1. Formulate the principle of mathematical induction.

2. State the definition of a non-Cauchy sequence.

3. State the axiom of completeness of \( \mathbb{R} \).

4. Let \( A = (0, 1) \cup [2, 4) \) and \( B = \{ \frac{1}{2} \} \cup (3, 5] \). Find \( A \cup B \), \( A \cap B \), \( A \setminus B \), \( B \setminus A \), \( A \triangle B \).

5. Let \( f(x) = \sin(x) \). Determine \( \sup_{[0, \pi/3]} f \), \( \inf_{[0, \pi/3]} f \), \( \max_{[0, \pi/3]} f \), \( \min_{[0, \pi/3]} f \), if the latter two exist.

\[ \sqrt{3}/2, \ 0, \ DNE, 0. \]

6. Find the set of all values \( x \) for which
\[
\frac{x + 1}{3x - 1} \geq 1,
\]
answer: \((1/3, 1]\).

\[
\frac{(x - 1)^2(2x + 3)}{(3x - 2)^3x} \leq 0.
\]
answer: \((-\infty, -3/2] \cup (0, 2/3)\).

7. Prove using the definition of the limit:
\[
\lim_{n \to \infty} \frac{n + 2}{n + 3} = 1
\]
\[
\lim_{n \to \infty} 2 - \frac{(-1)^n}{n} = 2
\]
\[
\lim_{n \to \infty} \frac{n^2}{n^3 - 1} = 0.
\]

First:
\[
\left| \frac{n + 2}{n + 3} - 1 \right| = \frac{1}{n + 3} < \varepsilon
\]
implies $n > -3 + 1/\varepsilon$

Choose $N = [1/\varepsilon] - 2$

or 1 if the latter is negative.

Second: $\left| 2 - \frac{(-1)^n}{n} - 2 \right| = \frac{1}{n} < \varepsilon$

implies $n > 1/\varepsilon$

Choose $N = [1/\varepsilon] + 1$.

Third $\frac{n^2}{n^3 - 1} < \frac{n^2}{n^3 - \frac{n^3}{2}} = \frac{2}{n} < \varepsilon$


8. Compute the limits

$$\lim_{n \to \infty} \sqrt[n]{n}$$

$$\lim_{n \to \infty} \frac{2^{2/n}}{1 + n}$$

$$\lim_{n \to \infty} \frac{3n^2 - 1}{5n^2 - \sqrt{3}}$$

First: consider the same approach as in the limit $a^{1/n} \to 1$. Let

$$\sqrt[n]{n} = 1 + h_n$$

Then $n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2}h_n^2 + \ldots > \frac{n(n-1)}{2}h_n^2$

So,

$$\frac{2}{n - 1} > h_n^2 > 0$$

By the Sandwich Theorem, $h_n \to 0$. 

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Second: $2^{2/n} = 4^{1/n} \to 1$. So, the limit is 0.
Third: multiply the top and the bottom by $1/n^2$, the limit is $3/5$.

9. Show directly that every Cauchy sequence is bounded without the use of the Cauchy theorem.
Let $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have

$$|a_n - a_m| < 1$$

Fix some $m = m_0 > N$. Then for all $n > N$ we have

$$a_{m_0} - 1 < a_n < a_{m_0} + 1$$

So, the sequence is bounded by $\max\{a_1, ..., a_N, a_{m_0} + 1\}$ above and below by $\min\{a_1, ..., a_N, a_{m_0} - 1\}$.

10. Show that the sequence

$$a_n = 1 - \frac{1}{2} + \frac{1}{4} - ... + (-1)^n \frac{1}{2^n}$$

has a limit. Compute it.
This sequence is the sequence of partial sums of the geometric progression

$$\sum_{n=0}^{\infty} (-1/2)^n$$

So its limit is the sum of the series, which is $2/3$.

11. Compute the sums of the series

$$\sum_{n=4}^{\infty} (-1)^n \frac{3^n}{3^n}, \sum_{n=2}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

First:

$$\sum_{n=4}^{\infty} (-1)^n \frac{3^n}{3^n} = (-1/3)^4 \sum_{n=0}^{\infty} (-1/3)^n = \frac{1}{81} \cdot \frac{3}{4} = 1/108.$$  
Second: this is a telescoping series:

$$= \lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{1}{4} - \frac{1}{9} + \frac{1}{9} - \frac{1}{16} + ... + \frac{1}{N^2} - \frac{1}{(N+1)^2} = \lim_{N \to \infty} \frac{1}{4} - \frac{1}{(N+1)^2} = 1/4.$$
12. Determine convergence of the series

\[ \sum_{n=1}^{\infty} \frac{2^{1/n}}{2^n} \]

Converges by LCT with \(1/2^n\).

\[ \sum_{n=1}^{\infty} \frac{n^5 - n + 1}{n^6 + n - 1} \]

Diverges by LCT with \(1/n\).

\[ \sum_{n=1}^{\infty} \frac{\sqrt{n}(n^3 + 1)}{\cos(1/n)(n^5 - 1)} \]

Converges by LCT with \(1/n^2\)

\[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1} \]

Converges by Leibnitz.

\[ \sum_{n=1}^{\infty} \frac{3^{2n}}{n!} \]

Converges by Ratio Test.

\[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \]

Ratio Test gives limit \(1/4\). So, converges.