Solutions to second problem set  
Math 414

1. (a) For every \( n \geq 1 \) and every \( x \in (0, \infty) \), we have

\[
0 < \frac{1}{(x+n)^2} < \frac{1}{n^2},
\]

and the series of positive terms

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]

is convergent by the integral test. Hence the Weierstrass \( M \)-test shows that

\[
\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}
\]

is uniformly convergent on \((0, \infty)\).

(b) For every \( n \geq 1 \) we have

\[
D_x \frac{1}{(x+n)^2} = \frac{-2}{(x+n)^3}
\]

for \( x > 0 \). Hence if

\[
\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}
\]

is pointwise convergent on \((0, \infty)\), and

\[
\sum_{n=1}^{\infty} \frac{-2}{(x+n)^3}
\]

is uniformly convergent on \((0, \infty)\), it will follow that

\[
f(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}
\]

is differentiable on \((0, \infty)\) and that

\[
f(x) = \sum_{n=1}^{\infty} \frac{-2}{(x+n)^3},
\]

Certainly

\[
\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}
\]
is pointwise convergent on $(0, \infty)$, since it is uniformly convergent by part (a). On the other hand, for every $n \geq 1$ and every $x \in (0, \infty)$, we have
\[
\left| \frac{-2}{(x + n)^3} \right| < \frac{2}{n^3},
\]
and the series of positive terms
\[
\sum_{n=1}^{\infty} \frac{2}{n^3}
\]
is convergent by the integral test. Hence the Weierstrass $M$-test shows that
\[
\sum_{n=1}^{\infty} \frac{-2}{(x + n)^3}
\]
is uniformly convergent on $(0, \infty)$.

2. (a) For every $n \geq 1$ and every $x \in (-\infty, \infty)$, we have
\[
|f_n(x)| = \frac{1}{n^2} |\sin nx| \leq \frac{1}{n^2},
\]
and the series of positive terms
\[
\sum_{n=1}^{\infty} \frac{1}{n^2}
\]
is convergent by the integral test. Hence the Weierstrass $M$-test shows that
\[
\sum_{n=1}^{\infty} f_n(x)
\]
is uniformly convergent on $(-\infty, \infty)$.

(b) For every $n \geq 1$ we have
\[
f_n'(0) = \frac{1}{n} \cos(n \cdot 0) = \frac{1}{n}.
\]
By the integral test, the series
\[
\sum_{n=1}^{\infty} f_n'(0) = \sum_{n=1}^{\infty} \frac{1}{n}
\]
is divergent. In particular, the series
\[
\sum_{n=1}^{\infty} f_n'(x)
\]
is not pointwise convergent on \((-\infty, \infty)\), and hence not uniformly convergent.

3. Define a sequence of functions \((g_n(x))\) on \([0, 1]\), for \(n \geq 1\), by \(g_1(x) = x\) and \(g_n(x) = x^n - x^{n-1}\) for \(n \geq 2\). Then we have

\[
\sum_{n=1}^{N} g_n(x) = x^N
\]

for every \(N \geq 1\). I pointed out in class that the sequence of functions \((x^N)_{N \geq 1}\) converges pointwise but not uniformly on \([0, 1]\). Hence the series

\[
\sum_{n=1}^{\infty} g_n(x)
\]

converges pointwise but not uniformly on \([0, 1]\).

4. (a) The series converges pointwise because

\[
\sum_{i=0}^{N} (-1)^n x^n = \frac{1 - (-x)^{N+1}}{1 + x},
\]

so that

\[
\sum_{i=0}^{\infty} (-1)^n x^n = \frac{1}{1 + x}
\]

for \(0 \leq x < 1\). Uniform convergence would mean that

\[
\sup_{0 \leq x < 1} \left| \frac{1 - (-x)^{N+1}}{1 + x} - \frac{1}{1 + x} \right| \to 0
\]

as \(N \to \infty\). But for every \(N\) we have

\[
\sup_{0 \leq x < 1} \left| \frac{1 - (-x)^{N+1}}{1 + x} - \frac{1}{1 + x} \right| = \sup_{0 \leq x < 1} \left| \frac{x^{N+1}}{1 + x} \right| = \frac{1}{2}
\]

since \(x^N/(1 + x)\) tends to \(1/2\) as \(x\) approaches 1 from the left. Hence the convergence is not uniform.

(b) For \(x = 1\) the series does not converge because the terms all have absolute value 1. Hence it does not converge pointwise on \([0, 1]\), and in particular it does not converge uniformly.

(c) Starting from the identity

\[
\sum_{i=0}^{N} (-1)^n t^n = \frac{1 + (-t)^{N+1}}{1 + t}
\]
and integrating from 0 to $x$, where $0 \leq x \leq 1$, we obtain

$$\sum_{i=0}^{N} (-1)^n \frac{x^{n+1}}{n+1} = \log(1 + x) + \int_{0}^{x} \frac{(-t)^{N+1}}{1 + t} dt.$$ 

Hence

$$\left| \sum_{i=0}^{N} (-1)^n \frac{x^{n+1}}{n+1} - \log(1 + x) \right| = \int_{0}^{x} \frac{t^{N+1}}{1 + t} dt \leq \int_{0}^{x} t^{N+1} dt = \frac{x^{N+2}}{N+2} \leq \frac{1}{N+2}.$$ 

It follows that

$$\sup_{0 \leq x \leq 1} \left| \sum_{i=0}^{N} (-1)^n \frac{x^{n+1}}{n+1} - \log(1 + x) \right| \leq \frac{1}{N+2}$$

and hence that

$$\sup_{0 \leq x \leq 1} \left| \sum_{i=0}^{N} (-1)^n \frac{x^{n+1}}{n+1} - \log(1 + x) \right| \to 0$$

as $N \to \infty$. This is uniform convergence.