Math313
Sample problems for Test 1.

1. Formulate the principle of mathematical induction.

2. State the definition of a non-Cauchy sequence.

3. State the axiom of completeness of $\mathbb{R}$.

4. Let $A = (0,1) \cup [2,4)$ and $B = \{\frac{1}{2}\} \cup (3,5]$. Find $A \cup B$, $A \cap B$, $A \backslash B$, $B \backslash A$, $A \bigtriangleup B$.

5. Let $f(x) = \sin(x)$. Determine $\sup_{[0,\pi/3)} f$, $\inf_{[0,\pi/3)} f$, $\max_{[0,\pi/3)} f$, $\min_{[0,\pi/3)} f$, if the latter two exist.

$$\sqrt{3}/2,\ 0,\ DNE, 0.$$

6. Find the set of all values x for which

$$\frac{x+1}{3x-1} \geq 1,$$

answer: $(1/3, 1]$.

$$\frac{(x-1)^2(2x+3)}{(3x-2)^3 x} \leq 0.$$

answer: $(-\infty, -3/2] \cup (0, 2/3)$.

7. Prove using the definition of the limit:

$$\lim_{n\to\infty} \quad \frac{n+2}{n+3} = 1$$

$$\lim_{n\to\infty} \quad 2 - \frac{(-1)^n}{n} = 2$$

$$\lim_{n\to\infty} \quad \frac{n^2}{n^3-1} = 0.$$

First:

$$\left| \frac{n+2}{n+3} - 1 \right| = \frac{1}{n+3} < \varepsilon$$

implies

$$n > -3 + 1/\varepsilon$$

Choose

$$N = [1/\varepsilon] - 2$$

or 1 if the latter is negative.

Second:

$$\left|2 - \frac{(-1)^n}{n} - 2\right| = \frac{1}{n} < \varepsilon$$

implies

$$n > 1/\varepsilon$$

Choose

$$N = [1/\varepsilon] + 1.$$

Third

$$\frac{n^2}{n^3 - 1} < \frac{n^2}{n^3 - \frac{n^3}{2}} = \frac{2}{n} < \varepsilon$$

Choose

$$N = [2/\varepsilon] + 1.$$

8. Compute the limits

$$\begin{array}{ll} \lim_{n\to\infty} & \sqrt[n]{n} \\ \lim_{n\to\infty} & \dfrac{2^{2/n}}{1+n} \\ \lim_{n\to\infty} & \dfrac{3n^2 - 1}{5n^2 - \sqrt[n]{3}}. \end{array}$$

First: consider the same approach as in the limit $a^{1/n} \to 1$. Let

$$\sqrt[n]{n} = 1 + h_n$$

Then

$$n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + ... > \frac{n(n-1)}{2}h_n^2$$

So,

$$\frac{2}{n-1} > h_n^2 > 0$$

By the Sandwich Theorem, $h_n \to 0$.

Second: $2^{2/n} = 4^{1/n} \to 1$. So, the limit is 0.

Third: multiply the top and the bottom by $1/n^2$, the limit is $3/5$.

9. Show directly that every Cauchy sequence is bounded without the use of the Cauchy theorem.

   Let $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have

$$|a_n - a_m| < 1$$

   Fix some $m = m_0 > N$. Then for all $n > N$ we have

$$a_{m_0} - 1 < a_n < a_{m_0} + 1$$

   So, the sequence is bounded by $\max\{a_1, ..., a_N, a_{m_0} + 1\}$ above and below by $\min\{a_1, ..., a_N, a_{m_0} - 1\}$.

10. Show that the sequence

$$a_n = 1 - \frac{1}{2} + \frac{1}{4} - ... + (-1)^n \frac{1}{2^n}$$

   has a limit. Compute it.

   This sequence is the sequence of partial sums of the geometric progression

$$\sum_{n=0}^{\infty} (-1/2)^n$$

   So its limit is the sum of the series, which is $2/3$.

11. Compute the sums of the series

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{3^n}; \quad \sum_{n=2}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

   First:

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{3^n} = (-1/3)^4 \sum_{n=0}^{\infty} (-1/3)^n = \frac{1}{81} \cdot \frac{3}{4} = 1/108.$$

   Second: this is a telescoping series:

$$= \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1/4 - 1/9 + 1/9 - 1/16 + ... 1/N^2 - 1/(N+1)^2 = \lim_{N\to\infty} 1/4 - 1/(N+1)^2 = 1/4.$$

12. Determine convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^{1/n}}{2^n}$$

Converges by LCT with $1/2^n$.

$$\sum_{n=1}^{\infty} \frac{n^5 - n + 1}{n^6 + n - 1}$$

Diverges by LCT with $1/n$.

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}(n^3 + 1)}{\cos(1/n)(n^5 - 1)}$$

Converges by LCT with $1/n^2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1}$$

Converges by Leibnitz.

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{n!}$$

Converges by Ratio Test.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

Ratio Test gives limit $1/4$. So, converges.