A DIMENSION-DEPENDENT MAXIMAL INEQUALITY

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Abstract. In this short note we show that \( \sup \{ \| M_\nu \| : \nu \text{ is measure on } \mathbb{R}^n \} \), where \( \| M_\nu \| \) denotes the centered Hardy-Littlewood maximal operator, depends exponentially on \( n \).

1. Statement of the problem

Let \( \nu \) be a \( \sigma \)-finite measure on the Borel subsets of \( \mathbb{R}^n \). Define the Hardy-Littlewood centered maximal operator associated with \( \nu \) by

\[
M_\nu f(x) = \sup_{r>0} \frac{1}{\nu(B_r(x))} \int_{B_r(x)} |f|d\nu, \quad x \in \mathbb{R}^n.
\]

It was proved in [1, 2] that

\[
\| M_\nu f \|_{L^p(\mathbb{R}^n, \nu)} \leq C \| f \|_{L^p(\mathbb{R}^n, \nu)}, \quad 1 < p < \infty,
\]

where \( C \) does not depend on \( \nu \). We present a simple construction showing that \( C \) depends exponentially on \( n \). This answers the question posed in [2, 3].

2. Construction

Claim 2.1. There is an absolute constant \( \alpha > 1 \) such that one can find \([\alpha^n]\) points \( x_1, x_2, ..., x_{[\alpha^n]} \) on the euclidian sphere \( S^{n-1} \) such that

\[
\| x_i - x_j \| > 1, \quad i \neq j.
\]

The maximal value of \( \alpha \) is immaterial. A simple argument based on volume estimates yields \( \alpha \geq e^{(\pi/6)^{2/3}} \).

Let us fix \( x_1, x_2, ..., x_{[\alpha^n]} \) as in the claim and put

\[
\nu = \delta_{\{0\}} + \sum_i \delta_{\{x_i\}}.
\]

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Define $f = \delta_{\{0\}}$. Then $\|f\|_{L^p(\mathbb{R}^n, \nu)} = 1$. On the other hand,

$$(M_\nu f)(x_i) \geq \frac{1}{\nu(B_1(x_i))} \int_{B_1(x_i)} |f|d\nu = 1/2, \quad i = 1, \ldots, [\alpha^n].$$

Hence, $\|M_\nu f\|_{L^p(\mathbb{R}^n, \nu)} \geq \frac{1}{2}[\alpha^n]$.

This is the end of construction.

References

