Homework problems #8 and 9
Math 536
Due Monday, Nov. 9, 2009

8. Prove that the family of functions

\[
\left( \left( \frac{1}{((m - \frac{1}{2})\tau + (n - \frac{1}{2}))^2} - \frac{1}{(m\tau + n)^2} \right) \right)_{(0,0) \neq (m,n) \in \mathbb{Z} \times \mathbb{Z}}
\]

is locally uniformly absolutely summable on the upper half-plane \( \mathbb{H} \). (It was pointed out in class that this implies that the function \( e_3 \) is holomorphic on \( \mathbb{H} \). Slightly simpler calculations, which you do NOT have to write down, show that \( e_1 \) and \( e_2 \) are holomorphic.)

9. (a) Prove the following proposition, which was stated in class. Let \( f : X \to Y \) be a non-constant proper map between connected Riemann surfaces. Then for every point \( y \) the set \( f^{-1}(y) \subset X \) is finite. Furthermore, the integer

\[
d = \sum_{x \in f^{-1}(y)} \text{loc deg}_x(f),
\]

where \( \text{loc deg}_x(f) \) denotes the local degree of \( f \) at \( x \), is independent of the choice of the point \( y \in Y \). (Adapt the proof given in class for the compact case. You will need the fact that a proper map is closed.)

The integer \( d \) is called the degree of \( f \).

(b) Prove that under the above hypotheses there is a discrete, closed subset \( S \subset X \) such that for every point \( y \in Y \setminus f(S) \), the set \( f^{-1}(y) \) has cardinality \( d \). This is a corrected version of an assertion that I made in class.