IV. The Circle Method

Chapter 14
Analytic Theory of Partitions

Preliminary remarks. The chapters of Part III dealt only with formal power series in which the indeterminate could not be replaced. If we now, however, replace the indeterminate \( x \) by a complex variable \( \tau \), the formal power series become power series in the ordinary sense to which the concept of convergence, totally absent in Part III, applies. The convergent power series are analytic functions, and the formal identities become equations between analytic functions. This step opens the whole store of analytic tools for the treatment of arithmetical problems in additive number theory.

We have so far by means of formal power series obtained only relations between different values of \( \hat{p}(u) \) and some congruence properties of \( \hat{p}(u) \), but could determine \( p(u) \) itself only by recursion. This situation is changed through the application of the theory of analytic functions, as Hardy and Ramanujan showed in their epoch-making paper of 1917 [18, 59], pp. 276–309.

116. A Cauchy integral and a special path of integration

If in the identity

\[ f(x) = \sum_{n=1}^{\infty} \hat{p}(n) x^n = \sum_{n=1}^{\infty} \frac{f(1-x^n)}{n-1}, \]

we consider \( x \) a complex variable, the right-hand member shows that the infinite product and thus also the infinite series are convergent for \( |x| < 1 \).

Thus if we substitute

\[ x = e^{\pi i \tau}, \quad \text{Im} \tau > 0 \]

we remain in the domain of convergence and can apply Cauchy's integral with the result

\[ \hat{p}(n) = \int_{\tau_0}^{\tau_0+1} f(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau. \]

where \( \tau_0 \) is an arbitrary point in the upper half-plane, and any path from \( \tau_0 \) to \( \tau_0 + 1 \) in the upper half-plane is permissible.

The formula (116.1) gains significance through a comparison of (98.1) and (55.2) which shows that

\[ f(e^{2\pi i \tau}) = e^{\pi i \tau/12} \eta(\tau)^{-1}. \]

This makes available our theory of the function \( \eta(\tau) \) in Chapter 9, §1; it is worth while to note that

\[ f(e^{2\pi i \tau}) = \sum_{n=0}^{\infty} \hat{p}(n) e^{2\pi in\tau} \]

has no terms for \( \nu < 0 \). The integral in (116.1) makes sense also for \( \nu < 0 \) and has then the trivial value zero. We use this fact as a definition

\[ \hat{p}(n) = 0 \quad \text{for } n < 0. \]

In (116.1) we choose now a path the construction of which is based on the properties of the Farey series. These properties are well known [18], pp. 87–89, [23], vol. 2, pp. 135–156, [49], pp. 7–9, 41, 42, so that we can just state them:

1. The ordered sequence of all reduced proper fractions

\[ \frac{0}{1}, \frac{1}{N}, \frac{1}{N-1}, \ldots, \frac{N-1}{1}, \frac{N}{1}, \]

i.e. \( 0 \leq h/k \leq 1 \) with \( 0 \leq h \leq k \), \( \gcd(h,k) = 1 \), \( k \leq N \) is the "Farey sequence of order \( N \).

2. Two adjacent fractions \( h_1/k_1 < h_2/k_2 \) have the difference

\[ \frac{h_2}{k_2} - \frac{h_1}{k_1} = \frac{1}{k_1 k_2}. \]

3. The "median" \( (h_1 + h_2)/(k_1 + k_2) \) of the two fractions \( h_1/k_1, h_2/k_2 \) satisfies

\[ \frac{h_1}{k_1} < \frac{h_1 + h_2}{k_1 + k_2} < \frac{h_2}{k_2}, \]

is also a reduced fraction and belongs to a Farey sequence of higher order than \( N \). Clearly therefore,

\[ \left| \frac{h_1 + h_2}{k_1 + k_2} - \frac{h_1}{k_1} \right| = \frac{1}{k_1 (k_1 + k_2)} \]

and thus

\[ \frac{1}{2Nk_1} \leq \frac{h_1 + h_2}{k_1 (k_1 + k_2)} - \frac{h_1}{k_1} \leq \frac{1}{N k_1}. \]

(4) Starting from the Farey sequence of order 1: 0/1, 1/1, all Farey sequences of higher order can be obtained by successive inclusion of mediant.

(5) To each \( \frac{h}{k} \) there belongs a “Ford circle” \( C(h, k) \):

\[
\tau - \left( \frac{h}{k} + \frac{i}{2k^2} \right) = \frac{1}{2k^2}
\]

in the upper half-plane, tangent to the real axis at \( \tau = \frac{h}{k} \).

(6) Two Ford circles \( C(h, k) \) and \( C(i, m) \) do not intersect. They are tangent to each other if and only if they belong to fractions \( \frac{h}{k}, \frac{i}{m} \) which are adjacent in some Farey sequence.

(7) If \( \frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2} \) are three adjacent fractions in a Farey sequence, then \( C(h, k) \) touches \( C(h_1, k_1) \) and \( C(h_2, k_2) \) respectively in the points

\[
\frac{h}{k} + \zeta_{kh}^+ \quad \text{and} \quad \frac{h}{k} + \zeta_{kh}^-
\]

where

\[
\zeta_{kh}^+ = \frac{h_1}{h(k^2 + h_1^2)} + \frac{i}{k^2 + h_1^2},
\]

\[
\zeta_{kh}^- = \frac{h_2}{h(k^2 + h_2^2)} + \frac{i}{k^2 + h_2^2}.
\]

(116.4)

A further remark explains the configuration of the set of all Ford circles:

(8) The set of all Ford circles at all rational points \( \frac{h}{k}, -\infty < \frac{h}{k} < \infty \) is the set of the images of \( \tau = x + i \), \( x \) real, under all modular transformations.

Indeed the modular transformation

\[
\tau' = \frac{ax + b}{cx + d}
\]

can be written

\[
\epsilon \tau' - a = \frac{-1}{\epsilon x + d}.
\]

Putting here \( \tau = x + i \) we obtain

\[
\epsilon \tau' - a - \frac{i}{2\epsilon} = \frac{-i}{2\epsilon} \frac{a(x - i) + d}{c(x - i) + d}
\]

and thus

\[
|\epsilon \tau' - a - \frac{i}{2\epsilon}| = \frac{1}{2\epsilon}
\]

or

\[
\left| \tau' - \frac{\epsilon}{c} + \frac{\epsilon}{2d} \right| = \frac{1}{2\epsilon}.
\]

Now, to each reduced fraction \( \frac{a}{b} \) there exist infinitely many \( h, d \) which fulfill \( ad - bc = 1 \), which proves statement 8.

Let now a natural number \( N \) be given. We construct the Ford circles belonging to the Farey sequence of order \( N \). On each circle \( C(h, k) \) we choose that are \( \gamma_{hk} \) which connect the tangency points \( \frac{h}{k} + \zeta_{kh}^+ \) and \( \frac{n}{k} - \zeta_{kh}^- \) and which does not touch the real axis (the “upper” arc). The circles \( C(0, 1) \) and \( C(1, 1) \) are congruent. Because of the periodicity of \( f(e^{2\pi i}) \) we choose one half of each, from \( i \) to \( 0/1 + \zeta_{01}^+ \) on \( C(0, 1) \) and from \( i+1 \) to \( 1/1 - \zeta_{11}^- \) on \( C(1, 1) \), and call \( \gamma_{hk} \) the union of the two arcs. These arcs are attached to each other at the points of contact of Ford circles. They together form the path of integration in (115.1).

117. An expression for \( \rho(n) \)

We continue now from (116.1).

\[
\rho(n) = \int_{\gamma_{hk}} f(z) e^{-2\pi i n z} \, dz = \sum_{0 < k < \frac{1}{2} \zeta_{kh}} \int_{\gamma_{hk}} f(z) e^{-2\pi i n z} \, dz,
\]

where the dash in the summation means that \( |h, k| = 1 \).
We have in the integrals,

\[ b(n) = \sum_{\kappa = \kappa h + \kappa} e^{2\pi i h/k} \int_{\kappa h}^\kappa e^{2\pi i \kappa x} f(\kappa) e^{2\pi i \kappa x} \, dx \]

with \( \kappa_{h\kappa}, \kappa_{\kappa} \) from (116.4). In the integral belonging to \( h/\kappa \) the variable \( \xi \) runs on an arc of the circle \( |\xi - i(2h')| = 1/(2\kappa) \). We introduce in each integral the new variable \( z \) by \( \xi = i z/\kappa \) and obtain

\[ b(n) = \sum_{\kappa = \kappa h + \kappa} \frac{i}{\kappa} e^{2\pi i h/k} \int_{\kappa h}^\kappa e^{2\pi i \kappa x} f(\kappa) e^{2\pi i \kappa x} \, dx , \quad (117.1) \]

where \( z \) runs on the circle

\[ \left| z - \frac{1}{2} \right| = \frac{1}{2} \quad (117.2) \]

on that arc between \( \kappa_{h\kappa} = (1/2) \kappa h \kappa \) and \( \kappa_{\kappa} = 1/2 \kappa h \kappa \), which does not touch the imaginary axis. Thus always \( \Re z > 0 \). By virtue of (116.4) we have

\[ z_{h\kappa} = \frac{h^2}{h^2 + \kappa^2} + i \frac{h^2 \kappa}{h^2 + \kappa^2}, \quad z_{\kappa} = \frac{h^2}{h^2 + \kappa^2} - i \frac{h^2 \kappa}{h^2 + \kappa^2} . \quad (117.3) \]

The variable \( z \) runs in the negative direction on the circle (117.2).

118. Application of the transformation formula for \( \eta(\tau) \)

For the integral in (117.1) we use (116.2) with

\[ \tau = \frac{h'}{h} + \frac{i}{k} \]

and have

\[ f(e^{2\pi i h/k - 2\pi i h}) = e^{i\tau/12} e^{-\pi i/2} (\tau - \frac{1}{h})^{-1} . \quad (118.1) \]

We determine \( h' \) so that

\[ hh' \equiv -1 \pmod h' . \]

Then:

\[ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} h & h' - 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -h' & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (118.2) \]

is the matrix of a modular substitution, and we have, as a short computation shows,

\[ \eta = \frac{h'}{h} - 1 , \quad \frac{e^{i\tau/12} e^{-\pi i/2}}{\tau} \eta = \frac{h'}{h} + \frac{i}{k} . \]

The application of (74.11) to (118.1) yields then

\[ f(\tau) e^{2\pi i h/k - 2\pi i h} = e^{i\tau/12} e^{-\pi i/2} (\tau - \frac{1}{h})^{-1} \eta (\frac{h'}{h} + \frac{i}{k})^{-1} \quad (118.3) \]

where according to (74.12) and (74.21)

\[ e \eta (\tau) e^{i\tau/12} e^{-\pi i/2} (\tau - \frac{1}{h})^{-1} \eta (\frac{h'}{h} + \frac{i}{k})^{-1} . \quad (118.4) \]

Here \( s(-h', h') \) has been taken into account (see 68.5′). Thus we have

\[ f(\tau) e^{2\pi i h/k - 2\pi i h} = \omega_{h} e^{ih/k} e^{i\tau/12} e^{-\pi i/2} \eta (\tau - \frac{1}{h})^{-1} \]

with

\[ \omega_{h} = e^{i\tau/12} e^{-\pi i/2} \quad (118.4) \]

We replace \( z \) by \( z/h \) and get

\[ f(\tau) e^{2\pi i h/k - 2\pi i h} = \omega_{h} \Psi_{h}(\tau) f(\tau) e^{2\pi i h/k - 2\pi i h} \]

with the abbreviation

\[ \Psi_{h}(\tau) = \int_{\tau} \eta (\tau - \frac{1}{h})^{-1} \eta (\tau - \frac{1}{h})^{-1} \quad (118.5) \]

The application of (118.5) changes (117.1) into

\[ b(n) = \sum_{\kappa = \kappa h + \kappa} \frac{i}{\kappa} \omega_{h} \Psi_{h}(\tau) f(\tau) e^{2\pi i \kappa h k - 2\pi i \kappa} e^{2\pi i \kappa x} \, dx . \]

Now we remember

\[ f(\tau) = 1 + f(\tau) x + f(2) x^2 + \ldots . \]

In our case

\[ \tau = e^{2\pi i h/k - 2\pi i} . \quad (118.7) \]
We plan to make \( \text{Re} \, z \) small, so that \( \text{Re} \, t_{1/2} \) will become large and thus \(|x|\) small. Therefore, for the subsequent estimates, we compare \( f(z) \) with 1. For this purpose we write

\[
\hat{p}(n) = \sum_{0 \leq b < k \leq N} \frac{i}{\lambda_b} \phi_{b,1} e^{-2\pi i n b} \int_{i \lambda_b}^{i \lambda_b + 1} \Psi_k(z) e^{2\pi i n k/2} \, dz + \sum_{0 \leq b < k \leq N} \frac{i}{\lambda_b} \phi_{b,1} e^{-2\pi i n b} \int_{i \lambda_b}^{i \lambda_b + 1} \Psi_k(z) \{ f(z) - 1 \} e^{2\pi i n k/2} \, dz,
\]

where \( x \) is given by (118.7).

119. Estimates and evaluations

We consider now the integrals

\[
I_{h_k} = \int_{i \lambda_b}^{i \lambda_b + 1} \Psi_k(z) e^{2\pi i n k/2} \, dz \tag{1.19.1}
\]

and

\[
I_{h_k}^* = \int_{i \lambda_b}^{i \lambda_b + 1} \Psi_k(z) \{ f(e^{2\pi i n/k - 2\pi i k/2}) - 1 \} e^{2\pi i n k/2} \, dz \tag{1.19.2}
\]

separately, beginning with (119.2). The variable \( z \) in running on the circle (117.2), avoiding the point \( z = 0 \). In and on the circle we have therefore

\[
0 < \text{Re} \, z \leq 1, \quad \text{Re} \, \frac{1}{z} \geq 1. \tag{1.19.3}
\]

Thus

\[
\left| \Psi_k(z) \{ f(e^{2\pi i n/k - 2\pi i k/2}) - 1 \} e^{2\pi i n k/2} \right| \leq 2^{1/2} e^{2\pi i n k/2} \sum_{m=1}^{\infty} \phi(m) e^{-2\pi i k(m-1/24)} e^{2\pi i n k/2} \leq 2^{1/2} e^{2\pi i n k/2} \sum_{m=1}^{\infty} \phi(m) e^{-2\pi i k(m-1/24)} \leq 2^{1/2} e^{2\pi i n k/2} \sum_{m=1}^{\infty} \phi(m) e^{-2\pi i k(m-1/24)} = C e^{2\pi i n k/2} \left( \frac{\lambda}{2} \right)^{1/2} \tag{1.19.4}
\]

The integrand in \( I_{h_k}^* \) is regular inside the circle (117.2). We integrate therefore from \( z_{h_b} \) to \( z_{h_b}^* \) along a chord of the circle. On this chord we have

\[
|z| \leq \max \{ |z_{h_b}|, |z_{h_b}^*| \}.
\]

and according to (117.3)

\[
|z_{h_b}^*|^2 - |z_{h_b}|^2 = \frac{k^4 + k^2 h_k^4}{k^2 + h_k^4} - \frac{k^2}{k^2 + h_k^4} \leq \frac{2k^2}{k^2 + h_k^4} \leq \frac{2k^2}{k^2 + h_k^4} \leq \frac{2k^2}{N^2}.
\]

The same estimate holds for \( |z_{h_b}^*|^2 \). This furnishes

\[
|z| \leq \frac{1/2}{N}, \tag{119.5}
\]

on the chord \( z_{h_b} \). Similarly for the real parts

\[
0 < \text{Re} \, z \leq \max \left( \frac{k^2}{k^2 + h_k^4}, \frac{k^2}{k^2 + h_k^4} \right)
\]

and thus

\[
0 < \text{Re} \, z < \frac{2k^2}{N^2} \tag{119.6}
\]

on the chord. The length of the chord, i.e. the length of the path of integration, is \( \leq |z_{h_b}^*| + |z_{h_b}^*| < 2 |z_{h_b}^*| N \).

Using (119.4) we obtain thus

\[
|I_{h_k}^*| \leq C \left( \frac{2k^2}{N^2} \right) e^{2\pi i n k/2} = O(k^2 N^{-3/2})
\]

for any fixed \( n \geq 0 \).

120. Continuation of estimates and evaluations. The final formula for \( \hat{p}(n) \)

If we extend the integration in \( I_{h_b}^* \) along the whole circle \( k \) given by (117.2) we have

\[
I_{h_k} = \int_{-1}^{1} \Psi_k(z) e^{2\pi i n k/2} \, dz - \int_{0}^{i \lambda_h} \frac{d}{dz} \left( \frac{1}{z} \right) \, dz = \int_{0}^{i \lambda_h} \frac{d}{dz} \left( \frac{1}{z} \right) \, dz - \int_{0}^{i \lambda_h} \frac{d}{dz} \left( \frac{1}{z} \right) \, dz \tag{120.1}
\]

where the rotation \( k \) is meant to indicate that the circle is travelled in the negative sense. The latter two integrals have to be understood as improper integrals with limits of integration not equal to zero but tending to zero.

On \( k \) we have

\[
\text{Re} \, \frac{1}{z} = 1, \quad 0 < \text{Re} \, z \leq 1. \tag{120.2}
\]
The length of the arc from 0 to \( z_j \) is less than
\[
\frac{\sqrt{2}}{2} \left| z_j \right| < \frac{\sqrt{2} \delta h}{N},
\]
and also, on that arc, we have
\[
|z| < \frac{\sqrt{2} \delta h}{N}.
\]

Moreover, (119.5) and (115.5) are valid so that
\[
\left| \int_{I_{m}} \Psi_k(z) e^{2\pi i n z} dz \right| = O \left( \delta \frac{h}{N} \frac{N}{2} e^{2\pi n |z| N} \right)
\]
and similarly for the last integral in (120.1). These estimates are the same as (119.7) for \( I_{m} \). We have now (118.8) in the notation (119.1), (119.2) as
\[
\rho(n) = \sum_{\sigma \in A} \frac{h}{2\pi} \omega_{\alpha} e^{-2\pi i n \sigma h} \left( I_{\alpha} + I_{\beta} \right).
\]
Collecting now (119.7), (120.1) and (120.3) we obtain
\[
P(n) = \sum_{\sigma \in \mathcal{A}} \frac{h}{2\pi} \omega_{\alpha} e^{-2\pi i n \sigma h} \int_{K^{(1)}} \Psi_k(z) e^{2\pi i n z} d\zeta + R,
\]
where
\[
R = O \left( \frac{\sum_{\sigma \in \mathcal{A}} \left| \sigma \right|^{1/2} e^{-2\pi |n|^{1/2}} \right) e^{2\pi n |z| N} = O \left( \sum_{\sigma \in \mathcal{A}} e^{2\pi |n|^{1/2} N^{-1/2}} \right)
\]
and thus
\[
P(n) = \frac{1}{2\pi} \int_{K^{(1)}} \Psi_k(z) dz + C \left( e^{2\pi |n|^{1/2} N^{-1/2}} \right)
\]
with
\[
A_k(n) = \sum_{\lambda \in \Lambda} \omega_{\alpha} e^{-2\pi i n \lambda h}.
\]

This summation over \( \lambda \) could not be carried out earlier since in the previous steps the paths of the integrals depended on \( h \).

The left-hand side does not depend on \( N \). We can therefore let \( N \) tend to \( \infty \) and must obtain a convergent series, since the remainder term goes to 0:
\[
\rho(n) = \frac{1}{2\pi} \int_{K^{(1)}} \Psi_k(z) e^{2\pi i n z} dz.
\]

This is an exact formula for \( \rho(n) \). Incidentally, the absolute convergence of (120.5) is evident, since after (113.5) and (120.2) we find that
\[
\left| \int_{K^{(1)}} \right| \leq 2 \pi e^{2\pi |n|^{1/2}},
\]
and from (120.5) we obtain the trivial estimate
\[
\left| A_k(n) \right| \leq h,
\]
so that the sum in (120.6) is majorized by
\[
2 \pi e^{2\pi |n|^{1/2}} \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}.
\]

We now have to compute the integral in (120.5). If we put
\[
w = \frac{1}{z}, \quad d\zeta = -\frac{1}{w^2} dw,
\]
we obtain
\[
\rho(n) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \int_{K^{(1)}} \Psi_k(z) e^{2\pi i n z} d\zeta = \int_{-\infty}^{\infty} \left( \sum_{k=1}^{\infty} A_k(n) \right) e^{2\pi i n z} d\zeta + R,
\]
and further, through \( \pi w/12 = 1 \),
\[
\rho(n) = \left( \frac{\pi}{12} \right)^{3/2} \int_{-\infty}^{\infty} \left( \sum_{k=1}^{\infty} A_k(n) \right) e^{2\pi i n z} d\zeta + R
\]
with \( c > 0 \). The integral here is known in terms of Bessel functions. Indeed, from (25.3) we infer, if we take into account also the remark about bending the path of integration,
\[
\rho(n) = 2 \pi \left( \frac{\pi}{12} \right)^{3/2} \int_{-\infty}^{\infty} \left( \sum_{k=1}^{\infty} A_k(n) \right) e^{2\pi i n z} d\zeta + R
\]
\[
\times \left( \frac{\pi}{2h} \right)^{3/2} \left( \sum_{k=1}^{\infty} A_k(n) \right) \left( \frac{\pi}{h} \right)^{3/2} \left( \frac{\pi}{12} \right)^{1/2} \left( n + \frac{1}{2} \right)^{1/2}.
\]

This formula can be simplified through some lemmas about Bessel functions. We treat together the two sorts of Bessel functions defined by (25.1) and (25.3) by writing, with \( \iota = \pm 1 \),
\[
J_{\iota}^{(1)}(z) = \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^{3\iota + p} n! \Gamma(p + x - 1),
\]
so that
\[
J_{\iota}^{(2)}(z) = J_{\iota}(z), \quad J_{\iota}^{(1)}(z) = I_{\iota}(z).
\]
Lemma I.

\[
\frac{d}{dz} \left( z^n J_{\nu}^{(1)}(z) \right) = z^n J_{\nu+1}^{(1)}(z),
\]
\[
\frac{d}{dz} \left( z^{-\nu} J_{\nu}^{(1)}(z) \right) = z^{-\nu} J_{\nu+1}^{(1)}(z).
\]

The proofs, consisting only of straightforward applications of (120.8), can be omitted here. The case \( \nu = 1/2 \) leads to elementary functions as expressed by

Lemma II.

\[ J_{1/2}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \sin z, \quad I_{1/2}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \cosh z. \]

These formulae can be read off directly from (120.3) and (120.9). With the abbreviation

\[ \lambda_n = \sqrt{\frac{\pi}{2}} \sqrt{\frac{n}{2}}, \quad K = \frac{\pi}{2} \sqrt{\frac{2}{3}}, \]

we have therefore, after application of Lemmas I and II,

\[ \frac{d}{dz} \lambda_n = \frac{\sqrt{\pi} \sqrt{K} \lambda_n}{2} \frac{\sinh K \lambda_n}{\lambda_n}, \]

If we insert this result in (120.7) we obtain

\[ p(n) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} k^{1/2} A_k(n) \frac{\sin \frac{\pi}{K} \sqrt{\frac{2}{3}} \lambda_n}{\lambda_n}, \]

(120.10)

our final formula for \( p(n) \).

We can again check easily that the infinite series is absolutely convergent. Indeed, with \( C = \pi \sqrt{2/3} \) we have

\[ \frac{\sin \frac{C}{k} \lambda_n}{\lambda_n} = \frac{C}{k} + \frac{1}{6} \left( \frac{C}{k} \right)^3 \lambda_n^2 + \cdots, \]

\[ \frac{d}{dz} \frac{\sin \frac{C}{k} \lambda_n}{\lambda_n} = \frac{1}{6} \left( \frac{C}{k} \right)^3 \lambda_n^2 + \cdots = O(k^{-1}), \]

\[ \sum_{k=1}^{\infty} k^{1/2} A_k(n) \frac{d}{dz} \frac{\lambda_n}{\lambda_n} = O \left( \sum_{k=1}^{\infty} k^{-3/2} \lambda_n^2 \right). \]

121. A partial sum with error term

Using the explicit evaluations of the previous section one can rewrite (120.10) as

\[ p(n) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n} k^{1/2} A_k(n) \frac{\sin \frac{\pi}{K} \sqrt{\frac{2}{3}} \lambda_n}{\lambda_n} + O(e^{2nN^2} N^{-1/2}). \]

If we choose here, as Hardy and Ramanujan do, \( N = \lfloor \sqrt{n/2} \rfloor \), with fixed \( \alpha \), then

\[ p(n) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\lfloor \sqrt{n/2} \rfloor} k^{1/2} A_k(n) \frac{\sin \frac{\pi}{K} \sqrt{\frac{2}{3}} \lambda_n}{\lambda_n} + O(n^{-1/4}). \]

This is not quite yet the Hardy-Ramanujan formula. We have

\[ \sinh \frac{\pi}{K} \sqrt{\frac{2}{3}} \lambda_n = \frac{e^{\lambda_n} - e^{-\lambda_n}}{2}. \]

and thus

\[ p(n) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\lfloor \sqrt{n/2} \rfloor} k^{1/2} A_k(n) \frac{e^{\frac{\pi}{K} \sqrt{\frac{2}{3}} \lambda_n}}{\lambda_n} + R - O(n^{-1/4}) \]

with

\[ R = O \left( \sum_{k=1}^{\lfloor \sqrt{n/2} \rfloor} k^{1/2} \frac{e^{\frac{\pi}{K} \sqrt{\frac{2}{3}} \lambda_n}}{\lambda_n} \right). \]

Now with \( C = \pi \sqrt{2/3} \) for abbreviation,

\[ \frac{d}{dz} \frac{\sin \frac{C}{k} \lambda_n}{\lambda_n} = -e^{-\left( \frac{C}{k} \lambda_n \right)} \left( \frac{C}{2 \lambda_n} + \frac{1}{2 \lambda_n^2} \right) \]

so that

\[ R = O \left( \sum_{k=1}^{\lfloor \sqrt{n/2} \rfloor} \frac{1}{\lambda_n} \left( \frac{1}{k} + \frac{1}{\lambda_n^2} \right) \right) \]

\[ = O \left( n^{-1/2} \sum_{k=1}^{\lfloor \sqrt{n/2} \rfloor} k^{1/2} \right) + O \left( n^{-3/2} \sum_{k=1}^{\lfloor \sqrt{n/2} \rfloor} k^{3/2} \right) \]

\[ = O(n^{-1/4}) + O(n^{-3/4} \cdot n^{3/4}) = O(n^{-1/4}). \]

and the Hardy-Ramanujan formula with error term \( O(n^{-1/4}) \) emerges.

For numerical computation by means of (120.10) an estimation of the error term after \( N \) terms is necessary, with a known constant implied by the symbol \( O(n^{-1/4}) \). For this purpose it would be possible to follow the
estimation in \( \S \) 19, 120 in detail. A better way is to go to the convergent series (120.10).

We write

\[
p(n) = \frac{1}{\pi/2} \sum_{k=1}^{N} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh C \lambda_n}{\lambda_n} + R_{\lambda},
\]

where

\[
R_{\lambda} = \frac{1}{\pi^2/2} \sum_{k=N+1}^{\infty} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh C \lambda_n}{\lambda_n}
\]

with \( C = \sqrt{2/3} \).

We are going to use now

\[
|A_k(n)| < 2k^{3/4},
\]

which we shall prove later. Then

\[
|R_{\lambda}| < \frac{\sqrt{2}}{\pi} \sum_{N+1}^{\infty} k^{3/4} \left| \frac{d}{dn} \frac{\sinh C \lambda_n}{\lambda_n} \right|
\]

\[
= \frac{\sqrt{2}}{\pi} \sum_{N+1}^{\infty} k^{3/4} \sum_{r=0}^{\infty} \frac{C^{2r+1}}{\lambda_n} \left( \frac{1}{2r+1} \right) \left( \frac{1}{2r+2} \right)
\]

\[
< \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \int_{N}^{\infty} \frac{d\lambda}{\lambda^{2r+3/4}}
\]

\[
= \frac{\sqrt{2}}{\pi} \left( \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \right) \left( \frac{\lambda_n^{2r+2}}{2r+4} \right)
\]

\[
< \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \left( \frac{\lambda_n^{2r+2}}{2r+4} \right)
\]

\[
< \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \left( \frac{\lambda_n^{2r+2}}{2r+4} \right)
\]

The last product increases with \( \lambda_n N \), and therefore, since \( \lambda_n < n^{3/4} \),

\[
R_{\lambda} < \frac{\sqrt{2}}{\pi} \left( \frac{10}{9} \right) C^2 N^{-3/4} + \frac{8}{11} N^{-3/4} \left( \frac{N}{\sqrt{n}} \right)^3 \left( \sinh \frac{C V_n}{N} - \frac{C V_n}{N} \right)
\]

We replace also

\[
\frac{\sinh C \lambda_n}{\lambda_n} \text{ by } \frac{C \lambda_n}{\lambda_n} \exp \left( -\frac{C \lambda_n}{\lambda_n} \right)
\]

in the finite sum of (121.1), producing thereby an additional error:

\[
R_{\lambda} = \frac{1}{2\pi} \sum_{k=1}^{N} k^{1/2} A_k(n) \frac{d}{dn} \frac{\sinh C \lambda_n}{\lambda_n} \exp \left( -\frac{C \lambda_n}{\lambda_n} \right)
\]

By virtue of (121.3) we obtain thus the estimate

\[
|R_{\lambda}| < \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \left( \frac{\lambda_n^{2r+2}}{2r+4} \right)
\]

\[
< \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \left( \frac{\lambda_n^{2r+2}}{2r+4} \right)
\]

\[
< \frac{\sqrt{2}}{\pi} \sum_{r=0}^{\infty} \frac{\lambda_n^{2r+1}}{(2r+1)! (2r+2)!} \left( \frac{\lambda_n^{2r+2}}{2r+4} \right)
\]

The final formula with remainder term is therefore

\[
\delta(n) = \frac{1}{2\pi} \sum_{k=1}^{N} k^{1/2} A_k(n) \frac{d}{dn} \frac{\exp \left( -\frac{C \lambda_n}{\lambda_n} \right)}{\lambda_n} + g N^{-3/4} \left( S_1 + S_2 \right)
\]

with \( g \), \( < 1 \) and

\[
S_1 = \frac{10}{9\pi} C^2 + \frac{8}{11} \pi \left( \frac{N}{V_n} \right)^2 \left( \sinh \frac{C V_n}{N} - \frac{C V_n}{N} \right)
\]

\[
S_2 = 2 \exp \left( -\frac{C V_n}{N} \right) \left( \frac{N + 1}{n - 1} \right) \left( \frac{1}{8\pi^3} + \frac{1}{9\pi^2} \frac{N + 1}{\sqrt{n - 1}} \right)
\]
It is clear that for $|\eta|/N$ bounded, $S_1$ and $S_2$ will remain bounded, and the error term will be of order $O(N^{-3/4})$ or $O(N^{-1/4})$. The error will thus become less than $1/2$ for suitably large $n$ and $N$. On the other hand, $\rho(n)$ is an integer, so that if the formula (121.4) $\rho(n)$ appears as the nearest integer to the sum $\sum_{k=1}^n \ldots$

Let us take in particular

$$N = \left\lfloor 2 \sqrt{n}/3 \right\rfloor$$

(121.5)

and consider only $n \geq 24^2 = 576$, since $\rho(n)$ is tabulated for $n \leq 600$ (MacMahon [38], Gupta [15]). We have under these conditions

$$\frac{3}{2} \leq \frac{3N}{2} \leq \frac{3}{2}, \quad \frac{3N}{2} \leq 15 \sqrt{23}, \quad \frac{N + 1}{V_n - 1} \leq \frac{17}{5\sqrt{23}}.$$

We observe also that $x^{-4}(\sin x - x)$ is monotone increasing with $x$.

Numerical computations give

$$S_1 < 3.3077, \quad S_2 < 0.3029,$$

and we see that the error term in (121.4) under these conditions is in absolute value less than

$$N^{-3/4} \cdot 3.3105 < 1/2 \quad \text{for} \quad n \geq 576,$$

(121.6)

which suffices to compute $\rho(n)$ exactly by (121.4) if $N$ is chosen according to (121.5).

Closer inspection of $S_1$ and $S_2$ shows that even $N = \left\lfloor 2 \sqrt{n}/3 \right\rfloor$ terms in (121.4) will make the error term tend to zero for increasing $n$.

We now have (121.4), also for a rather rough asymptotic determination of $\rho(n)$ by fixing $N = 2$. We obtain after a short computation, putting the summand for $k = 2$ into the error term,

$$\rho(n) = \frac{1}{2\pi \sqrt{2}} \cdot \frac{d}{dn} \cdot \exp \left( \frac{C}{n} \cdot \frac{\lambda_i}{\lambda_i} \right) + O\left( \frac{n^{1/2}}{n} \right),$$

and still less precisely,

$$\rho(n) \sim \frac{1}{4} \sqrt{\frac{n}{\pi}}, \quad \lambda_n = \sqrt{n - \frac{1}{24}}, \quad C = \pi \sqrt{\frac{3}{2}}.$$

122. Discussion of the sums $A_k(n)$. A new expression for $\omega_{nk}$

For the actual computation of $\rho(n)$ by means of the formula (121.4), the $A_k(n)$ are of utmost importance. They are defined through (125.3) and are in principle known, since the $\omega_{nk}$ are determined by (118.41), where the Dedekind sums can be computed by means of formulae in §§ 68–70. However, this would be a rather laborious computation. D. H. Lehmer [35] has given a factorization of the $A_k(n)$ into rather simple factors, which make computations feasible. Later A. Selberg gave a new formula for the $A_k(h)$ from which Lehmer’s results can then be obtained easily (Whitteman [78], Rademacher [51]).

Our discussion of the $\omega_{nk}$, following Dedekind, is based on the product expansion of $\eta(t)$, whereas Selberg’s formula has its origin in the infinite series for $\eta(t)$.

Using (118.1), (118.3), and (118.42) we have

$$\eta\left( \frac{\gamma}{h} + \frac{i}{k} \right) = \omega_{nk} \cdot e^{\pi i / 24} \cdot \frac{\eta}{\eta} \cdot \eta\left( \frac{h}{k} + \frac{i}{k} \right),$$

(122.1)

with $[h, k] = 1, \# \gamma = -1 \mod k$. We know also, e.g., from (118.42) or from Theorem § 65, that $\omega_{nk}$ is a root of unity, so that

$$\omega_{nk}^{-1} = \omega_{nk}.$$

(122.2)

Now we have, after Euler’s pentagonal number formula,

$$\eta(\tau) = e^{\pi i / 12} \sum_{k=-\infty}^{\infty} \left( 1 - e^{2\pi i (\tau - 1)} \right) e^{\pi i / 12} \sum_{k=-\infty}^{\infty} (-1)^k e^{2\pi i (\tau - 1)k}$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k e^{2\pi i (\tau - 1)k}.$$

We put here $\tau = \gamma/k - i\beta/k$, $\lambda = 2k\beta + j$, $\beta > 0$, and obtain

$$\eta\left( \frac{h}{k} + \frac{i}{k} \right) = \sum_{j=-\infty}^{\infty} \sum_{j=0}^{2k-1} (-1)^j e^{2\pi i (\lambda k + j - \frac{1}{2})} \sum_{j=-\infty}^{\infty} (-1)^j e^{2\pi i (\lambda k - \frac{3}{2})}$$

$$= \sum_{j=0}^{2k-1} (-1)^j e^{2\pi i (\lambda k - \frac{3}{2})} \sum_{j=-\infty}^{\infty} (-1)^j e^{2\pi i (\lambda k - \frac{3}{2})}.$$

(122.3)

We use now (36.2) on the inner sum here with

$$j = 12kz, \quad x = \frac{1}{2k} \cdot \left( j - \frac{1}{2} \right)$$

with the result

$$\sum_{j=-\infty}^{\infty} e^{-\pi i / 3k \cdot (2q + j - \frac{1}{2})} = \frac{1}{2} e^{\frac{\pi i}{3k}} \sum_{q=-\infty}^{\infty} e^{-\frac{2\pi i}{3k} \left( \frac{2q + j - \frac{1}{2}}{x} \right)}$$

$$= \frac{1}{2} e^{\frac{\pi i}{3k}} \sum_{q=-\infty}^{\infty} e^{-\frac{2\pi i}{3k} \left( \frac{2q + j - \frac{1}{2}}{x} \right)}.$$
so that (122.3) goes over into

\[ \eta \left[ \frac{\alpha}{\beta} + \frac{i \varepsilon}{\kappa} \right] = \frac{1}{2} \sqrt{3 \pi \varepsilon} \sum_{j=0}^{\infty} \frac{(-1)^j \xi^j}{e^{j^2/6 \varepsilon} \left( j + \frac{1}{2} \right)} \sum_{n=-\infty}^{\infty} \xi^{n^2/2} e^{-n^2/2 \varepsilon} \left( \frac{j}{\kappa} \right). \]

(122.4)

We apply now (122.4) on the right-hand side of (122.1) and (122.3) on the left-hand side; in the latter application we have, of course, to change in (122.3) \( \xi \) into \( h \) and \( \alpha \) into \( 1 \xi \). After some cancellation we obtain thus

\[ \sum_{j=0}^{2h-1} (-1)^j \xi^j \sum_{n=-\infty}^{\infty} \xi^n \xi^{n^2/2} e^{-n^2/2 \varepsilon} \left( \frac{j}{\kappa} \right) \]

\[ = \omega_{hh} \frac{1}{2} \sqrt{3 \pi \varepsilon} \sum_{j=0}^{2h-1} (-1)^j \xi^j \sum_{n=-\infty}^{\infty} \xi^n \xi^{n^2/2} e^{-n^2/2 \varepsilon} \left( \frac{j}{\kappa} \right). \]

(122.5)

Here we have in both sides power series in \( Z = e^{-n^2/2 \varepsilon} \). The first power of \( Z \) appears on both sides, namely for \( q = j = 0 \) on the left and \( \alpha = \pm 1 \) on the right. Since the coefficients must agree, we infer

\[ 1 = \omega_{hh} \frac{1}{2} \sqrt{3 \pi \varepsilon} \sum_{j=0}^{x^2} (-1)^j \xi^j \sum_{n=-\infty}^{\infty} \xi^n \xi^{n^2/2} e^{-n^2/2 \varepsilon} \left( \frac{j}{\kappa} \right). \]

(122.5)

123. A lemma by Whiteman and the Selberg sum

It is not obvious that the right-hand member of (122.5) represents a root of unity. To show this one would have to use the theory of Gaussian sums [cf. (5)].

Of course, in the definition of \( \omega_{hh} \) it is always assumed that \( (h, k) = 1 \). However, the right-hand member of (122.5) also makes sense for \( (h, k) \neq 1 \). This case is described by the

**Lemma (Whiteman [78]).** If \( (h, k) = d > 1 \) then

\[ \omega_{hh} = \sum_{j \equiv m \pmod{k}} (-1)^j e^{-\frac{2\pi j i}{k} \alpha j} = 0. \]

**Proof** We put

\[ h = h', k = K d, (h', K) = 1 \]

and

\[ j = 2 K' - r, 0 \leq l < d, 0 \leq r < 2 K'. \]

Thus

\[ \omega_{hh} = \sum_{l=0}^{d-1} \sum_{r=0}^{K-1} (-1)^l e^{-\frac{2\pi l i}{k} \alpha (h' - l)} e^{-\frac{2\pi r i}{k} \alpha} = \frac{1}{2} \sum_{l=0}^{d-1} e^{-\frac{2\pi l i}{k} \alpha} \]

\[ = \sum_{r=0}^{K-1} (-1)^r e^{-\frac{2\pi r i}{k} \alpha} = 0 \]

since the inner sum vanishes.

Thus the sum in the right member of (122.5) represents \( \omega_{hh} \) when \( (h, k) = 1 \) and 0 otherwise. If we therefore replace \( \omega_{hh} \) by this sum in (120.5), we can sum over a full residue system modulo \( h \) and have thus

\[ A_k(n) = \frac{1}{2} \sqrt{3 \pi h} \left\{ \sum_{j \equiv 0 \pmod{k}} \left( \frac{2h}{h} \right) - \sum_{j \equiv 1 \pmod{k}} \left( \frac{2h}{h} \right) \right\} \]

\[ + \sum_{j \equiv 0 \pmod{k}} \left( \frac{2h}{h} \right) \sum_{j \equiv 1 \pmod{k}} \left( \frac{2h}{h} \right) \]

\[ = \frac{1}{2} \sqrt{3 \pi h} \left\{ \sum_{j \equiv 0 \pmod{k}} \left( \frac{2h}{h} \right) \sum_{j \equiv 1 \pmod{k}} \left( \frac{2h}{h} \right) \right\} \]

\[ + \sum_{j \equiv 0 \pmod{k}} \left( \frac{2h}{h} \right) \sum_{j \equiv 1 \pmod{k}} \left( \frac{2h}{h} \right) \]

\[ = \frac{1}{2} \sqrt{3 \pi h} \left\{ \sum_{j \equiv 0 \pmod{k}} \left( \frac{2h}{h} \right) \sum_{j \equiv 1 \pmod{k}} \left( \frac{2h}{h} \right) \right\} \]

and finally

\[ A_k(n) = \sqrt{\frac{h}{2}} \frac{1}{2} \sum_{j \equiv 0 \pmod{k}} \left( \frac{2h}{h} \right) \sum_{j \equiv 1 \pmod{k}} \left( \frac{2h}{h} \right) \]

\[ \left( \frac{2h + 1}{h} \right) \cos \left( \frac{2h - 1}{h} \right) \frac{2h}{h} \]

\[ \left( \frac{2h + 1}{h} \right) \cos \left( \frac{2h - 1}{h} \right) \frac{2h}{h} \]

This is the formula of A. Selberg (Whiteman [78]). It makes it evident that the \( A_k(n) \) are real numbers. It is a little more convenient for the discussion of the \( A_k(n) \) to write the sum in a slightly different way. The congruence condition

\[ \frac{2h}{h} \left( \frac{2h + 1}{h} \right) \cos \left( \frac{2h - 1}{h} \right) \frac{2h}{h} \]

\[ \left( \frac{2h + 1}{h} \right) \cos \left( \frac{2h - 1}{h} \right) \frac{2h}{h} \]
is equivalent with
\[(6j - 1)^2 \equiv -24v + 1 \pmod{24k}.
\]
Let us put here and in the sequel
\[v = -24w + 1.
\]
(123.3)

Then (122.1) goes over into
\[A_k(w) = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{j \equiv 0 \pmod{2k}} (-1)^j e^{\frac{\pi i}{6k}(6j-1)} \]
\[\quad - \sum_{j \equiv 1 \pmod{24k}} (-1)^j e^{\frac{\pi i}{6k}(6j-1)}\]
where we have in the second sum \(j\) replaced by \(2k - j\), so that, in brief,
\[A_k(w) = \frac{1}{2} \sqrt{\frac{\pi}{3}} \sum_{j \equiv 1 \pmod{24k}} (-1)^j e^{\frac{\pi i}{6k}(6j-1)}.
\]
(124.1)

Let us put
\[6j \equiv 1 \equiv i
\]
so that always \(\ell, 6j = i\). From \(\ell\) we return to \(j\) by
\[j = \frac{i \pm 1}{6} \quad \text{or} \quad j = \left\{\frac{1}{6}\right\},
\]
where by \{x\} we mean the nearest integer to \(x\). In this notation we obtain
\[A_k(u) = \frac{1}{4} \sqrt{\frac{\pi}{3}} \sum_{j \equiv 1 \pmod{24k}} (-1)^j j e^{\frac{\pi i}{6k}(6j-1)}.
\]
(123.4)

The condition \(\ell, 6j = i\) is here already implied by \(j^2 \equiv v \pmod{24k}\) in view of the definition (123.3). We shall from now on write
\[B_k(v) = A_k(u),
\]
(123.5)
since the sum in (123.4) contains explicitly only the variable \(u\). We remember that always
\[v \equiv 1 \pmod{24k}.
\]

124. Different cases of \(B_k(v)\) according to \(k\)

The function \((-1)^{[v/6]} \cdot [\ell, 6] = 1\), is evidently periodic of period 12. We have the list

<table>
<thead>
<tr>
<th>(\frac{1}{4})</th>
<th>(\frac{1}{3})</th>
<th>(\frac{1}{2})</th>
<th>(\frac{1}{1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-1)^{[v/6]})</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

We compare this with the following Legendre-Jacobi symbols for the same arguments 1, 5, 7, 11:

<table>
<thead>
<tr>
<th>(\ell(3))</th>
<th>(\ell(5))</th>
<th>(\ell(7))</th>
<th>(\ell(11))</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

These tables show that
\[\frac{(-1)^{[v/6]} \cdot [\ell, 6]}{((-1)^{[v/6]} \cdot [\ell, 6])} = \left(\frac{1}{3}\right)
\]
so that we can write
\[B_k(v) = \frac{1}{4} \sqrt{\frac{k}{3}} \sum_{i = [v \pmod{24k}]} (-1)^i \left[\frac{1}{3}\right] e^{\frac{\pi i}{6k}}
\]
\[= \frac{1}{4} \sqrt{\frac{k}{3}} \sum_{i = [v \pmod{24k}]} \left(\frac{1}{3}\right) e^{\frac{\pi i}{6k}}.
\]
(124.2)

We define now
\[a' = (24, b'),
\]
which clearly takes only the four values 1, 3, 8, 24. We write also
\[c = \frac{2a'}{a}, \quad e = 24, 8, 3, 1,
\]
so that
\[\langle c, e \rangle \equiv (e, h) = 1.
\]
We utilize the numbers \(a'\) and \(c\) to break the condition of summation in the right-hand member of (124.2) into the two conditions
\[F \equiv v \pmod{d'k},
\]
\[F \equiv v \equiv 1 \pmod{c},
\]
where in turn the second condition can be replaced by \((i, \epsilon) = 1\). If we put now \(k = \epsilon r + d\epsilon j\), then the condition \(\bar{\nu} \equiv \nu \mod 24k\) is equivalent to \(\bar{\nu} \equiv \nu \mod d^2 k\) and \(\bar{\nu} \equiv \nu \mod \epsilon r\) so that

\[
B_{\nu}(r) = \frac{1}{4} \sqrt{\frac{k}{3}} \sum_{(\nu, n) = 1} \sum_{\nu \equiv \nu \mod 24k} \left( \frac{\nu + \epsilon kj}{3} \right) \left( -\frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}}.
\]

where

\[
S_{\nu}(r) = \sum_{\nu \equiv \nu \mod 24k} \left( \frac{\nu + \epsilon kj}{3} \right) \left( -\frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}}
\]

which yields

\[
B_{\nu}(r) = \frac{1}{i} \left( -\frac{1}{3} \right) \sqrt{k} \sum_{\nu \equiv \nu \mod 24k} \left( \frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}}, \quad (5, 6) = 1. \tag{124.4}
\]

(III) \(d = 8, \epsilon = 3, i.e. (6, 6) = 2\).

\[
S_{\nu}(r) = \frac{1}{4} \sum_{\nu \equiv \nu \mod 24k} \left( \frac{2\nu + \epsilon kj}{3} \right) \left( \frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}} = \left( \frac{2}{3} \right) \left( \frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}} \quad (5, 6) = 2. \tag{124.5}
\]

(IV) Finally for \(d = 24, \epsilon = 1, i.e. 6\) \(d\), the sum \(S_{\nu}\) reduces to a single term

\[
S_{\nu}(r) = \left( \frac{5}{3} \right) \left( -\frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}}
\]

with the consequence

\[
B_{\nu}(r) = \frac{1}{4} \sqrt{k} \sum_{\nu \equiv \nu \mod 24k} \left( \frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k}}, \quad (5, 6) = 2. \tag{124.6}
\]

where nothing is gained over (124.2).

125. Multiplicativity of \(B_{\nu}(r)\)

We are going to establish now a rule

\[
B_{\nu}(r) = B_{\nu}(r_1) \cdot B_{\nu}(r_2)
\]

for \((r_1, r_2) = 1\) and suitable \(r_1, r_2\). Two cases have to be considered.

(A) At least one of the \(k_1, k_2\) is prime to 5, say \((k_1, 5) = 1\).

(B) Both \(k_1\) and \(k_2\) have a factor in common with 5, in which case we may assume \(2|k_1, 3|k_2\).

In case (A) we use (124.3) for \(B_{\nu}(r_1)\) and the general equation (124.2) for \(B_{\nu}(r_2)\), so that we have

\[
B_{\nu}(r_1) \cdot B_{\nu}(r_2) = \frac{1}{4} \sqrt{k_1 k_2} \sum_{(\nu, 5) = 1} \sum_{(\nu, k_1, k_2) = 1} \left( \frac{1}{\nu + dj} \right) e^{\frac{\pi nk}{3k_1 k_2}}.
\]
If we put
\[ t = 24k_2 r + k_1 l \pmod{24k_1 k_3}, \]
we have
\[ \beta \equiv 24^* k_2^2 r^2 - k_1^2 \beta^2 \pmod{24k_1 k_3}. \]

We define now a number \( \gamma \) satisfying
\[ \gamma = \frac{h_2^0 r_1}{\gamma_3} (\pmod{k_1}), \quad \gamma = \frac{h_2^0 r_2}{\gamma_3} (\pmod{24k_4 k_1}). \quad (125.2) \]

Then the conditions of summation in (125.1) can be contracted into
\[ \ell \equiv \gamma (\pmod{24k_1 k_3}). \]

Moreover
\[ \left( \frac{3}{h_2} \right) = \left( \frac{3}{24k_2 r + k_1 l} \right) = \left( \frac{3}{r} \right). \]

so that (125.3) goes over into
\[ B_k(r_1) \cdot B_k(r_2) = \frac{1}{4} \sqrt{\frac{k}{3}} \sum_{\gamma = (24k_3)}^{3} \left( \frac{3}{r} \right) e^{2\pi \gamma \ell} = B_k(r), \quad (125.3) \]

with \( k = k_1 k_3 \). This equation can also be read in the opposite sense: if we start with \( B_k(r) \), and break \( k \) into coprime factors, \( k = h_1 h_2 \) with \( (h_1, h_2) = 1 \), we can determine \( r_1 \) and \( r_2 \) by (125.2) so that (125.3) holds. This formula gives thus a factorization of \( B_k(r) \).

In case (B), \( 2 | h_1, 3 | h_3 \), we apply (124.5) to \( B_k(r_1) \) and (124.4) to \( B_k(r_2) \), obtaining thus
\[ B_k(r_1) \cdot B_{k_1}(r_2) = \frac{1}{4} \sqrt{\frac{k}{3}} \sum_{\gamma = (24k_3)}^{3} \left( \frac{3}{r} \right) e^{2\pi \gamma \ell \cdot \frac{h_1 h_2}{24k_1 k_3}}. \]

Let us put
\[ t = 3k_2 r + 8k_1 l \pmod{24k_1 k_3}. \]
Then each pair \( r, l \) in the ranges modulo \( 8k_1 \) and \( 3k_2 \) respectively determines one \( t \) modulo \( 24k_1 k_3 \), and conversely. From
\[ t' \equiv (3k_2 r)^2 + (8k_1 l)^2 \pmod{24k_1 k_2}, \]
we infer that the summation conditions of the above formula are equivalent with
\[ t' \equiv \gamma (\pmod{24k_1 k_3}) \]

if and only if
\[ \gamma_2^2 r_1 \equiv v (\pmod{8k_1}), \quad \gamma_2^2 r_2 \equiv v (\pmod{3k_2}). \quad (125.4) \]

Moreover
\[ \left( \frac{3}{t} \right) = \left( \frac{-1}{3} \right) \left( \frac{t}{2} \right) = \left( \frac{-1}{3} \right) \left( \frac{3k_2 r + 8k_1 l}{3} \right) = \left( \frac{-1}{h_2^0} \right) \left( \frac{h_1 l}{3} \right) \]
so that
\[ B_k(r_1) \cdot B_{k_1}(r_2) = \frac{1}{4} \sqrt{\frac{k}{3}} \sum_{\gamma = (24k_3)}^{3} \left( \frac{3}{t} \right) e^{2\pi \gamma \ell} = B_k(r), \quad (125.5) \]

with \( k = h_1, k_2 \) under the conditions (125.4) for \( \gamma \).

The definition (124.2) shows that \( B_k(r) \) depends on \( r \) only modulo \( 24k \). However, the equations (124.3), (124.4), and (124.5) give the more precise information that \( B_k(r) \) depends on \( r \) only modulo \( d k \), where \( d \) is defined through (124.21). In view of this fact we can contract the statements (125.3), (125.5) for the cases (A1 and B) respectively into the following:

**Theorem.** Let \( k = k_1 k_3 \). \( (k_1, k_3) = 1 \) and \( d, d_1, d_2 \) be defined by \( d = (24, k_3), d_1 = (24, k_2^0), d_2 = (24, k_3^2) \) (so that \( d = d_1 d_2 \)). Let moreover
\[ r \equiv r_1 \equiv r_2 \equiv 1 (\pmod{24}) \]

Then
\[ B_k(r_1) \cdot B_k(r_2) = B_k(r) \quad (125.6) \]

if
\[ r \equiv \gamma_2^2 r_3 \pmod{d_1 d_2 k_1}, \quad r \equiv \gamma_2^2 r_2 \pmod{d_1^2 d_2 k_2}. \]

For the application of this theorem to \( A_k(r) \), we remember that \( A_k(r) - B_k(1 - 24r) \).

### 126. Evaluation of \( B_k(r) \) for a prime power:

Let \( k = p^\lambda \), where we assume first \( p > 3 \), so that \( (p, \ell) = 1 \). We apply (124.3). If, first, \( (\ell/p) = -1 \), then the condition of summation cannot be fulfilled for any \( r \), so that we find
\[ B_{p^\lambda}(r) = 0. \quad (126.1) \]

If, secondly, \( (\ell/p) = 1 \), then
\[ x^2 \equiv r (\pmod{p^\lambda}) \quad (126.2) \]

has exactly 2 solutions, as can be seen by induction with respect to \( \lambda \).
Indeed suppose
\[ x_0^2 \equiv 1 \quad (\text{mod } p^4) \]
and assume
\[ (x_0 + p^3 y_0)^2 \equiv 1 \quad (\text{mod } p^{4+1}) . \]
This yields for \( x \) the condition
\[ x_0^2 + 2x_0 p^3 y \equiv 0 \quad (\text{mod } p^{3+1}) \]
or
\[ \frac{x_0^2 - y}{p^3} + 2x_0 y \equiv 0 \quad (\text{mod } p) , \]
which gives just one solution for \( y \). Since the hypothesis of the existence of exactly 2 solutions \( x \) is fulfilled for \( \lambda = 1 \), it follows now for all \( p^3 \).

Thus
\[ (24r)^2 \equiv v \quad (\text{mod } p^4) \]
has two solutions, which we may write as \( \pm r \). In this case we obtain thus from (124.3)
\[ B_\lambda (v) = 2 \left( \frac{3}{p} \right)^{3\lambda/2} \cos \frac{4\pi r}{p^3} . \]
Thirdly we may have \( (r|p) = 0 \), i.e. \( p | r \). For \( \lambda = 1 \)
\[ (24r)^2 \equiv 0 \quad (\text{mod } p) , \]
has only the solution \( r = 0 \) so that
\[ B_\lambda (v) = \left( \frac{3}{p} \right)^{3\lambda/2} . \]
For \( \lambda \geq 2 \) the congruence
\[ (24r)^2 \equiv v \quad (\text{mod } p^4) \]
may not have any solution\(^1\), in which case
\[ B_\lambda (v) = 0 . \]
If it has a solution \( r \), then \( p | r \), so that we have \( r = pr_1 \). But then
\[ p^3 (r_1 + j p^{3-2}) , \quad j = 0, 1, \ldots , p - 1 \]
are all also solutions of (126.21), and the exponential sum in (124.3) becomes
\[ e^{r_1 / p^{3-1}} \sum_{j=0}^{p-1} e^{4\pi i j/p^{3}} = 0 , \]
which again leads to (126.22).

Summarizing we state the

\textbf{Theorem.} For \( \lambda = p^3 \), \( k \geq 1 \), \( p > 3 \) the following possibilities appear
\[ a_k (n) = B_\lambda (v) = \left\{ \begin{array}{cl} 0 & \text{for } \left( \frac{v}{p} \right) = -1 , \\
\left( \frac{3}{p} \right)^{3\lambda/2} \cos \frac{4\pi r}{p^3} & \text{for } \left( \frac{v}{p} \right) = +1 , \end{array} \right. \]
\[ \lambda = 1 , \quad \text{for } \left( \frac{v}{p} \right) , \lambda > 1 . \]

Here \( r \) is a solution of \( (24r)^2 \equiv v \quad (\text{mod } p^4) \).

There remain the cases \( \lambda = 3^4 \), \( k = 2n \) to be investigated. We take first \( \lambda = 3^4 \). Since by definition \( v \equiv 1 \quad (\text{mod } 3) \), we have always \( (r|3) = 1 \), and the arguments connected with \( (r|3)^{\lambda} \) carry over completely. That means we have always exactly two solutions of \( (24r)^2 \equiv v \quad (\text{mod } 3^4) \), which we can write as \( \pm r \) with \( (\pm r|3) \). Since we have here \( (k | 2) = 2 \), we employ (124.4) with the result
\[ B_\lambda (v) = (\pm (-1)^{\lambda/2} \left( \frac{3}{3^4 + 1} - \frac{3^4}{3^4 + 1} \right) \right) , \]
which yields the

\textbf{Theorem.}
\[ a_n (n) = B_\lambda (v) = 2 (-1)^{\lambda/2} 3^{\lambda/2} \cos \frac{4\pi r}{3^4 + 1} \]
\[ \left( 3^{1/2} \right)^{3^3 / 2} . \]

With\(^2\)
\[ (24r)^2 \equiv v \quad (\text{mod } 3^4) . \]

Finally, with \( k = 2 \), we have \( (k, 6) = 2 \) and have to consider, according to (124.5) the solutions of
\[ (3r)^2 \equiv v \quad (\text{mod } 2^{2+1}) . \]

Now
\[ n^2 \equiv v \quad (\text{mod } 2^{2+1}) \]

\(^1\) E.g., in the case that \( r \) is divisible only by the first power of \( p. \)
has exactly four solutions. Indeed,
\[ x^2 = \nu \pmod{2^3} \]
has the solutions \( x = 1, 3, 5, 7 \), since by definition \( \nu \equiv 1 \pmod{8} \). We put then with an unknown \( \nu \)
\[ (x + 4y)^2 \equiv \nu \pmod{2^4} \]
or
\[ \frac{x^3 - x}{8} + xy \equiv 0 \pmod{2^3} \].
This shows that to each \( x \) modulo \( 2^3 \) there is exactly one \( y \) modulo 2, so that we obtain one solution \( x' = x + y \pmod{2^4} \). This argument can be repeated indefinitely, and in particular up to the modulus \( 2^{4-3} \).

There are thus exactly 4 solutions \( \nu \) of (126.5). If \( \nu \) is one of them, then all four can be written as
\[ x = (x + \nu \cdot 2^{4-3}) \cdot h, \quad h = 0, 1. \]
This yields after (124.5) the expression
\[ B_4(\nu) = \frac{1}{4^4} \frac{1 - \nu}{\nu} 2^{\nu/2} \left( -1 \right)^{\nu/2} \frac{\nu - 1}{\nu - 2^3} \]
\[ \times \left\{ 1 + \frac{\nu e^{\pi \nu/2}}{2^{\nu/2}} + \frac{\nu^2 e^{\pi \nu/2}}{2^{\nu/2}} - e^{\pi \nu/2} \right\} \]
and thus the
**Theorem.** For \( k = 2^3 \) we have
\[ B_4(\nu) = \frac{1}{4^4} \frac{1 - \nu}{\nu} 2^{\nu/2} \sin \frac{\pi \nu}{2^{\nu/2}} \]
with
\[ (3x)^2 \equiv x \pmod{2^{4-3}}. \]
We are now in a position to compute easily \( A_k(n) \) for given \( k \) and \( n \). As an example we choose
\[ A_{16}(7) = E_{16}(1 - 2^{4-4}) = B_{16}(-167) = B_{16}(\nu_1 \cdot B_3(\nu_2) \]
with
\[ \nu_1 \equiv \nu_2 \equiv 1 \pmod{24} \]
and
\[ 3^2\nu_1 \equiv -167 \pmod{5}, \quad 5^2\nu_2 \equiv -167 \pmod{24 \cdot 3}, \]
which have solutions
\[ \nu_1 = -23, \quad \nu_2 = 25. \]
Thus
\[ A_{16}(7) = E_{16}(-23) \cdot B_3(25). \]
Now
\[ \left\{ \frac{-23}{5} \right\} = -1, \]
and therefore after (125.3), \( E_8(-23) = 0 \), so that
\[ A_{16}(7) = 0. \]
This example corrects a statement about \( A_{16} \) in Ramanujan's Collected Papers [59], p 367.

### 127. Estimations of \( A_k(n) \)

The results (126.3), (126.4) and (126.6) show that
\[ |A_{kp}(n)| < 2p^{1/2}, \quad p > 3, \]
\[ |A_{2^k}(n)| < \frac{2}{3} 2^{k/2}, \]
\[ |A_{4^k}(n)| < 2^{k/2}, \quad (127.2) \]
from which we conclude
\[ |A_k(n)| < \frac{1}{2} \prod_{p | k} p \leq \frac{1}{2} 2^{k/2} \sigma_1(k), \quad (127.2) \]
since
\[ \sigma_1(k) = \sigma_1(p^1) \ldots \sigma_1(p^r) = (1 + \ldots + \lambda_r), \quad \lambda_i + 1 \geq 2^r. \]
Now it can be proven quite simply [19], Th. 315, p. 269, that \( \sigma_1(k) = O(k^{1/2}) \), which yields then
\[ A_k(n) = O(k^{1/2 + 1/2}). \]
(127.3)

If we choose a special value for \( k > 0 \), we can easily supply a numerical constant in this estimate. We choose \( k = 1/4 \).

**Theorem.** For all \( k \) and \( n \) we have
\[ |A_k(n)| < 2^{k/4}. \quad (127.4) \]
Proof. Let us have
\[ \phi = 2x^{3/4} \beta_1^{1/4} \cdots \beta_r^{1/4} \]
with
\[ x \geq 0, \quad \beta \geq 0, \quad \lambda_1, \lambda_2 \cdots \lambda_r > 0. \]
Then, in view of (127.1) we have to prove only
\[ \frac{2}{\beta^{1/4}} \leq 2^r \phi_1^{1/4} \cdots \phi_r^{1/4} \quad \text{for} \quad \beta > 0 \]
and
\[ 2^r < 2 \cdot \phi_1^{1/4} \cdots \phi_r^{1/4} \quad \text{for} \quad \beta = 0. \]
Since \( 2/\sqrt{3} < 3^{1/4} \), it suffices to verify only the latter inequality, and that one only for the critical case \( \lambda_1 = \cdots = \lambda_r = 1 \), i.e.
\[ \frac{2}{\beta^{1/4}} \leq 1, \quad \text{i.e.} \quad \beta \geq 16 \]
can be dismissed, and we have to verify only the case of the 4 primes \( 3 < \beta < 16 \) i.e. \( 2^4 < 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \) \( 1/4 \), which comes down to
\[ 2^{12} = 4096 < 5005 = 5 \cdot 7 \cdot 11 \cdot 13. \]
This proves (127.4) of the theorem, to which we have already referred in formula (121.3).

128. The generating function \( f(x) \) for \( p(n) \)

The explicit value for \( p(n) \) in (121.1) can now be used for a new study of the generating function
\[ f(x) = \prod_{h=1}^{\infty} \frac{1}{1-x^h} = \sum_{s=0}^{\infty} p(s) x^s. \]  

For brevity we write (123.10) as
\[ f(x) = 2 \pi \left( \frac{\pi}{12} \right)^{1/4} \sum_{k=1}^{\infty} A_k \pi^{k-3/2} L_{3/2} \left( \left( \frac{\pi}{12k} \right)^2 (24k - 1) \right) \]  

with
\[ L_{3/2} \left( \left( \frac{\pi}{12k} \right)^2 (24k - 1) \right) = \sum_{a=0}^{\infty} \frac{\pi^a}{a!} T(a/2 - q - 1) \]  

(128.1).

We remember, incidentally, from (116.3), that (120.10) remains also true for \( n \leq 0 \), giving \( \hat{p}(0) = 1 \) and \( \hat{p}(n) = 0 \) for \( n < 0 \) Putting now (128.1) into the series for \( f(x) \) we obtain
\[ f(x) = 2 \pi \left( \frac{\pi}{12} \right)^{1/4} \sum_{a=0}^{\infty} \sum_{k=1}^{\infty} A_k \pi^{k-3/2} L_{3/2} \left( \left( \frac{\pi}{12k} \right)^2 (24k - 1) \right) \]  

(128.2).

Now
\[ L_{3/2} \left( \left( \frac{\pi}{12k} \right)^2 (24k - 1) \right) \leq \sum_{a=0}^{\infty} \frac{\pi^a}{a!} T \left( (a/2) - 1/2 \right) \frac{\left( \left( \frac{\pi}{6} \right)^{1/2} \right)^a (a + 1)/24}{(a + 1/2)^a} \]
\[ = \frac{2}{\sqrt{\pi}} \sum_{a=0}^{\infty} \frac{C^a}{a!} \left( \left( \frac{\pi}{6} \right)^{1/2} (a + 1/2)^a \right) \]
\[ \leq \frac{2}{\sqrt{\pi}} \sum_{a=0}^{\infty} \frac{C^a}{a!} \left( \left( \frac{\pi}{6} \right)^{1/2} (a + 1/2)^a \right) \]
\[ \leq \frac{1}{\sqrt{\pi}} e^{\pi/2 \sqrt{\pi}^{1/2}}, \]

where as before \( C = \pi \rightarrow 2/3. \) This estimate is valid also for \( n < 0 \), although we do not need this at this moment. Therefore, the series (128.2) with
\[ A_k \pi^{k-3/2} L_{3/2} \left( \left( \frac{\pi}{12k} \right)^2 (24k - 1) \right) \]

as majorized by
\[ K \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \pi^{3/2} e^{3/2 \sqrt{\pi}^{1/2}}, \]

which converges absolutely for \( \sigma | < 1 \). Thus the summation in \( \sigma \) and \( k \).
can be interchanged:

\[
f(v) = 2\pi i \left( \frac{2}{12} \right)^{\frac{3}{2}} \sum_{k=1}^{\infty} k^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{\mu n}{k-L_{2/k}} \left( \frac{-\pi}{12k} \right)^2 (24n-1)
\]

\[
= 2\pi i \left( \frac{2}{12} \right)^{\frac{3}{2}} \sum_{k=1}^{\infty} k^{-\frac{1}{2}} \sum_{n=0}^{\infty} \Phi \left( x, \frac{2n}{k} \right) \left( \frac{-\pi}{12k} \right)^2
\]

with

\[
\Phi \left( x, \frac{2n}{k} \right) = \sum_{n=0}^{\infty} L_{2/k} \left( \left( \frac{\pi}{6k} \right)^{\frac{3}{2}} \left( \frac{2n}{k} \right) \right) \left( \frac{-\pi}{12k} \right)^2, \quad \left| z \right| < 1, \quad \alpha = \frac{1}{24}
\]  \hspace{1cm} (128.51)

129. Discussion of \( \Phi \) \( \left( z, \alpha \right) \)

We shall prove that \( \Phi \) \( \left( z, \alpha \right) \) exists in the whole plane, with the exception of \( z = 1 \), as an entire function of \( 1/iz - 1 \). For this assertion we could quote a theorem of Wigert [8], \( L_{2/k}(w) \) being an entire function of order less than 1, namely of order \( 1/2 \). We shall, however, give the explicit Laurent expansion of \( \Phi \) \( \left( z, \alpha \right) \) about \( z = 1 \). For a further purpose we need also the expansion at \( z = \infty \).

We have

\[
\Phi \left( z, \alpha \right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma \left( \frac{3}{2} + \frac{\alpha}{2} \right)}{\Gamma \left( \frac{3}{2} + \frac{\alpha}{2} \right)} \left( \frac{\pi}{6k} \right)^{\frac{3}{2}} (n-\alpha)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\Gamma \left( \frac{3}{2} + \frac{\alpha}{2} \right)}{\Gamma \left( \frac{3}{2} + \frac{\alpha}{2} \right)} \psi \left( z, \alpha \right), \quad \left| z \right| < 1
\]  \hspace{1cm} (129.1)

where

\[
\psi \left( z, \alpha \right) = \sum_{n=0}^{\infty} (n-\alpha)^n z^n, \quad \left| z \right| < 1
\]  \hspace{1cm} (129.11)

The function \( \psi \left( z, \alpha \right) \) will turn out to be a polynomial of degree \( \alpha + 1 \) in \( 1/(z - 1) \). Indeed,

\[
\psi \left( z, \alpha \right) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{1}{z-1}
\]  \hspace{1cm} (129.2)

Since

\[
z \psi' \left( z, \alpha \right) = \sum_{n=0}^{\infty} (n-\alpha) z^n,
\]

we find

\[
\psi' \left( \frac{1}{z}, \alpha \right) = \sum_{n=0}^{\infty} (n-\alpha)^n \frac{1}{z^n} = \left( 1 + (z-1) \psi' \left( z, \alpha \right) \right) - \psi \left( z, \alpha \right)
\]

which together with (129.2) proves our assertion through induction. Thus the function

\[
\psi \left( \mu, \alpha \right) = (1 \rightarrow 1)^{\alpha + 1} \psi' \left( \frac{\mu}{\mu}, \alpha \right)
\]

obtained by the substitution

\[
z = \frac{\mu + 1}{\mu}, \quad \mu = \frac{1}{z - 1}
\]

is a polynomial of degree \( \alpha + 1 \) in \( u \).

**Theorem:** The generating function of the \( \psi \left( z, \alpha \right) \) is

\[
\sum_{\alpha=0}^{\infty} \psi \left( \alpha \right) \frac{w^\alpha}{\alpha!} = \frac{e^{-aw}}{1 - z e^w}
\]  \hspace{1cm} (129.4)

It follows that

\[
\left( -1 \right)^\alpha \psi \left( \frac{1}{z}, \alpha \right) = a^\alpha - \psi \left( z, -\alpha \right)
\]  \hspace{1cm} (129.5)

and for

\[
\left| 1 - z \right| < \delta, \quad 0 < \delta < 1,
\]

\[
\left| \psi \left( z \right) \right| < K \cdot \left( \frac{3}{2} \right)^{\alpha \alpha}
\]

**Proof:** We have

\[
\sum_{\alpha=0}^{\infty} \psi \left( \alpha \right) \frac{w^\alpha}{\alpha!} = \sum_{\alpha=0}^{\infty} \sum_{\alpha=0}^{\infty} \frac{(u-\alpha)^n}{n!} \frac{z^n}{\alpha!} = \sum_{\alpha=0}^{\infty} \frac{e^{-z w}}{1 - z e^w}
\]

which is (129.4). Replacing here \( z \) by \( 1/z \) we obtain

\[
\sum_{\alpha=0}^{\infty} \psi \left( \frac{1}{z}, \alpha \right) \frac{w^\alpha}{\alpha!} = \frac{e^{-z w}}{z - e^w} = -\frac{e^{-z w}}{z - e^w} = e^{-z w} \left( 1 - z e^w \right)
\]

Comparison with (129.4) shows

\[
\sum_{\alpha=0}^{\infty} \psi \left( \frac{1}{z}, \alpha \right) \frac{w^\alpha}{\alpha!} = \sum_{\alpha=0}^{\infty} \frac{w^\alpha}{\alpha!} - \sum_{\alpha=0}^{\infty} \psi \left( z, -\alpha \right) \frac{(-w)^\alpha}{\alpha!}
\]

\[\text{The use of the generating function for the discussion of the } \psi \left( z, \alpha \right), \text{ I owe to a remark by Professor C. Pisot.}\]
The coefficient of \( x^k \) on both sides gives then (129.5). Now the expression on the right side of (129.4) exhibits a meromorphic function in \( z \) with the simple poles at \( z = \log \frac{1}{2} \) (for \( z = 0 \) it is an entire function). The power series in \( z \) converges, therefore, certainly for \(|x| < |\log z|\), where the smallest possible value of \(|\log z|\) is taken. In particular for
\[
|z - 1| \geq \delta, \quad 0 < \delta \leq \frac{1}{2}
\]  
(129.6)
we have for any determination of \( \log z \)
\[
|\log z| > \frac{\delta}{2}
\]  
(129.61)
since, as can be seen easily, the circle
\[
|z - 1| < \frac{1}{2}
\]
contains the domain
\[
1 - \frac{\delta}{2} < |z| < \frac{1 + \delta}{2}, \quad |\arg z| < \frac{\delta}{2}
\]
outside of which (129.61) is fulfilled.

The power series
\[
\sum_{q=0}^{\infty} \psi_q(z, \alpha) \frac{x^q}{q!}
\]
converges, therefore, certainly for \(|x| < \delta/2\) if \( z \) lies in the domain (129.6). This has as a consequence
\[
\lim_{q \to \infty} \left| \frac{\psi_q(z, \alpha)}{q!} \right|^{1/q} = \frac{2}{\delta}
\]
and thus
\[
|\psi_q(z, \alpha)| < K \left( \frac{3}{\delta} \right)^q \cdot q!
\]
with a suitable constant \( K \). This concludes the proof of the theorem.

The series in (129.1) is therefore majorized by
\[
K_{\psi} \sum_{q=0}^{\infty} \frac{1}{q^{5/2} + q^2} \left( \frac{\pi}{\lambda/6} \right)^{2q} \left( \frac{3}{5} \right)^q,
\]
(129.62)
which means that it converges uniformly outside the circle \(|z - 1| = \delta\).

In other words \( \Phi_{\alpha}(x) \) has only the point \( z = 1 \) as an essential singularity. The functions \( \psi_{\alpha}(z, \alpha) \) all vanish at \( z = \infty \) and so does \( \Phi_{\alpha}(x) \), which is regular at \( z = \infty \).

For the explicit expansion of \( \Phi_{\alpha}(x) \) about the point \( z = 1 \), we utilize (129.3) and prove the

**Theorem.** With the designation
\[
A_{\alpha}f(x) = f(x + \alpha) - f(x),
\]
\[
A_{\alpha}^j f(x) = \sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} f(x - j - \nu)
\]
(the \( j \)-fold application of \( A_{\alpha} \) on \( f(x) \)) and
\[
A_{\alpha}^j f(x) = f(x),
\]
we have
\[
\Psi_{\alpha}(\mu, \alpha) = \sum_{j=0}^{\infty} \frac{A_{\alpha}^j(\alpha + 1; \mu)}{q!} \mu^{q+1},
\]
(129.7)

**Proof.** We use again the method of generating series. From the definition (129.3) and from (129.4) follows
\[
\sum_{q=0}^{\infty} \psi_q(\mu, \alpha) \frac{x^q}{q!} = - \sum_{q=0}^{\infty} \psi_q \left( \frac{\mu + 1}{\mu}, \alpha, \frac{|x|}{q!} \right) = - \frac{\mu e^{\alpha x}}{\mu - (\mu + 1)x} - \frac{\mu e^{\alpha x} (\mu + 1)x}{\mu - (\mu + 1)x - 1 + \mu e^{\alpha x} - 1},
\]
On the other hand, we have the generating function for \( \sum_{\nu=0}^{\infty} \frac{A_{\alpha}^\nu(\alpha + 1; \mu)}{q!} \mu^{q+1} \):
\[
\sum_{q=0}^{\infty} \frac{A_{\alpha}^\nu(\alpha + 1; \mu)}{q!} \mu^{q+1} = \sum_{q=0}^{\infty} \left( -1 \right)^{\nu} \binom{j}{\nu} \frac{\mu + 1 + j - \nu}{q!} \frac{x^q}{q!}
\]
we multiply both sides by \( x^{j+1} \), and under the assumption \(|x| < 1\), sum over \( j \):
\[
\sum_{j=0}^{\infty} \mu^{j+1} \sum_{q=0}^{\infty} \frac{A_{\alpha}^j(\alpha + 1; \mu)}{q!} \frac{x^q}{q!} = \frac{\mu e^{\alpha x + 1; \mu}}{1 - \mu e^{\alpha x - 1}} = \sum_{q=0}^{\infty} \psi_q(\mu, \alpha) \frac{x^q}{q!},
\]
Now the left-hand member can be written

\[ \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} A_{q}^j [x + 1]^{q-j} \frac{u^q}{q!} \]

since \( A_{q}^j [(x - 1)^q] = 0 \) for \( q < j \). Interchange of the summations in \( j \) and \( q \) leads to

\[ \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} A_{q}^j [(x + 1)^q] \mu^{j+1} \frac{u^q}{q!} = \sum_{q=0}^{\infty} \mu^q \psi_{q} [x, \alpha] \frac{u^q}{q!}, \]

which implies the statement \((129.7)\) of the theorem. The just-performed interchange of the order of summations can be justified by the majorization

\[ |A_{q}^j [(x + 1)^q]| \leq q! (q - 1) \ldots (q - j + 1) |x|^{j+1}, \quad (129.8) \]

which follows from a \( j \)-fold application of the mean-value theorem.

We have now available three expansions of \( \Phi_k(z) \), viz. about \( z = 0 \), \( z = \infty \), and \( z = 1 \).

(I) Equation (128.32) gives \( \Phi_k(z) \) as a power series in \( z \):

\[ \Phi_k(z) = \sum_{n=0}^{\infty} \frac{(\pi^2/6)^n (n - \alpha)}{q! \Gamma \left( \frac{5}{2} + q \right)} z^n, \quad |z| < 1. \quad (129.91) \]

(II) Using (129.51), after replacing \( \alpha \) by \( -\alpha \), we obtain from (129.6) and (129.1)

\[ \Phi_k(z) = \sum_{q=0}^{\infty} \frac{-\pi^2}{6 \cdot q!} \frac{1}{\Gamma \left( \frac{5}{2} + q \right)} \Psi_q (1, z, -\alpha) \]

\[ = -\sum_{q=0}^{\infty} \frac{-\pi^2}{6 \cdot q!} \frac{1}{\Gamma \left( \frac{5}{2} + q \right)} \sum_{n=1}^{\infty} (n + \alpha) z^n \]

\[ = -\sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{q=0}^{\infty} \frac{-\pi^2}{6 \cdot q!} \frac{1}{\Gamma \left( \frac{5}{2} + q \right)} \]

\[ \Phi_k(z) = -\sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{-\pi^2}{6 \cdot \Gamma \left( \frac{5}{2} + q \right)} (n + \alpha) z^n, \quad |z| > 1, \quad (129.92) \]

the power series expansion about \( z = \infty \).

(III) Equations (129.3) and (129.7) applied to (129.1) give

\[ \Phi_k(z) = -\sum_{q=0}^{\infty} \frac{(\pi^2/6)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \Psi_q \left( \frac{1}{x-1} \right) \]

\[ = -\sum_{q=0}^{\infty} \frac{(\pi^2/6)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} A_{q}^j [x, \alpha]^{j+1} \frac{1}{(x-1)^{j+1}} \]

\[ = -\sum_{j=0}^{\infty} \frac{1}{(x-1)^{j+1}} \sum_{q=j}^{\infty} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{q=0}^{\infty} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} A_{q}^j [x, \alpha]^{j+1} \frac{1}{(x-1)^{j+1}} \]

where the estimate \((125.8)\) permits the interchange of the summations for large enough \(|z-1|\), as a short computation shows. The inner sum over \( q \) can now be extended to \( 0 < q \leq j \), and the difference symbol be pulled out of the infinite sum:

\[ \Phi_k(z) = -\sum_{j=0}^{\infty} \frac{1}{(z-1)^{j+1}} \sum_{q=j}^{\infty} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} A_{q}^j [x, \alpha]^{j+1} \frac{1}{(x-1)^{j+1}} \]

\[ = \sum_{j=0}^{\infty} \frac{(x-1)^{-j-1} A_{q}^j [x, \alpha]^{j+1}}{q! \Gamma \left( \frac{5}{2} + q \right)} \]

or

\[ \Phi_k(z) = -\sum_{j=0}^{\infty} (z-1)^{-j-1} A_{q}^j [x, \alpha]^{j+1} \frac{1}{q! \Gamma \left( \frac{5}{2} + q \right)} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} A_{q}^j [x, \alpha]^{j+1} \frac{1}{(x-1)^{j+1}} \]

\[ \Phi_k(z) = -\sum_{j=0}^{\infty} (z-1)^{-j-1} A_{q}^j [x, \alpha]^{j+1} \frac{1}{q! \Gamma \left( \frac{5}{2} + q \right)} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} \frac{(-\pi^2)^q}{q! \Gamma \left( \frac{5}{2} + q \right)} \sum_{j=0}^{\infty} A_{q}^j [x, \alpha]^{j+1} \frac{1}{(x-1)^{j+1}} \]

This equation is at first valid only for large enough \(|z-1|\), as we observed, but since \( \Phi_k(z) \) has only the point \( z = 1 \) as singularity, it gives the Laurent expansion about \( z = 1 \) in the whole \( z \)-plane with the exception of \( z = 1 \). This expansion is the common analytic continuation of \((129.91)\) and \( \Phi_k(z) \).

130. Decomposition of \( f(x) \) into partial fractions

We utilize now our formulae for \( \Phi_k(z) \) for a further, more detailed expression of \((128.31)\). For all \( x \) with \(|x| < 1 - \delta \), as well as \(|x| > 1 + \delta \) we have

\[ |x e^{-\frac{2\pi x}{k}} - 1| > \delta. \]
In these cases the estimation (129.62) applies
\[ \Phi \left( x e^{-\frac{2\pi i t^*}{k}} \right) \leq K_3 \sum_{k=1}^{\infty} \frac{1}{\Gamma \left( \frac{5}{2} + \frac{3}{q} \right)} \left( \frac{\pi^2}{6k^2} \right)^q \]
for \( k = 1, 2, 3, \ldots \). Therefore, the power series (130.2) has only vanishing coefficients, and we conclude
\[ F(x) = 0 \quad \text{for} \quad |x| > 1. \]

The series (128.31) is therefore majorized by
\[ 2\pi \left( \frac{\pi}{12} \right)^{3/2} H(\delta) \sum_{k=1}^{\infty} k^{-3/2} \]
and converges uniformly in \( |x| < 1 - \delta \) and \( |x| > 1 + \delta \). Since \( \delta \) is arbitrary, the series obtained by inserting (125.93) in the right member of (128.31), namely,
\[ F(x) = -2\pi \left( \frac{\pi}{12} \right)^{3/2} \sum_{k=1}^{\infty} k^{-3/2} \sum_{\nu \equiv 0 \pmod{k}} \sum_{j=0}^{\infty} A_k \left( \frac{\pi^2}{6k^2} (\nu + 1) \right) \]
represents an analytic function for \( |x| < 1 \) as well as for \( |x| > 1 \). For \( |x| < 1 \) we know from our deduction that
\[ F(x) = f(x). \]
The circumstance \( |x| = 1 \) is a natural boundary for \( f(x) \), since (118.4) shows that \( f(x) \) goes to infinity near all roots of unity, which is seen by letting \( x \) go through positive values to \( +\infty \). We have to determine \( F(x) \) for \( |x| > 1 \). In this case we apply (129.52) and obtain
\[ F(x) = -2\pi \left( \frac{\pi}{12} \right)^{3/2} \sum_{k=1}^{\infty} k^{-3/2} \sum_{\nu \equiv 0 \pmod{k}} \sum_{j=0}^{\infty} A_k \left( \frac{\pi^2}{6k^2} (\nu + 1) \right) x^{-\nu} a_{\nu} a_{\frac{\pi}{k}}^{\text{div}} \]
with \( A_k(-n) \) in the meaning of (120.5). Here the coefficient of \( x^{-n} \), \( n = 1, 2, \ldots \), is
\[ -2\pi \left( \frac{\pi}{12} \right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} \sum_{\nu \equiv 0 \pmod{k}} \sum_{j=0}^{\infty} A_k \left( \frac{\pi^2}{6k^2} (\nu + 1) \right) \]
which is \( \rho(-n) \) after (128.1). But we know by (116.3) that \( \rho(-n) = 0 \) for \( n = 1, 2, 3, \ldots \). Therefore, the power series (130.2) has only vanishing coefficients, and we conclude
\[ F(x) = 0 \quad \text{for} \quad |x| > 1. \]

The series (130.1) defines two completely different analytic functions for \( |x| < 1 \) and \( |x| > 1 \):
\[ F(x) = \begin{cases} \sum_{n=0}^{\infty} f(n) x^n, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \]
(130.3)
We can look upon (130.1) as partial fraction decomposition around the singularities
\[ x = e^{\frac{2\pi i}{k}}, \]
which, however, in distinction from more elementary cases, lie everywhere dense on the line \( |x| = 1 \).

There is a much simpler expansion into partial fractions which behave like \( F(x) \) in (130.3). That is the expansion
\[ \lim_{x \to 1} \frac{1}{(1-x)(1-x^2) \cdots (1-x^{N-1})} \to 0 \]
Indeed, this expression fulfills, as we know from the start of these discussions, the first line of (130.3). For \( |x| > 1 \) we see that
\[ \frac{1}{N} \prod_{n=1}^{N} \frac{1}{(1-x^n)} \to 0, \]
(150.4)
which corresponds to the second statement in (130.3).

Let us write the unique algebraic partial fraction decomposition of (130.4) as
\[ \frac{1}{\prod_{n=1}^{N} (1-x^n)} = \sum_{k=1}^{N} \frac{C_{nk}(N)}{(x - e^{\frac{2\pi i}{k}})} \]
(130.5)
The \( C_{nk}(N) \) can be obtained algebraically as expressions containing roots of unity, although the actual computation becomes very cumbersome with increasing \( N \). No explicit formula for \( C_{nk}(N) \) is known, not even for the simplest case \( k = 0, b = 1, l = 1 \), and variable \( N \).

I conjecture now that the partial fraction decomposition (130.5) converges termwise to the expansion (130.1). More explicitly, I propose the
Conjecture.

\[ \lim_{\delta \to 0} C_{41}\{N, \delta\} = -2\pi \left( \frac{\pi}{12} \right)^{3/2} \frac{\omega}{k^2} \delta^{1/4} \left( \frac{\omega}{k^2} \right)^{1/2} \left( 1 - \frac{\pi^2}{6\alpha} (\alpha + 1) \right), \]

\[ \alpha = \frac{1}{24}. \]  

(130.6)

These numbers can easily be computed, since they reduce to trigonometric functions. Some simple calculations starting from (128.12) show that

\[ L_{\omega/k^2} = \frac{1}{2\pi y} \frac{d}{dy} \left( \frac{\sin 2y}{y} \right) = \frac{1}{2\pi y^2} \left( 2 \cos 2y - \frac{\sin 2y}{y} \right). \]

The following table contains some values for the \( C_{41}\{N\} \), the algebraic ones from (130.5) as well as the transcendental ones from (130.6):

<table>
<thead>
<tr>
<th>( C_{31} )</th>
<th>( C_{31} )</th>
<th>( C_{31} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-0.25</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>-0.3251</td>
<td>0.25</td>
</tr>
<tr>
<td>4</td>
<td>-0.2361</td>
<td>0.286</td>
</tr>
<tr>
<td>5</td>
<td>-0.23832</td>
<td>0.1875</td>
</tr>
</tbody>
</table>

(Im \( \frac{\omega}{k^2} > 0 \) and possessing the two properties:

1. Homogeneity of dimension \( r \)

\[ F (\omega_1, \omega_2) = \beta^r F (\omega_1, \omega_2). \]

2. Modular invariance

\[ F (a\omega_1 + b\omega_2, c\omega_1 - d\omega_2) = (c\tau + d)^{-r} F (1, \tau). \]

Application of the Circle Method to Modular Forms of Positive Dimension

131. Generalized modular forms

The circle method, which led to an explicit formula for the Fourier coefficients \( \sigma(n) \) of \( e^{-\pi i n/12} \pi(\tau)^{-1} \), can be generalized to deal with general modular forms of positive dimension.

We have already encountered examples of modular forms, especially the Eisenstein series \( C_{2k}(w_1, w_2) \) of dimension \(-2k\) and \( A(w_1, w_2) \) of dimension \(-12\). We repeat here the definition of a modular form.

Definition. A modular form is an analytic function \( F(w_1, w_2) \) of two variables \( w_1, w_2 \) defined for

\[ \text{Im} \left( \frac{w_1}{w_2} \right) > 0 \]

and possessing the two properties:

1. Homogeneity of dimension \( r \)

\[ F (\omega_1, \omega_2) = \beta^r F (\omega_1, \omega_2). \]

2. Modular invariance

\[ F (a\omega_1 + b\omega_2, c\omega_1 - d\omega_2) = (c\tau + d)^{-r} F (1, \tau). \]

where \( ad - bc = 1 \).

Application of property (1), at first only for integer \( r \), permits us to write (2) in the inhomogeneous manner

\[ F \left( 1, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{-r} F (1, \tau). \]

with

\[ \tau = \frac{w_2}{w_1}, \quad \text{Im} \tau > 0. \]

From now on we write simply \( F(\tau) \) instead of \( F(1, \tau) \) and shall speak of \( F(\tau) \), although written in the inhomogeneous manner.

Definition. A function \( F(\tau) \), analytic for \( \text{Im} \tau > 0 \), with at most poles as singularities, is called a modular form of dimension \( r \) if it satisfies

\[ F \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{-r} F(\tau). \]