1. (a) Suppose that \( f \) is a continuous function on an open interval \( I \). Let \( a \) be any point of \( I \), and define a function \( F \) on \( I \) by setting
\[
F(x) = L^x_a f
\]
for every \( x \in I \). Then \( F \) is differentiable on \( I \), and \( F' = f \).

(b) Suppose that \( f \) is a continuous function on an open interval \( I \), and that \( G \) is a primitive of \( f \). Then for any two points \( a, b \in I \), we have
\[
L^b_a f = G(b) - G(a).
\]

(c) Suppose that \( f \) is a continuous function on the open interval \( I \), that \( G \) is a primitive of \( f \), and that \( a, b \in I \) are given. According to the first form of the fundamental theorem, the function \( F \) defined on \( I \) by \( F(x) = L^x_a f \) is differentiable and satisfies \( F' = f \). By the definition of a primitive, \( G \) is also differentiable, and it satisfies \( G' = f \). It is a consequence of the mean value theorem that if two functions on an open interval are differentiable and have the same derivative, they differ by a constant. Hence there is a constant \( C \) such that \( G(x) = F(x) + C \) for every \( x \in I \).

In particular, we have
\[
G(a) = F(a) + C = L^a_a f + C = 0 + C = C,
\]
so that the constant \( C \) is \( G(a) \). Hence
\[
L^b_a f = F(b) = G(b) - C = G(b) - G(a).
\]

2. For any real number \( c \) we have \( c \leq |c| \) and \( -c \leq |c| \). Hence for every \( x \in [a, b] \) we have \( f(x) \leq |f(x)| \) and \( -f(x) \leq |f(x)| \). By one of the basic properties of the integral it follows that
\[
\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx
\]
and
\[
\int_a^b -f(x) \, dx \leq \int_a^b |f(x)| \, dx.
\]
Since $f$ is continuous we know that $\int_a^b Cf = C \int_a^b f$ for every constant $C$, and in particular $\int_a^b -f(x) \, dx = -\int_a^b f(x) \, dx$. Hence

$$- \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$  

It now follows that

$$|\int_a^b f(x) \, dx| = \pm \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.$$

3(a) One defines

$$T_{a,n}(x) = \sum_{i=0}^n \frac{(x-a)^i}{i!} f^{(i)}(a).$$

(b) Assume that for a given $n \geq 0$ we have

$$f(x) = T_{a,n}(x) + R_{a,n}(x),$$

where

$$R_{a,n}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) \, dt.$$  

Define $g$ on $I$ by

$$g(t) = -\frac{(x-t)^{n+1}}{(n+1)!},$$

and note that

$$g'(t) = \frac{(x-t)^n}{n!}.$$  

Hence

$$R_{a,n}(x) = \int_a^x f^{(n+1)}(t) g'(t) \, dt.$$  

Integrating by parts, we obtain

$$R_{a,n}(x) = \left[f^{(n+1)}(t) g(t)\right]_a^x dt - \int_a^x f^{(n+2)}(t) g(t) \, dt.$$

Since $g(x) = 0$, we have

$$\left[f^{(n+1)}(t) g(t)\right]_a^x = -f^{(n+1)}(a) g(a) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a).$$
On the other hand we have

\[- \int_a^x f^{(n+2)}(t) g(t) \, dt = \int_a^x \frac{(x-t)^{n+1}}{(n+1)!} f^{(n+2)}(t) \, dt = R_{a,n+1}(x).\]

It follows that

\[R_{a,n}(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + R_{a,n+1}(x),\]

and the induction hypothesis gives

\[f(x) = T_{a,n}(x) + R_{a,n}(x)\]

\[= \sum_{i=0}^{n} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + R_{a,n+1}(x)\]

\[= \sum_{i=0}^{n+1} \frac{(x-a)^i}{i!} f^{(i)}(a) + R_{a,n+1}(x)\]

\[= T_{a,n+1}(x) + R_{a,n+1}(x).\]