Math313
Sample problems for Test 2.

1. Review all definions and statements of theorems from Chapters 3,4. There will be two questions on those.

2. Determine all points of discontinuity of $f(x) = \frac{\tan(x)}{1-x}$.

   $x = 1, \frac{\pi}{2} + \pi n.$

3. Show that f is continuous by definition

$$f(x) = \frac{x^2 - x + 1}{x + 2}, \text{ at } x = 0.$$

Fix $\varepsilon > 0$. Then

$$|f(x) - 1/2| = \left|\frac{x(2x-3)}{2(x+2)}\right|.$$

Let $\delta < 1$, then $|x| < \delta$ implies $|x + 2| > 1$. So, continuing

$$\leq |x|5/1 = 5|x| < \varepsilon.$$

So, choose $\delta = \min\{1, \varepsilon/5\}$.

$$f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ x^2, & x \notin \mathbf{Q} \end{cases} \quad \text{at } x = 1.$$

Fix $\varepsilon > 0$,

$$|f(x) - 1| = \begin{cases} |x-1|, & x \in \mathbf{Q} \\ |x^2-1|, & x \notin \mathbf{Q} \end{cases}$$

We have $|x^2 - 1| = |x - 1||x + 1| \leq 2|x - 1|$ provided $\delta < 1$. So, whether x in $\mathbf{Q}$ or not, we have

$$|f(x) - 1| \leq 2|x - 1|.$$

So, choose $\delta = \min\{1, \varepsilon/2\}$.

4. Prove that the Heine-Borel Theorem is not valid for the interval $(0, 1)$.

   Take open cover $I_n = (1/n, 1)$. It covers the interval, yet every finite subcover does not.

5. Prove that if $f'(a) = f''(a) = 0$ and $f'''(a) \neq 0$, then a is not a point of local expremum. Assume that the function is four times differentiable.

   Use Taylor theorem:

$$f(a+h) = f(a) + f'''(a)h^3/6 + f^{(4)}(c)h^4/24.$$

   for all h, where c depends on h and between a and $a+h$. Let $M > 0$ be a bound on $|f^{(4)}|$. Then for all $|h| < 2|f'''(a)|/M$, we have

$$|f'''(a)h^3/6| \geq 2|f^{(4)}(c)h^4/24|$$

   Thus, by trianble inequality

$$1/2 f'''(a)h^3/6 \leq f(a+h) - f(a) \leq 3/2 f'''(a)h^3/6.$$

   Now choose h negative and h positive. It implies that $f(a+h) - f(a)$ changes sign as h passes through zero. So, $f(a)$ is not a local extremum.

6. Prove that if $f'''(x) = 0$ for all x then f is a quadratic function.

   Again follows from Taylor's theorem.

7. Find all local extrema of

$$f(x) = \frac{x^2+1}{x^2-1}.$$

   $f'(x) = 0$ at $x = 0$ and dne at $x = 1$. $f''(0) < 0$, so 0 is a local max. And $f \to \infty$ as $x \to 1$, so 1 is not a local expremum.

8. Let f be differentiable on $[a, b]$. Show that for every dissection of $[a, b]$ with points $x_0, x_1, ..., x_n$ there are points $c_i \in (x_{i-1}, x_i)$ such that

$$f(b) - f(a) = \sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}).$$

   Use $f(b) - f(a) = \sum_{i=0}^{n}(f(x_i) - f(x_{i-1}))$ and apply the MVT.