**Homework problems ##6 and 7**
**Math 536**
**Due Monday, Oct. 19, 2009**

**6. (a)** Let $n$ be any positive integer. Prove that the map $z \to z^n$ from $\mathbf{C} \setminus \{0\}$ to itself is a covering map, using the method that was used in class to show that $\exp : \mathbf{C} \to \mathbf{C} \setminus \{0\}$ is a covering map. You will need to find an appropriate action of the group $\mathbf{Z}/n\mathbf{Z}$ on $\mathbf{C} \setminus \{0\}$. (A direct proof that the map is a covering map is possible, but part of the point of this problem is to adapt the method used in class.)

**(b)** Let $f$ be a nowhere-vanishing holomorphic function on a simply connected open set in $\mathbf{C}$. Give a precise statement about the existence of an $n-th$ root of $f$ that follows from part (a).

**7.** Let $\Omega \subset \mathbf{C}$ be a lattice, let $p : \mathbf{C} \to \mathbf{C}/\Omega$ denote the orbit map, and let $V$ denote the vector space consisting of all meromorphic functions on $\mathbf{C}/\Omega$ which have poles at most at $\bar{0} = p(0)$. Let $W$ denote the vector space of all polynomials in $1/z$ with zero constant term, and let $T$ denote the linear map defined by taking $T(f)$ to be the principal part of $f \circ p$ at 0. It is essentially a special case of a result proved in class that the kernel of $T$ consists of constant functions. (In class I considered only functions for which the order of the pole is subject to some bound; this guarantees finite-dimensionality but is not needed for injectivity.)

**(a)** Let $W_0$ denote the subspace of $W$ consisting of polynomials for which the coefficient of $1/z$ is 0. Use residue calculus to show that the image of $T$ is contained in $W_0$. (This was sketched in class on September 4. Fill in details.)

**(b)** Show that $T : V \to W_0$ is surjective by showing that for every polynomial $A = A(1/z) \in W_0$ there is a two-variable polynomial $Q_0$ such that $T(Q_0(\overline{\mathcal{P}}, \overline{\mathcal{P}'})) = A$. From this argument and the description of the kernel of $T$ given above, deduce that every function in $V$ has the form such that $T(Q(\overline{\mathcal{P}}, \overline{\mathcal{P}'}))$ for some two-variable polynomial $Q$.