parameter ω, so is absolute.
3. "u is countable" is Σ_1 in parameter ω; it is upwards, but not downwards, absolute.
4. Neither "$u = \omega_1$" nor "u is the set of real numbers" is upwards, or downwards, absolute.

As the last example demonstrates, different models may disagree about the identities of basic mathematical objects. We therefore use the notation ν^M to refer to the object in M uniquely defined by the relativization of the notion ν to M. For example,

1. $\mathbb{R}^M$ denotes the reals of M and coincides with $\mathbb{R} \cap M$ when M is transitive (in particular, is countable when M is).
2. ω_1^M is the least ordinal of M not surjected onto by ω^M.
3. BC^M is the set of Borel codes (Definition 9.1) of M.
4. When $c \in \mathrm{BC}^M$, $(A^c)^M$ is M's version of the Borel set coded by c.

We next turn to projective statements. Recall that $\mathbf{\Sigma}^1_n$ statements involve quantifiers over $\mathbb{R}$; for models of set theory, we have a natural notion of satisfaction for projective predicates by interpreting the real quantifiers in the obvious way (that is, bounding by $\mathbb{R}^M$). Even though $\mathbb{R}$ is highly non-absolute, sufficiently simple projective statements are. Let us prove absoluteness for analytic predicates.

Theorem 12.6 (Mostowski absoluteness). *Let A be a $\mathbf{\Sigma}^1_1$ set of reals. Then membership in A is absolute between any two transitive models of set theory containing the parameters used in the definition of A.*

More precisely, whenever T is a tree on $\omega \times \omega$ and $\phi(y)$ is $(\exists x \in \omega^\omega)\langle x, y\rangle \in [T]$ (so that $A = \{y \in \omega^\omega \mid \phi(y)\} = \{y \in \omega^\omega \mid T_y \text{ is illfounded}\}$), we have, for all transitive models M of ZFC containing T,

$$M \models \phi(y) \iff V \models \phi(y)$$

for all $y \in \omega^\omega \cap M$.

Proof. Let T be a tree witnessing A is $\mathbf{\Sigma}^1_1$ and suppose $y, T \in M$. Suppose $M \models \phi(y)$ where ϕ is as above. Then T_y is illfounded in M; this witnessing branch $x \in [T_y]$ in M really is a branch in V, so $\phi(y)$ holds in V. Similarly if $M \models \neg\phi(y)$ then T_y is wellfounded in M, so there is a rank function $\rho : T_y \to \omega_1^M$ in M. But since M is transitive ω_1^M is an ordinal, so this rank function really is a rank function, and $\neg\phi(y)$ must hold in V. ⊣

The point of the proof is that membership in A is equivalent to a Δ_1 statement over ZFC. In fact, ZFC in M is rather more than we need, since the strongest axiom used for the existence of rank functions is Σ_1-Replacement.

Note that as an immediate consequence, BC and WO are absolute.

Our next goal is a similar absoluteness result for $\mathbf{\Sigma}^1_2$.

TO-DO:

- SHOENFIELD ABSOLUTENESS
- RELATIVE CONSTRUCTIBILITY
- MEASURABLE CARDINALS
- ANALYTIC DETERMINACY
- MEASURABILITY IN HOD
- FORCING
- FAILURE OF CHOICE

- MARTIN'S AXIOM
- CARDINAL CHARACTERISTICS

§13. An application of Martin's Axiom to Lebesgue measure. For ease of notation we let $\mathcal{C}$ be the collection of finite unions of open intervals with rational endpoints. Note that $\mathcal{C}$ is countable. We will show that open sets can be approximated closely in measure by members of $\mathcal{C}$.

PROPOSITION 13.1. *Let U be an open set with $0 < \mu(U) < \infty$. For every $\varepsilon > 0$ there is a member $Y \in \mathcal{C}$ such that $Y \subseteq U$ and $\mu(U \setminus Y) < \varepsilon$.*

PROOF. Let $\varepsilon > 0$ and assume that $\mu(U)$ is some positive real number m. Write $U = \bigcup_{n<\omega}(a_n, b_n)$ where the collection $\{(a_n, b_n) \mid n < \omega\}$ is pairwise disjoint. We choose $N < \omega$ such that $\sum_{n \geq N}(b_n - a_n) < \frac{\varepsilon}{2}$. For each $n < N$ we choose rational numbers q_n, r_n such that $a_n < q_n < r_n < b_n$ and

$$\mu((a_n, b_n) \setminus (q_n, r_n)) = |b_n - r_n| + |q_n - a_n| < \frac{\varepsilon}{2} \cdot 2^{-n-1}$$

We set $Y = \bigcup_{n<N}(q_n, r_n) \in \mathcal{C}$. An easy calculation shows that this works. ⊣

We are ready for our application of MA to Lebesgue measure.

THEOREM 13.2. MA(κ) *implies the union of κ-many measure zero sets is measure zero.*

PROOF. Let $\varepsilon > 0$. Define a poset $\mathbb{P}$ to be the collection of open $p \in \mathcal{L}$ such that $\mu(p) < \varepsilon$ and set $p_0 \leq p_1$ if and only if $p_0 \supseteq p_1$. As usual we need to show that $\mathbb{P}$ is ccc.

Towards showing that $\mathbb{P}$ is ccc, we let $\{p_\alpha \mid \alpha < \omega_1\}$ be a collection of conditions from $\mathbb{P}$. For each α we know that $\mu(p_\alpha) < \varepsilon$, so there is an $n_\alpha < \omega$ such that $\mu(p_\alpha) < \varepsilon - \frac{1}{n_\alpha}$. By the pigeonhole principal we may assume that there is an n such that $n = n_\alpha$ for all $\alpha < \omega_1$.

Now for each α we choose $Y_\alpha \in \mathcal{C}$ such that $Y_\alpha \subseteq p_\alpha$ and $\mu(p_\alpha \setminus Y_\alpha) < \frac{1}{2n}$. Since $\mathcal{C}$ is countable we may assume that there is a $Y \in \mathcal{C}$ such that $Y = Y_\alpha$ for all $\alpha < \omega_1$. Now let $\alpha < \beta < \omega_1$, we have

$$\mu(p_\alpha \cup p_\beta) \leq \mu(p_\alpha \setminus Y) + \mu(p_\beta \setminus Y) + \mu(Y) < \frac{1}{2n} + \frac{1}{2n} + \varepsilon - \frac{1}{n} = \varepsilon.$$

So p_α and p_β are compatible.

We use this poset to prove the theorem. Let $\{A_\alpha \mid \alpha < \kappa\}$ be a collection of measure zero sets. We want to show that the measure of the union is zero. Let $\varepsilon > 0$ and $\mathbb{P}$ be defined as above. We claim that $E_\alpha = \{p \in \mathbb{P} \mid A_\alpha \subseteq p\}$ is dense for each $\alpha < \kappa$. Let $q \in \mathbb{P}$. Since $\mu(A_\alpha) = 0$ we can find an open set r such that $A_\alpha \subseteq r$ and $\mu(r) < \varepsilon - \mu(q)$. Clearly $p = q \cup r \in E_\alpha$. So E_α is dense.

Now we apply MA to $\mathbb{P}$ and the collection of $\{E_\alpha \mid \alpha < \kappa\}$ to obtain G. We claim that $U = \bigcup G$ is an open set containing the union of the A_α and $\mu(U) \leq \varepsilon$. Clearly U is open since it is the union of open sets. Clearly it contains the union of the A_α, since G meets each E_α. It remains to show that $\mu(U) \leq \varepsilon$.

We claim that if $\{p_n \mid n < \omega\}$ is a subset of G, then $\mu(\bigcup_{n<\omega} p_n) \leq \varepsilon$. Note that since each $p_n \in G$, $p_0 \cup \cdots \cup p_n \in G$. Hence $\mu(p_0 \cup \cdots \cup p_n) < \varepsilon$. If we define $q_n = p_n \setminus (p_0 \cup \cdots \cup p_{n-1})$, then we have $\mu(q_0 \cup \cdots \cup q_n) = \mu(p_0 \cup \cdots \cup p_n) < \varepsilon$. So we have

$$\mu\left(\bigcup_{n<\omega} p_n\right) = \mu\left(\bigcup_{n<\omega} q_n\right) = \sum_{n<\omega} \mu(q_n) \le \varepsilon$$

since each partial sum is less than ε. This finishes the claim.

To finish the proof it is enough to show that there is a countable subset $B \subseteq G$ such that $\bigcup B = U$. Suppose that $x \in U$. Then $x \in p$ for some $p \in G$. So we can find $q_x \in \mathcal{C}$ such that $x \in q_x \subseteq p$. Since G is a filter $q_x \in G$. So $G = \bigcup_{x\in U} q_x$. But $\mathcal{C}$ is countable so $B = \{q_x \mid x \in U\}$ is as required. $\dashv$

Corollary 13.3. *If* ZFC *is consistent, then so is* $\mathsf{ZFC} + \omega_1 = \omega_1^L +$ *all* $\mathbf{\Sigma}^1_2$ *sets are Lebesgue measurable.*

Proof. Suppose $M \models V{=}L + \mathsf{ZFC}$; there is a ccc poset so that for M-generics G we have that $M[G]$ satisfies $2^{\aleph_0} > \aleph_1 + \mathrm{MA}_{\aleph_1}$. Note that then $\aleph_1^M = \aleph_1^{M[G]} = (\aleph_1^L)^{M[G]}$. Work in $M[G]$.

If A is a $\mathbf{\Sigma}^1_2$ set, then there is a tree T on ω^3 so that

$$A = \{y \in \omega^\omega \mid (\exists x) T_{x,y} \text{ is wellfounded}\}.$$

Since wellfounded trees have rank at most ω_1, we set $B_\alpha = \{\langle x, y\rangle \in \omega^\omega \mid T_{x,y}$ is wellfounded with rank exactly $\alpha\}$. Recall WO_α, the wellorders with order-type α, is Borel for each $\alpha < \omega_1$; and B_α is the continuous preimage of WO_α, so each B_α is Borel also. Now

$$A = \exists^{\omega^\omega} \bigcup_{\alpha<\omega_1} B_\alpha = \bigcup_{\alpha<\omega_1} \exists^{\omega^\omega} B_\alpha$$

so A is the ω_1-union of $\mathbf{\Sigma}^1_1$ sets. Now $\mathbf{\Sigma}^1_1$ sets are Lebesgue measurable (see Kechris's book for a proof, or use the following argument to prove it from $\mathrm{MA}_{\aleph_1}$); let $C_\alpha = \exists^{\omega^\omega} B_\alpha \setminus \bigcup_{\xi<\alpha} \exists^{\omega^\omega} B_\xi$. Then each C_α is Lebesgue measurable, since Lebesgue measurable sets form a σ-algebra.

We have A is the disjoint union $\bigcup_{\alpha<\omega_1} C_\alpha$, with each C_α Lebesgue measurable. But for an uncountable pairwise disjoint family of measurable sets, all but countably many must be measure zero. Hence for some $\alpha < \omega_1$, $\mu(C_\xi) = 0$ for all $\xi \ge \alpha$, and

$$A = \bigcup_{\xi<\alpha} C_\xi \cup \bigcup_{\alpha\le\xi<\omega_1} C_\xi.$$

By $\mathrm{MA}_{\aleph_1}$ and the previous theorem, the second set in the union has measure zero, and the first is measurable. It follows that A is Lebesgue measurable. $\dashv$

§14. The random real forcing. The following poset is central to Solovay's proof of the consistency of $\mathsf{ZF}+$ All sets of reals are Lebesgue measurable. In what follows, $\mathbb{R}$ can be taken to be any of the various objects we refer to as "the reals" $(\mathbb{R}, 2^\omega, \omega^\omega)$ with the appropriate notion of Lebesgue measure μ.

Definition 14.1. The **random real forcing** is

$$\mathbb{B} = \{p \in \mathcal{B}(\mathbb{R}) \mid \mu(p) > 0\},$$

ordered by $p \le q$ iff $p \subseteq q$.

Here as usual $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$. It's easy to see that $p \perp q$ iff $\mu(p \cap q) = 0$; and $\mathbb{B}$ has the ccc.

Recall from 9.1 the coding of Borel sets by $c \in \mathrm{BC}$ and decoding A^c. In order to analyze the random real forcing, we will need some absoluteness properties of this coding.

Proposition 14.2. *Let $c, d, e \in M \cap \omega^\omega$, where M is a transitive model of* ZF*, and let $x \in \mathbb{R}$. The following are absolute between M and all transitive models of* ZF *containing the relevant real parameters.*

1. $c \in \mathrm{BC}$
2. $x \in A^c$
3. $A^c \neq \varnothing$
4. $A^c \subseteq A^d$
5. $A^c \cap A^d = \varnothing$
6. $A^c \cap A^d = A^e$
7. $A^c = \bigcup_n A^{(d)_n}$; $A^c = \bigcap_n A^{(d)_n}$
8. $\mu(A^c) = x$

Note the absolute statements are just statements about the real parameters c, d, e, x; the interpretations of the sets A^c etc. of course will depend on the reals of the model in which these are taken; the point is that the relationships listed do not.

Proof. The set BC of Borel codes is $\mathbf{\Pi}^1_1$ (check this!) so by Mostowski absoluteness is absolute between all transitive models of ZF. The other items are clearly seen to be absolute when the A^c are basic open neighborhoods N_i; the claim follows for all Borel codes by a straightforward induction. Let us check the final item on Lebesgue measure.

Again note that the measure of the basic open neighborhoods is absolute. If $c \in \mathrm{BC}_1$ (c is an open code) there is a code d definable from c that expresses $A^c = A^d$ as a countable disjoint union of some N_{i_k}. Thus $\mu(A^c) = \mu(A^d) = \sup_n \mu(\bigcup_{k \le n} N_{i_k})$ is absolute between all models containing c.

Note that it is immediate that $\mu(A^c)$ is correctly computed iff $\mu(\mathbb{R} \setminus A^c)$ is.

Now suppose $c \in \mathrm{BC}_\alpha$; so there is a sequence of $c_n \in \mathrm{BC}_{\gamma_n}$, computable from c, with $\gamma_n < \alpha$, so that $A^c = \bigcup_{n \in \omega} \mathbb{R} \setminus A^{c_n}$. Then we may compute codes $d_n \in \mathrm{BC}_{\gamma_n}$ so that

$$A^{d_n} = \bigcap_{i \le n} A^{c_i};$$

in particular, the A^{d_n} are decreasing. Thus we have inductively that

$$\mu(A^c) = \mu(\bigcup_n \mathbb{R} \setminus A^{c_n}) = \mu(\bigcup_n \mathbb{R} \setminus A^{d_n}) = \sup_n \mu(\mathbb{R} \setminus A^{d_n})$$

is correctly computed in all models containing c.

$\dashv$

We return to our study of $\mathbb{B}$. Let $\{N_i\}_{i \in \omega}$ be some simple fixed basis of open neighborhoods in $\mathbb{R}$. The following two claims can be deduced from inner regularity of Lebesgue measure, i.e., $\mu(A) = \sup\{\mu(K) \mid K \subset A, K \text{ compact}\}$ for all Lebesgue measurable A.

Claim. *For all ε, the set*

$$D_\varepsilon = \{C \in \mathbb{B} \mid C \textit{ is closed and } (\exists i) C \subseteq N_i \wedge \operatorname{diam}(N_i) < \varepsilon\}$$

is dense in $\mathbb{B}$.

CLAIM. *For each Lebesgue measurable $A \subseteq \mathbb{R}$, the set*

$$E_A = \{C \in \mathbb{B} \mid C \textit{ is closed and either } C \subseteq A \vee C \cap A = \varnothing\}$$

is dense in $\mathbb{B}$.

LEMMA 14.3. *Let G be $\mathbb{B}$-generic over V. Then there is a unique real r such that for all Borel codes c in V,*

$$r \in (A^c)^{V[G]} \iff (A^c)^V \in G.$$

Hence $V[r] = V[G]$.

PROOF. Note that for all Borel codes $c \in V$ we have $(A^c)^V \subseteq (A^c)^{V[G]}$. Now

$$\mathcal{F} = \{(A^c)^{V[G]} \mid c \in V, (A^c)^V \in G\}$$

is a collection of closed sets, and has the finite intersection property: if $(A^{c_1})^V, \dots, (A^{c_n})^V \in G$, then because G is a filter, there is a single d with $A^d \subseteq \bigcap_{i=1}^n A_{c_i}$, $(A^d)^V \in G$. By a compactness argument, $\bigcap \mathcal{F}$ is non-empty; and by Claim 14, it must be a singleton. Let $\bigcap \mathcal{F} = \{r\}$.

We have the right-to-left direction by definition of r. For the forward implication, suppose $r \in (A^c)^{V[G]}$ for some $c \in \mathrm{BC}^V$. Then E_{A^c} as in Claim 14 is dense, and so we have a closed code $d \in \mathrm{BC}^V$ with $(A^d)^V \in G$ and either $A^d \subseteq A^c$ or $A^d \cap A^c = \varnothing$. Clearly the latter case is impossible since $r \in A^d \cap A^c$ in $V[G]$, and non-emptiness of this intersection is absolute. So we must have $(A^c)^V \in G$ as needed.

For the final conclusion, note that G is definable from r as

$$G = \{p \in \mathbb{B}^V \mid (\exists c \in \mathbb{R}^V) c \text{ is a closed code, } r \in A^c \text{, and } A^c \cap \mathbb{R}^V \subseteq p\}.$$

$\dashv$

We say a real r as in the conclusion of the lemma is **random over V**.

THEOREM 14.4. *Let $M \subseteq V$ be a transitive ZF-model. Then r is random over M iff $r \notin A^c$ for all Borel codes $c \in M$ such that $\mu(A^c) = 0$.*

PROOF. For the forward direction, suppose r is random over M and let G be the associated M-generic filter. Let $c \in M$ be a code with $\mu(A^c) = 0$. In M, $\{p \in \mathbb{B} \mid p \cap A^c = \varnothing\}$ is dense, by Claim 14. By the lemma let $d \in \mathrm{BC}^M$ such that $(A^d)^M \in G$ and $A^d \cap A^c = \varnothing$; then $r \in (A^d)^{M[G]}$ implies $r \notin A^c$.

For the converse, it's sufficient to show that whenever $D \in M$ is a dense subset of $\mathbb{B}^M$, then there is a closed code $c \in M$ such that $r \in A^c$ and $(A^c)^M \subseteq p$ for some $p \in D$. So working in M, fix a maximal antichain

$$\mathcal{A} \subseteq \{C \subseteq \mathbb{R} \mid C \text{ is closed and } C \subseteq p \text{ for some } p \in D\}.$$

$\mathbb{B}$ is ccc, so $\mathcal{A}$ is countable. Say $\mathcal{A} = \{A^{c_n}\}_{n\in\omega}$ for some Borel codes c_n. Since $\mathcal{A}$ is maximal, $\bigcup \mathcal{A}$ has full Lebesgue measure, and so there is a Borel code d so that $A^d = \mathbb{R} \setminus \bigcup A^{c_n}$ is null.

Back in V, we have by assumption that $r \notin (A^d)^M$, and so $r \in A^{c_n}$ for some $n \in \omega$, as needed. $\dashv$

COROLLARY 14.5. *Let M be a transitive ZFC-model with $\mathbb{R}^M$ countable. Then*

$$\{x \in \mathbb{R} \mid x \textit{ is not random over } M\}$$

has Lebesgue measure zero.

Proof. By the previous theorem, this set is precisely $\bigcup\{A^c \mid c \in \mathrm{BC}^M \wedge \mu(A^c) = 0\}$, a countable union of Lebesgue null sets. $\dashv$

§15. The Levy collapse.

Definition 15.1. For λ a cardinal and X a non-empty set, $\mathrm{Col}(\lambda, X)$ is the poset

$$\{p \subseteq \lambda \times X \mid |p| < \lambda \text{ and } p \text{ is a function}\},$$

ordered by reverse inclusion.

$\mathrm{Col}(\lambda, X)$ is the poset to collapse $|X|$ to λ. The following is immediate.

Proposition 15.2. *If λ is regular, then $\mathrm{Col}(\lambda, X)$ is λ-closed.*

As an example, let G be V-generic for $\mathrm{Col}(\omega_1, \mathbb{R})$. Since this poset is ω_1-closed, it adds no countable sequences of ordinals, and so $\mathbb{R}^V = \mathbb{R}^{V[G]}$ and $\omega_1^V = \omega_1^{V[G]}$. By standard density arguments, $\bigcup G$ is a surjection of ω_1 onto $\mathbb{R}$. Thus the continuum hypothesis holds in $V[G]$.

An **embedding** of posets $e : \mathbb{P} \to \mathbb{Q}$ is a map that preserves (left-to-right) $\leq$ and $\perp$. An embedding is **dense** if its range is dense in $\mathbb{Q}$, and **complete** if maximal antichains in $\mathbb{P}$ are mapped to maximal antichains in $\mathbb{Q}$; note that dense embeddings are complete. When there is a dense embedding $e : \mathbb{P} \to \mathbb{Q}$, then generics for $\mathbb{P}$ add generics for $\mathbb{Q}$ and vice versa; thus forcing with $\mathbb{P}$ is the same as forcing with $\mathbb{Q}$ as far as generic extensions are concerned.

Recall a poset $\mathbb{P}$ is **separative** if for all $p, q \in \mathbb{P}$, $p \not\leq q$ implies $r \perp q$ for some $r \leq p$. Separativity is equivalent to the condition $p \leq q \iff p \Vdash \check{q} \in \dot{G}$ for all $p, q \in \mathbb{P}$, where $\dot{G}$ is the name for the generic filter.

The next lemma tells us that forcing with any separative poset of size $\leq \alpha$ that collapses α to be countable is equivalent to forcing with $\mathrm{Col}(\omega, \alpha)$.

Lemma 15.3. *Suppose $\mathbb{P}$ is a separative poset, $|\mathbb{P}| \leq |\alpha|$, and $\mathbb{1} \Vdash_{\mathbb{P}} \exists f : \omega \to \check{\alpha}$ onto with $f \notin \check{V}$. Then there is a dense set $D \subseteq \mathrm{Col}(\omega, \alpha)$ and an injective dense embedding $e : D \to \mathbb{P}$.*

Proof. Working in the ground model, let $\nu = |\alpha|$. Note that for all $p \in \mathbb{P}$, there exists a ν-sized antichain of conditions below p; this is because if $\nu = \omega$, then the fact that we add a new function $f : \omega \to \alpha$ tells us every $p \in \mathbb{P}$ has incompatible extensions (so that in fact $\mathbb{P}$ is equivalent to $\mathrm{Add}(1, \omega)$; and if ν is uncountable then the fact that we collapse ν to ω tells us the ν-cc fails densely often.

Let $\dot{g}$ be a name so that $\mathbb{1} \Vdash_{\mathbb{P}} \dot{g} : \check{\omega} \to \dot{G}$ is onto. D will be the dense set

$$D := \{p \in \mathrm{Col}(\omega, \alpha) \mid \mathrm{dom}(p) \in \omega\},$$

and we define $e : D \to \mathbb{P}$ by induction on $\mathrm{dom}(p)$. $e(\mathbb{1}_{\mathrm{Col}(\omega,\alpha)}) = e(\varnothing) = \mathbb{1}_{\mathbb{P}}$. Suppose $e(p)$ is defined, $\mathrm{dom}(p) = n$; use the above remarks to obtain a maximal-below-p antichain $\langle a^p_\xi \rangle_{\xi<\alpha}$ of conditions strictly below p, such that for each $\xi < \alpha$, a^p_ξ decides the value of $\dot{g}(n)$, i.e. for each $\xi < \alpha$ there is some $r \in \mathbb{P}$ so that $a^p_\xi \Vdash_{\mathbb{P}} \dot{g}(\check{n}) = \check{r}$. Then set $e(p \cup \langle n, \xi \rangle) = a^p_\xi$ for all $\xi < \alpha$.

Clearly $e : D \to \mathbb{P}$ is one-to-one and an embedding; it is easy to verify by induction that for all n, $e[\{p \in \mathrm{Col}(\omega, \alpha) \mid \mathrm{dom}(p) = n\}] = \{a^p_\xi \mid p \in D, \mathrm{dom}(p) = n, \xi < \alpha\}$ is a maximal antichain in $\mathbb{P}$. Let us see that e is dense.

Suppose $r \in \mathbb{P}$; note that $r \Vdash_{\mathbb{P}} \check{r} \in \dot{G}$, so we may find $s \leq r$ so that $s \Vdash_{\mathbb{P}} \dot{g}(\check{n}) = \check{r}$ for some n. There is then a unique $p \in \mathrm{Col}(\omega, \alpha)$ so that $\mathrm{dom}(p) = n + 1$ and $e(p) \| s$. By our construction, also $e(p) \Vdash_{\mathbb{P}} \dot{g}(n) = \check{r}$. Since it is forced that g maps into $\dot{G}$, we have $e(p) \Vdash \check{r} \in \dot{G}$, whence $e(p) \leq r$ by separativity. ⊣

DEFINITION 15.4. Let X be a non-empty set. The **Levy collapse** of X is the poset $\mathrm{Col}(\lambda, {\in}X)$ with set of conditions

$$\{p \subseteq (\lambda \times X) \times X \mid p \text{ is a function, } |p| < \lambda, \text{ and } (\forall \xi, a \in \mathrm{dom}(p)) p(\xi, a) \in a\},$$

ordered by reverse inclusion. We write $\mathrm{Col}(\lambda, {<}\alpha)$ for the Levy collapse when $X = \alpha$ is an ordinal.

Note that by the previous lemma, forcing with $\mathrm{Col}(\omega, {<}\alpha + 1)$ is the same as forcing with $\mathrm{Col}(\omega, \alpha)$. It's also clear that $\mathrm{Col}(\lambda, {<}X)$ is λ-closed whenever λ is regular.

LEMMA 15.5. *Suppose $\kappa > \lambda$ and both λ, κ are regular; and that either κ is inaccessible or $\lambda = \omega$. Then $\mathrm{Col}(\lambda, {<}\kappa)$ has the κ-cc.*

PROOF. ⊣

TO-DO:

- REST OF LEVY COLLAPSE
- SOLOVAY MODEL

We have already seen that the perfect set property for all $\boldsymbol{\Sigma}^1_2$ sets implies $\aleph_1$ is inaccessible in L (and remark in passing that actually $\boldsymbol{\Pi}^1_1$ is enough), so we have an equiconsistency in the case of the perfect set property. We have also seen that Lebesgue measurability of all $\boldsymbol{\Sigma}^1_2$ sets is relatively consistent with just ZFC. Our next goal is to prove a result of Shelah that shows we can't carry this to the next level without an inaccessible: If all $\boldsymbol{\Sigma}^1_3$ sets are Lebesgue measurable, then $\aleph_1$ is inaccessible in L. In particular, we have an equiconsistency in the case of Lebesgue measurability. The proof we give here is due to Raisonnier.

§16. Filters as pathological sets. In this section $\mathcal{F}$ is always a filter on ω that extends the Frechet filter. We regard $\mathcal{F}$ as a subset of Cantor space by identifying it with

$$\{\chi_a \mid a \in \mathcal{F}\},$$

where $\chi_a : \omega \to 2$ is the characteristic function of a.

THEOREM 16.1. *Let $\mathcal{F}$ be a filter. Then*

1. *$\mathcal{F}$ is Lebesgue measurable iff $\mu(F) = 0$.*
2. *$\mathcal{F}$ has the Baire property iff $\mathcal{F}$ is meager.*
3. *If $\mathcal{F}$ is an ultrafilter, then $\mathcal{F}$ is not measurable, and does not have the Baire property.*

PROOF. Note that by our assumption that $\mathcal{F}$ extends the Frechet filter, $\mathcal{F}$ is a tail set. We show (2); (1) is similar.

So suppose $\mathcal{F}$ has the Baire property. Then by Theorem 6.4, $\mathcal{F}$ is either meager or comeager. We claim comeager is impossible. For letting $T : 2^\omega \to 2^\omega$ be the toggle map,

$$T(x)(n) = 1 - x(n)$$

we have that T is a category-preserving homeomorphism of Cantor space. Then $T[\mathcal{F}]$ is comeager iff $\mathcal{F}$ is. So if $\mathcal{F}$ is comeager, then $\mathcal{F} \cap T[\mathcal{F}]$ is non-empty. But since T takes χ_a to $\chi_{\omega \setminus a}$, this contradicts $\mathcal{F}$ a filter.

For item (3), just note that if $\mathcal{F}$ is an ultrafilter then $2^\omega = \mathcal{F} \sqcup T[\mathcal{F}]$, and so $\mathcal{F}$ cannot be meager or measure zero. ⊣

Given a filter $\mathcal{F}$, let $E_{\mathcal{F}} = \{e_a \in \omega^\omega \mid a \in \mathcal{F}\}$, where for $a \in [\omega]^\omega$ the function e_a enumerates the elements of a in increasing order.

Theorem 16.2. *Let $\mathcal{F}$ be a filter. Then $\mathcal{F}$ has the Baire property iff $E_{\mathcal{F}}$ is a bounded family.*

Proof. For right-to-left, let $f : \omega \to \omega$ be such that $e_a \leq^* f$ for all $a \in \mathcal{F}$. Then for all n,

$$A_n = \{a \subseteq \omega \mid (\forall k \geq n) e_a(k) \leq f(k)\}$$

is nowhere dense in $\mathcal{P}(\omega)$. And $\mathcal{F} \subseteq \bigcup_{n \in \omega} A_n$ by assumption.

For left-to-right, we will need a definition and lemma that give a nice characterization of meagerness.

Definition 16.3. A **chopped real** is a pair (x, Π) such that $x \in 2^\omega$ and $\Pi = \langle I_n \rangle_{n \in \omega}$ is a partition of ω into finite intervals. We say $y \in 2^\omega$ **matches** (x, Π) is $\{n \mid x \restriction I_n = y \restriction I_n\}$ is an infinite set.

Lemma 16.4. *A set $A \subseteq 2^\omega$ is meager iff*

$$A \subseteq N^{(x,\Pi)} := \{y \in 2^\omega \mid y \textit{ does not match } (x, \Pi)\}.$$

Proof of Lemma. Note that $y \in N^{(x,\Pi)}$ iff $(\exists n)(\forall i \geq n) x \restriction I_n \neq y \restriction I_n$, and clearly this inner condition is nowhere dense, so $N^{(x,\Pi)}$ is indeed meager.

For the converse, let $A \subseteq \bigcup_n F_n$ with each F_n nowhere dense. Define sequences $\langle k_n \rangle_{n \in \omega}$ and $\langle t_n \rangle n \in \omega$, $k_n \in \omega$ and $t_n \in 2^{<\omega}$, as follows:

- $k_0 = 0$;
- t_0 satisfies $N_{t_0} \cap F_0 = \varnothing$.

Then, having defined k_n,

- t_{n+1} satisfies $(\forall s \in 2^{k_n})(\forall i \leq n) N_{s^\frown t_{n+1}} \cap F_i = \varnothing$;
- $k_{n+1} = k_n + |t_{n+1}|$.

Note that the first condition is possible since the F_i are nowhere dense; we get t_{n+1} by extending $n \cdot 2^{k_n}$ many times, dodging F_i with $s^\frown t$ at the $\langle i, s \rangle$-th step.

Now set $x = t_0^\frown t_1^\frown t_2^\frown \ldots$, and $\Pi = \langle I_n \rangle_{n \in \omega} = \langle [k_n, k_{n+1}) \rangle_{n \in \omega}$. By construction, if $y \in F_n$, then $\forall m > n$ and $\forall s \in 2^{k_{m-1}}$, $y \notin N_{s^\frown t_m}$; hence $t_m = x \restriction I_m \neq y \restriction I_m$, so that $F_n \subseteq N^{(x,\Pi)}$ for all n as needed. ⊣

We resume the proof of the forward direction of the theorem. So suppose for a contradiction that $\mathcal{F}$ is meager but $E_{\mathcal{F}}$ is an unbounded family. By the lemma we have a chopped real (x, Π) so that $\mathcal{F} \subseteq N^{(x,\Pi)}$. Say $\Pi = \langle [f(n), f(n+1)) \rangle_{n \in \omega}$ with f increasing. Since $E_{\mathcal{F}}$ is unbounded we have some $a \in \mathcal{F}$ so that $e_a(n) \not\leq f(2n)$ for infinitely many n. It follows that $a \cap I_n = \varnothing$ for infinitely many n.

Now we are free to define $b \subset \omega$ such that

$$\chi_b \restriction I_n = \begin{cases} x \restriction I_n & \text{if } a \cap I_n = \varnothing, \\ \chi_a \restriction I_n & \text{otherwise.} \end{cases}$$

Clearly $b \supset a$, hence $b \in \mathcal{F}$. But χ_b matches (x, Π), contradicting $\mathcal{F} \subseteq N^{(x,\Pi)}$. $\dashv$

DEFINITION 16.5. A filter $\mathcal{F}$ is **rapid** if for all increasing $f : \omega \to \omega$, there is $a \in \mathcal{F}$ such that for all n,

$$|a \cap f(n)| \leq n.$$

This is equivalent to saying that for all increasing interval partitions $\langle I_n \rangle_{n\in\omega}$, at most k many elements of a appear among the first k-many intervals; thus no matter what f we choose, there is an $a \in \mathcal{F}$ so that e_a grows *rapidly*. Note that then if $\mathcal{F}$ is rapid, $E_{\mathcal{F}}$ is a dominating family (i.e. for all $f \in \omega^\omega$ there is $a \in \mathcal{F}$ with $f \leq^* e_a$); in particular, by the previous theorem, rapid filters do not have the Baire property.

LEMMA 16.6. *The following are equivalent, for filters $\mathcal{F}$:*

1. *$\mathcal{F}$ is rapid.*
2. $(\exists f_0 \in \omega^\omega)(\forall f \in \omega^\omega)(\exists a \in \mathcal{F})(\forall n \in \omega)|a \cap f(n)| \leq f_0(n)$.
3. *For all sequences $\langle \varepsilon_n \rangle_{n\in\omega}$ of reals with $\lim_{n\to\infty} \varepsilon_n = 0$, there is $a \in \mathcal{F}$ such that $\Sigma_{n\in a}\varepsilon_n < \infty$.*

PROOF. Trivially, (1) implies (2).

Suppose f_0 witnesses (2). Fix $\langle \delta_k \rangle_{k\in\omega}$ so that $\Sigma_{n\in\omega}\delta_n \cdot f_0(n) < \infty$. Then let f be increasing such that $\varepsilon_n < \delta_k$ whenever $n > f(k)$. If $a \in \mathcal{F}$ is as in (2), we have

$$\sum_{n\in a} \varepsilon_n < \sum_{k\in\omega} \delta_k \cdot f_0(k) < \infty,$$

and so (2) implies (3).

Finally, assume (3); let $f : \omega \to \omega$ increasing. Let $\varepsilon_k = 1/n$ whenever $k \in [f(n), f(n+1))$. Then by (3), we may fix $a \in \mathcal{F}$ so that $\Sigma_{n\in a}\varepsilon_n < 1$. Clearly then $|a \cap f(n)| \leq n$ for all n, so we have (1). $\dashv$

THEOREM 16.7. *Suppose $\mathcal{F}$ is a rapid filter. Then $\mathcal{F}$ is not Lebesgue measurable.*

We freely use the following standard lemma from measure theory.

LEMMA 16.8 (Lebesgue density lemma). *For Lebesgue measurable $A \subseteq 2^\omega$, set*

$$\Phi(A) := \{x \in 2^\omega \mid \lim_{n\to\infty} \frac{\mu(A \cap N_{x\restriction n})}{2^n} = 1\}.$$

Then $\mu(\Phi(A) \triangle A) = 0$.

PROOF OF THEOREM 16.7. Suppose $\mathcal{F}$ rapid. We will show for every closed $K \subseteq 2^\omega$ with positive measure that $\mathcal{F} \cap K \neq \varnothing$; clearly then $\mathcal{F}$ must be nonmeasurable, since otherwise $\mathcal{F}$ has measure zero and so $\mathcal{F} \cap (2^\omega \setminus \mathcal{F})$ is non-empty, a contradiction.

So let $\mu(K) > 0$ with K closed. Fix a tree $T \subseteq 2^{<\omega}$ such that $K = [T]$.

Notice that by Lebesgue density we have for every $s \in 2^{<\omega}$ and $i \in \omega$, that there are arbitrarily long extensions t of s such that $\mu(N_t \cap K) > (1 - 2^{-i})\mu(N_t)$. We can use this to define a sequence of maximal $\supseteq$-antichains $\langle A_i \rangle_{i\in\omega}$ in T, satisfying

- $s \in A_i$ and $t \in A_{i+1}$ implies $|s| < |t|$,
- For all $s \in A_i$, $\mu(N_s \cap K) > (1 - 2^{-i})\mu(N_s)$.

Note each A_i is necessarily finite, so the first item can be satisfied; and maximality of the A_i implies each $t \in A_{i+1}$ extends a unique $s \in A_i$. Now set $n_i = \sup_{s \in A_i} |s| + 1$ and let

$$B = \{x \in 2^\omega \mid (\exists i)(\exists k) x \restriction k \in A_i\}.$$

Clearly $B \subseteq [T] = K$.

By rapidity, there exists $a \in \mathcal{F}$ such that $|a \cap n_{i+2}| \leq i$ for all i. We now inductively define an increasing sequence $\langle s_i \rangle_{i \in \omega}$ satisfying

1. $s_i \in A_i$,
2. $N_{s_i} \cap B \neq \varnothing$,
3. $s_i(n) = 1$ whenever $n \in a \cap n_i$.

The key item is (3), which ensures that the $\bigcup_{i \in \omega} s_i$ that we build is the characteristic function of a superset of a.

So suppose we have s_i as above. Let

$$H = \{x \in 2^\omega \mid x(n) = 1 \text{ for all } n \in a \cap [|s_i|, n_{i+1}]\}.$$

Now $|a \cap [|s_i|, n_{i+1}]| \leq i - 1$ by choice of a, hence

$$\mu(H \cap N_{s_i}) \geq \frac{1}{2^{i-1}} \mu(N_{s_i})$$

by the definition of Lebesgue measure as "coin-flipping" measure. It now follows from our definition of the A_i that $H \cap K \cap N_{s_i}$ is non-empty, so let $x \in H \cap K$ with $s_i \subseteq x$. Since A_{i+1} is a maximal antichain, we have some unique $s_{i+1} \subseteq x$ in A_{i+1}. So the induction proceeds.

Finally, let $b = \{n \mid (\exists i) s_i(n) = 1\}$. Then $a \subseteq b$, whence $b \in \mathcal{F}$. But also $b \in B \subseteq K$ since B is closed. This completes the proof. ⊣

We next identify a special filter that is simply definable and, under the right circumstances, is rapid.

Definition 16.9. Let $X \subseteq 2^\omega$; define $S(X) \subseteq \omega$, the **set of splitting levels of** X,

$$S(X) = \{n \in \omega \mid (\exists x, y \in X) x \restriction n = y \restriction n \text{ and } x(n) \neq y(n)\}.$$

Note that $X \subseteq Y$ implies $S(X) \subseteq S(Y)$, and $S(X) = S(\overline{X})$, where $\overline{X}$ is the closure in 2^ω of X.

Definition 16.10. Let $X \subseteq 2^\omega$ be uncountable. The **Raisonnier filter** for X, denoted $\mathcal{R}_X$, is

$$\mathcal{R}_X := \{b \subseteq \omega \mid \text{For some } \{Y_n\}_{n \in \omega} \subseteq \mathcal{P}(2^\omega),\ X \subseteq \bigcup_n Y_n \text{ and } b \supseteq \bigcup_n S(Y_n)\}.$$

Note that by the preceding remarks, we may always assume the Y_n witnessing membership of b in $\mathcal{R}_X$ are pairwise disjoint. Also note that if X were countable, then we could take $Y_n = \{x_n\}$ where $X = \{x_n\}_{n \in \omega}$, which would witness $\varnothing \in \mathcal{R}_X$; thus the definition would be vacuous for countable X.

Proposition 16.11. *$\mathcal{R}_X$ is a cofinite filter.*

Proof. Note $\varnothing \notin \mathcal{R}_X$ by uncountability of X, since the only sets without splitting levels have size at most 1.

To see $\mathcal{R}_X$ extends the Frechet filter, suppose $n \in \omega$; then letting $Y_s = N_s \cap X$, we get $\{Y_s\}_{s\in 2^n}$ witnesses $\omega \setminus n \in \mathcal{R}_X$.

The only thing left to check is closure under intersection. Let $b_1, b_2 \in \mathcal{R}_X$; let this be witnessed by partitions $\{Y^1_n\}_{n\in\omega}, \{Y^2_n\}_{n\in\omega}$ of X. Then their common refinement, $\{Y^1_i \cap Y^2_j\}_{i,j\in\omega}$, witnesses that $b_1 \cap b_2 \in \mathcal{R}_X$. $\dashv$

PROPOSITION 16.12. *$b \in \mathcal{R}_X$ iff there is a function $f : 2^{<\omega} \to 2$ so that for all $x \in X$, we have $f(x \restriction n) = x(n)$ for all but finitely many $n \notin b$.*

PROOF. For the forward direction, suppose $\langle Y_n\rangle_{n\in\omega}$ is a witness to $b \in \mathcal{F}_X$. We may assume $Y_n = [T_n]$ for trees $T_n \subseteq 2^{<\omega}$. For $s \in 2^{<\omega}$, let n_s be the least n such that $s \in T_n$ if such exists, and undefined otherwise. Put

$$f(s) = \begin{cases} \text{the unique } i \text{ such that } s^\frown\langle i\rangle \in T_{n_s} & \text{if } |s| \notin b \text{ and } n_s \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

We claim f is as desired. For if $x \in X$, then let n be least such that $x \in [T_n]$. $n_{x\restriction i}$ is a nondecreasing sequence that converges to n. It follows that for all sufficiently large $n \notin b$, $x(n) = f(x \restriction n)$.

For the converse, let $b \subseteq \omega$ be such that some f as above exists. Define, for all $s \in 2^{<\omega}$,

$$Y_s = \{x \in 2^\omega \mid x \restriction |s| = s, (\forall n \notin b)x(n) = f(x \restriction n)\}.$$

Clearly $S(Y_s) \subseteq b$ for all $s \in 2^{<\omega}$, and by assumption on f, if $x \in X$ then letting n_0 be such that $f(x \restriction n) = x(n)$ for all $n \geq n_0$, we have $x \in Y_{x\restriction n_0}$; hence $X \subseteq \bigcup_{s\in 2^{<\omega}} Y_s$ as needed. $\dashv$

COROLLARY 16.13. *If X is $\mathbf{\Sigma}^1_n$, then $\mathcal{R}_X$ is $\mathbf{\Sigma}^1_{n+1}$.*

For our consistency strength lower bound, we will need a deep fact connecting combinatorics of $(([\omega]^{<\omega})^\omega$ to Lebesgue measurability. We first need to introduce a special subset $\mathcal{C}$ of $([\omega]^{<\omega})^\omega$ and a partial order of $\mathcal{C}$ whose structure will coincide with that of the ideal of Lebesgue null sets. Put

$$\mathcal{C} := \{f : \omega \to [\omega]^{<\omega} \mid \sum_{n=1}^{\infty} \frac{|f(n)|}{2^n} < \infty\}$$

And for $f, g \in ([\omega]^{<\omega})^\omega$, write $f \subseteq^* g$ iff $f(n) \subseteq^* g(n)$ for all but finitely many n. (We remark that the choice of "2^n" in the denominator is somewhat arbitrary, and could be replaced with any function in n tending to infinity.)

LEMMA 16.14. *Let $a \in \omega^\omega$. Then the following are equivalent.*

1. *All $\Sigma^1_2(a)$ sets are Lebesgue measurable.*
2. *For set $\{x \in \mathbb{R} \mid x$ is not random over $L[a]\}$ is Lebesgue null.*
3. *$\bigcup\{A^c \mid c \in \mathrm{BC}^{L[a]}$ and $\mu(A^c) = 0\}$ is Lebesgue null.*
4. *There is $\phi \in \mathcal{C}$ so that $g \subseteq^* \phi$ for all $g \in \mathcal{C} \cap M$.*
5. *There is $\phi : \omega \to [\omega]^{<\omega}$ with $|\phi(n)| \leq n$ for all n, and for all $x \in \omega^\omega \cap L[a]$, $x(n) \in \phi(n)$ for all but finitely many n.*

We will return to the proof of this (with many parts left as exercises) in a bit. First let's prove the result assuming Lemma 16.14.

THEOREM 16.15 (Raisonnier). *Suppose all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable and that $\aleph_1^{L[a]} = \aleph_1$ for some $a \in \omega^\omega$. Then if we set $X = 2^\omega \cap L[a]$, $\mathcal{R}_X$ is a rapid filter.*

Proof. Note our assumption about $\aleph_1$ is necessary to ensure that X is uncountable, and so (by Proposition 16.11) is a filter.

Fix an increasing function $f : \omega \to \omega$. Let us write $I_n = [f(n), f(n+1))$ for $n \in \omega$. By our assumption and Lemma 16.14 for the model $L[a, f]$, we have the existence of a function $\phi : \omega \to [\omega]^{<\omega}$ as in (5), so that $x(n) \in \phi(n)$ for all but finitely many n, for every $x \in \omega^\omega \cap L[a, f]$.

It is easy to see that we can code any $x \in 2^\omega = \Pi_{n\in\omega} 2^{I_n}$ by a real y so that $y(n)$ is the index of $x \upharpoonright I_n$ in some enumeration of $2^{<\omega}$. So using our ϕ, we obtain a sequence of sets $G_n \subseteq 2^{I_n}$ so that $|G_n| \leq n$ for all n, and for every $x \in 2^\omega \cap L[a, f]$, $\{n \in \omega \mid x \upharpoonright I_n \notin G_n\}$ is finite.

For $s \in 2^{f(n)}$, set $Y_{n,s} = \{x \in N_s \mid (\forall k \geq n) x \upharpoonright I_n \in G_k\}$. By what was just said, $X \subseteq \bigcup_{n\in\omega, s\in 2^{f(n)}} Y_{n,s}$. We let $b = \bigcup_{n\in\omega, s\in 2^{f(n)}} S(Y_{n,s})$. By definition we have $b \in \mathcal{R}_X$.

We claim b is a witness to rapidity of $\mathcal{R}_X$ for f. To this end, we need to study the total number of splits among the $Y_{n,s}$ restricted to each interval I_n. If $i \in \bigcup_{n,s} S(Y_{n,s})$, then we have some n so that $i \in I_n \cap Y_{n,s}$; in particular, $s \in 2^{f(n)}$, and i is a splitting level of $Y_{n,s}$. But the various possible restrictions of elements of $Y_{n,s}$ to I_n all belong to G_n, and $|G_n| \leq n$. Since a split is witnessed by a pair $u, v \in G_n$ with $s^\frown u$ and $s^\frown v$ first disagreeing at i, there are at most $|G_n|^2 \leq n^2$ elements in $\bigcup S(Y_{n,s})$. We have

$$|b \cap f(n)| = |\bigcup S(Y_{n,s}) \cap f(n)| \leq \sum_{i=1}^{n} |\bigcup S(Y_{n,s}) \cap I_n| \leq \sum_{i=1}^{n} i^2 \leq n^4.$$

Then $f_0(n) = n^4$ is a witness to (2) in Lemma 16.6, so $\mathcal{R}_X$ is rapid. ⊣

Corollary 16.16. *Suppose all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable. Then if $\mathbf{\Sigma}^1_3$ sets all are Lebesgue measurable, or all have the Baire property, then $\aleph_1$ is inaccessible in $L[a]$ for all reals a.*

Proof. We show the contrapositive: if $\aleph_1$ is not inaccessible from reals, then $\aleph_1 = \aleph_1^{L[a]}$ for some $a \in \omega^\omega$, and if all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable, then by Theorem 16.15, $\mathcal{R}_X$ is a rapid filter with $X = 2^\omega \cap L[a]$; note X is $\mathbf{\Sigma}^1_2$, and so $\mathcal{R}_X$ is $\mathbf{\Sigma}^1_3$ by Corollary 16.13. By Theorems 16.7 and 16.2, $\mathcal{R}_X$ is neither Lebesgue measurable nor has the Baire property. ⊣

Proof of Lemma 16.14. Note that equivalence of (2) and (3) is immediate by Theorem 14.4. (4) implies (5) is also quite clear; that (5) implies (4) is left as an exercise.

For (1) implies (2), we define a preorder on reals not random over $L[a]$ as follows: For x not random over $L[a]$ let $\nu(a)$ be the height in the $\leq_{L[a]}$ ordering of the $\leq_{L[a]}$-least Borel code $c \in L[a]$ of a Lebesgue null set with $x \in A^c$. Then for x, y not random over $L[a]$, we let $x \preceq y$ iff $\nu(x) \leq \nu(y)$. Note that initial segments of $\preceq$ need not be countable, although each such initial segment is a union of $\preceq$-induced equivalence classes.

One can show that $\preceq$, regarded as a subset of $\mathbb{R} \times \mathbb{R}$, is $\Sigma^1_2(a)$-definable, and that $\preceq$ is either measure zero or nonmeasurable; furthermore, if $\preceq$ is measure zero then (2) holds. These are left as an exercise.

The proof that (2) implies (1) is likewise left as an exercise, but closely mirrors Solovay's proof that all sets of reals are Lebesgue measurable in $L(\mathbb{R})$ of the Levy collapse. The key step is finding the right Borel set to approximate the $\Sigma^1_2(a)$ set A under consideration; for that, one can use a maximal antichain in $\mathbb{B}$ (from the point of view of $L[a]$)

of conditions deciding membership in A, noting that $\mathbb{B}$ has the ccc. For preservation of the decision, appeal to Shoenfield absoluteness.

The main event, of course, is (3) iff (4). We will restrict our attention to Lebesgue measurability in Cantor space 2^ω. Let $\mathcal{N}$ be the set of Lebesgue null subsets of 2^ω.

Proposition 16.17. *There exist maps $\psi : \mathcal{C} \to \mathcal{N}$ and $\pi : \mathcal{N} \to \mathcal{C}$ such that whenever $f \in \mathcal{C}$ and $G \in \mathcal{N}$ are such that $\psi(f) \subseteq G$, we have $f \subseteq^* \pi(G)$.*

Proof. We first fix a family of open sets $\{G_{i,j}\}_{i,j\in\omega}$ that are independent, in the sense that $\mu(G_{i,j} \cap G_{i',j'}) = \mu(G_{i,j}) \cdot \mu(G_{i',j'})$ whenever the pairs $\langle i, j\rangle, \langle i', j'\rangle$ are distinct. We also require that $\mu(G_{i,j}) = 2^{-i}$ for all i, j. (This is easy enough to arrange—for example, take $\{I_{i,j}\}_{i,j\in\omega}$ to be a partition of ω with $|I_{i,j}| = i$ for all i, and let $x \in G_{i,j}$ iff $x \restriction I_{i,j}$ is the zero sequence.)

Now for $f \in \mathcal{C}$ let $\psi(f) = \bigcap_{n\in\omega} \bigcup_{m>n} \bigcup_{k\in f(m)} G_{m,k}$. Note that

$$\mu(\psi(f)) \leq \inf_n \sum_{m>n} |f(m)|\mu(G_{m,k}) = \inf_n \sum_{m>n} \frac{|f(m)|}{2^m},$$

and the right-hand side is zero by assumption, so that $\psi(f) \in \mathcal{N}$. We note also that $f \subseteq^* g$ implies $\psi(f) \subseteq \psi(g)$.

The map π is harder to define. For $G \in \mathcal{N}$, fix some $K^G \subseteq 2^\omega$ closed with positive measure such that $K^G \cap G = \varnothing$. Since K^G is closed we can further assume for all open U that $K^G \cap U \neq \varnothing$ iff $\mu(K^G \cap U) > 0$. Let $\{s^G_n\}_{n\in\omega}$ enumerate those $s \in 2^{<\omega}$ such that $K^G \cap N_s$ is nonempty. For $n, i \in \omega$ we put

$$A^G_{n,i} = \{j \in \omega \mid K^G \cap N_{s^G_n} \cap G_{i,j} = \varnothing\}.$$

By definition, then, we have that if $x \in K^G \cap N_{s^G_n}$, then for all $i \in \omega$ and $j \in A^G_{n,i}$, we have $x \notin G_{i,j}$. That is, for all n

$$K^G \cap N_{s^G_n} \subseteq \bigcap_{i\in\omega} \bigcap_{j\in A^G_{n,i}} 2^\omega \setminus G_{i,j}.$$

Thus by independence of the $G_{i,j}$, we have for all n

$$0 < \mu(K^G \cap N_{s^G_n}) \leq \prod_{i\in\omega} \prod_{j\in A^G_{n,i}} \mu(2^\omega \setminus G_{i,j}).$$

Taking logs, it follows that

$$\sum_i \frac{|A^G_{n,i}|}{2^i} < \sum_i |A^G_{n,i}| \ln(\frac{2^i}{2^i - 1}) < \infty.$$

In particular, the map $i \mapsto A^G_{n,i}$ is in $\mathcal{C}$ for all $n \in \omega$.

The following claim is left as an exercise.

Claim. *The order $(\mathcal{C}, \subseteq^*)$ is countably directed; that is, for all sequences $\langle f_n\rangle_{n\in\omega}$ of elements of $\mathcal{C}$, there is $g \in \mathcal{C}$ with $f_n \subseteq^* g$ for all n.*

Using the claim, we let $\pi(G)$ be any function in $\mathcal{C}$ so that for all n, $A^G_{n,i} \subseteq \pi(G)$ for all but finitely many i. We need to show ψ, π have the desired properties.

So let $f \in \mathcal{C}$ and suppose $G \in \mathcal{N}$ with $G \supseteq \psi(f)$. That is,

$$\bigcap_{n\in\omega} \bigcup_{m>n} \bigcup_{k\in f(m)} G_{m,k} \subseteq G.$$

Then $K^G \cap \psi(f) = \varnothing$. Notice that $\psi(f)$ is G_δ; by the Baire Category Theorem applied inside K^G, some set in this intersection must fail to be dense, i.e. for some n we have $N_{s_n^G} \cap K^G \neq \varnothing$ and all sufficiently large n,

$$N_{s_n^G} \cap K^G \cap \bigcup_{i \geq n} \bigcup_{k \in f(i)} G_{i,k} = \varnothing.$$

that is, for all sufficiently large n,

$$f(i) \subseteq A^G_{n,i} \subseteq \pi(G)(i),$$

as needed. ⊣

To complete the proof of Lemma 16.14, note that membership in $\mathcal{C}$ or in $\mathcal{N}$ is absolute between V and $L[a]$, as are the relations $\subseteq^*$ and $\subseteq$. It is clear from the definitions of ψ and π that we may demand that $\psi(f), \pi(A^c)$ belong to $L[a]$ whenever f, c do (where here c is a Borel code of a null set in $L[a]$).

Now if (3) holds, then there is a set $G \in \mathcal{N}$ so that $G \supseteq H$ for all Borel null sets H with Borel code in $L[a]$. Hence for all $f \in \mathcal{C} \cap L[a]$, we have $\pi(G) \supseteq^* \pi \circ \psi(f) \supseteq^* f$. So $\pi(G)$ is a witness to (4). ⊣

TO-DO:

- ITERATED ULTRAPOWERS