6. (a) Let \( n \) be any positive integer. Prove that the map \( z \to z^n \) from \( \mathbb{C} \setminus \{0\} \) to itself is a covering map, using the method that was used in class to show that \( \exp: \mathbb{C} \to \mathbb{C} \setminus \{0\} \) is a covering map. You will need to find an appropriate action of the group \( \mathbb{Z}/n\mathbb{Z} \) on \( \mathbb{C} \setminus \{0\} \). (A direct proof that the map is a covering map is possible, but part of the point of this problem is to adapt the method used in class.)

(b) Let \( f \) be a nowhere-vanishing holomorphic function on a simply connected open set in \( \mathbb{C} \). Give a precise statement about the existence of an \( n \)-th root of \( f \) that follows from part (a).

7. Let \( \Omega \subset \mathbb{C} \) be a lattice, let \( p: \mathbb{C} \to \mathbb{C}/\Omega \) denote the orbit map, and let \( V \) denote the vector space consisting of all meromorphic functions on \( \mathbb{C}/\Omega \) which have poles at most at \( \bar{0} = p(0) \). Let \( W \) denote the vector space of all polynomials in \( 1/z \) with zero constant term, and let \( T \) denote the linear map defined by taking \( T(f) \) to be the principal part of \( f \circ p \) at \( 0 \). It is essentially a special case of a result proved in class that the kernel of \( T \) consists of constant functions. (In class I considered only functions for which the order of the pole is subject to some bound; this guarantees finite-dimensionality but is not needed for injectivity.)

(a) Let \( W_0 \) denote the subspace of \( W \) consisting of polynomials for which the coefficient of \( 1/z \) is 0. Use residue calculus to show that the image of \( T \) is contained in \( W_0 \). (This was sketched in class on September 4. Fill in details.)

(b) Show that \( T: V \to W_0 \) is surjective by showing that for every polynomial \( A = A(1/z) \in W_0 \) there is a two-variable polynomial \( Q_0 \) such that \( T(Q_0(\overline{P}, \overline{P})) = A \). From this argument and the description of the kernel of \( T \) given above, deduce that every function in \( V \) has the form such that \( T(Q(\overline{P}, \overline{P})) \) for some two-variable polynomial \( Q \).