Homework problems ##2 and 3
Math 536
Due Monday, Sept. 14, 2009

2. Let U be an open subset of $\mathbf{C}$, and let $(f_t)_{t\in T}$ be a locally uniformly absolute summable family of meromorphic functions on U. Suppose that the family of derivatives $(f_t')_{t\in T}$ is also locally uniformly absolute summable. Let $S(z)$ denote the meromorphic function defined on U by

$$S(z) = \sum_{t\in T} f_t(z).$$

Prove that

$$S'(z) = \sum_{t\in T} f_t'(z).$$

(You will have to examine the proof given in class that $S(z)$ is meromorphic, and use Weierstrass's theorem that if $g_1, g_2, \ldots$ is a sequence of holomorphic functions on an open set $V \subset \mathbf{C}$ which converges uniformly on compact subsets of V to a function g_∞ then $g_1', g_2', \ldots$ converges to g_∞'.)

3. In class we showed that

$$\left(\frac{1}{(z-n)^2}\right)_{n\in\mathbf{Z}}$$

is a LUAS family on $\mathbf{C}$. The same argument shows that

$$\left(\frac{1}{(z-n)^k}\right)_{n\in\mathbf{Z}}$$

is LUAS for any $k \geq 2$.

(a) Using the formula

$$\sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)},$$

which was proved in class, and the result of problem #2 above, derive formulae for

$$\sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^3} \qquad \text{and} \qquad \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^4}.$$

(b) Using the formula for $\sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^4}$ that you derived in part (a) and taking $z = 1/2$, derive a formula for

$$\sum_{k=1}^{\infty} \frac{1}{k^4}.$$

(c) Explain what happens if you try to use the formula for $\sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^3}$ that you derived in part (a) to derive a formula for

$$\sum_{k=1}^{\infty} \frac{1}{k^3}.$$