Math313
Test 2, 100 points total.

1. (10pts) State the Heine-Borel Theorem.

2. (10pts) State the definition of a differentiable function on a set of reals S.

3. (15pts) Give an example of a continuous function on the interval $(0, 1)$, which is not uniformly continuous.

4. (20pts) Prove that the equation

$$x^4 = \cos x$$

has at least two solutions on the interval $[-2, 2]$.

5. (20pts) Let f be a differentiable function on $[0, 1]$. Suppose that $f(0) = 0$, and $f'(x) < 1$ for all $x \in [0, 1]$. Show that $f(x) < x$ for all $x \in (0, 1]$.

6. (25pts) Let f be twice differentiable on $\mathbb{R}$. Suppose $f''(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that the inequality

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right)$$

holds for all $x, y \in \mathbb{R}$. (Hint: use the Taylor Theorem about the point $(x + y)/2$).

7*. (extra credit 5 pts, no partial credit) Let f be a differentiable function on $[0, 1]$. Suppose that $f(0) = 0$, $f'(x) \leq 1$ for all $x \in [0, 1]$ and there exists $x_0 \in [0, 1]$ for which $f'(x_0) < 1$. Show that $f(1) < 1$.