Third problem set
Math 414
Due Monday, Feb. 18, 2008

1. (a) Prove that if \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series of strictly positive terms such that
\[
\frac{a_n}{a_{n-1}} \leq \frac{b_n}{b_{n-1}}
\]
for every sufficiently large \( n \), and if \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) also converges.

(b) Use the mean value theorem to prove the inequality
\[
n^{\alpha+1} - (n - 1)^{\alpha+1} \leq (\alpha + 1)n^\alpha
\]
for every positive real number \( \alpha \) and every positive integer \( n \).

(c) Let \( \alpha \) be a strictly positive real number which is not an integer. Combining the results of (a) and (b) with the fact that \( \sum_{n=1}^{\infty} n^{-(1+\alpha)} \) converges, show that
\[
\sum_{n=0}^{\infty} \binom{\alpha}{n}
\]
converges absolutely.

(d) Let \( \alpha \) be a strictly positive real number which is not an integer. Combine the result of (c) with the Weierstrass M-test and the version of the binomial theorem proved in class to show that
\[
\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n
\]
converges absolutely and uniformly to \((1 + x)^\alpha\) on the interval \(-1 \leq x \leq 1\).

2. (a) Using the version of the binomial theorem proved in class, give a power series expression for \(1/\sqrt{1 - x^2}\) on the interval \(-1 < x < 1\). You may leave the expression for the coefficients in terms of binomial coefficients.

(b) Combine the result of part (a) with a result about Taylor series proved in class to give (with justification) an expression for the 1,000-th derivative of \(1/\sqrt{1 - x^2}\) at \(x = 0\).

3. (a) Suppose that \( \sum_{n=0}^{\infty} a_n x^n \) is a power series with radius of convergence \( R \). Using the theorem on differentiation of power series proved in class, show that the series
\[
\sum_{n=0}^{\infty} a_n \frac{x^n}{n+1}
\]
has radius of convergence $R$; and that if $R > 0$, and $f$ is the function defined for $-R < x < R$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

then we have

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

for $-R < x < R$.

(b) Use the result of part (a) to give another proof of the identity

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

for $-1 < x < 1$.

(c) Starting from the formula

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1 - t^2}} dt \quad (-1 < x < 1)$$

and the results of problems 2(a) and 3(a), give and justify a power series expression for $\arcsin x$ for $-1 < x < 1$. 

2