THE LARGEST LINEAR SPACE OF OPERATORS SATISFYING THE DAUGAVET EQUATION IN $L_1$

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Abstract. We find the largest linear space of bounded linear operators on $L_1(\Omega)$ that, being restricted to any $L_1(A)$, $A \subset \Omega$, satisfy the Daugavet equation.

1. Introduction

Let $(\Omega, \Sigma, \mu)$ be an arbitrary measure space without atoms of infinite measure. Also let $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. If $A \in \Sigma^+$, $L_1(A)$ stands for the space of (classes of) real-valued $\mu$-integrable functions supported on $A$. If $T$ is a bounded linear operator on $L_1(\Omega)$ and $A \in \Sigma^+$, we denote by $T_A$ the restriction of $T$ onto $L_1(A)$. Finally, $\mathcal{L}(L_1(\Omega))$ denotes the space of all bounded linear operators on $L_1(\Omega)$.

The purpose of this note is to give an explicit description of the largest linear space $\mathcal{M}$ of operators $T \in \mathcal{L}(L_1(\Omega))$ satisfying the following identity:

$$\|Id_A + T_A\| = 1 + \|T_A\|,$$

for any set $A \in \Sigma^+$.

Originally, (1) was established by Daugavet for compact operators on $C[0, 1]$ (see [2]). The case of $L_1$ was first treated by Lozanovskii in his paper [6], where he proved Daugavet’s theorem for compact operators in $L_1[0, 1]$ (see also [1]). Later, Holub generalized this result for all weakly compact operators on an arbitrary atomless $L_1(\Omega)$ (see [4]). Plichko and Popov in their work [7] found a still broader (in case of atomless $\mu$) linear class of so-called narrow operators satisfying the Daugavet equation, and in fact their proof works for operators from $L_1(A)$ to $L_1(\Omega)$, whenever $A \in \Sigma^+$.

So, finding the largest class of such operators naturally completes this line of results.

We also refer the reader to papers [5] and [8] for recent developments and applications of the Daugavet theory.
2. Main result

In the sequel it is convenient to denote $\Sigma^+_A = \{B : B \subset A, B \in \Sigma^+\}$, whenever $A \in \Sigma^+$.

We define $\mathcal{M}$ as the set of all operators $T \in \mathcal{L}(L_1(\Omega))$ that meet the following condition:

For every $\varepsilon > 0$ and $A \in \Sigma^+$

(2) there is a $B \in \Sigma^+_A$ with $\mu(B) < \infty$ such that

$$\left\| \chi_B \cdot T \left( \frac{\chi_B}{\mu(B)} \right) \right\| < \varepsilon.$$ 

This condition simply means that the operator $T$ can shift sufficiently many functions from their supports.

Let us state our main result.

**Theorem 1.** Every linear space of operators satisfying (1) for any $A \in \Sigma^+$ is contained in $\mathcal{M}$, and $\mathcal{M}$ itself is a closed linear space consisting of such operators.

The main ingredient in the proof of this theorem is the following proposition.

**Proposition 2.** For an operator $T \in \mathcal{L}(L_1(\Omega))$ the following conditions are equivalent:

(i) $T$ and $-T$ satisfy (1) for all $A \in \Sigma^+$;

(ii) for every $\varepsilon > 0$ and $A \in \Sigma^+$ there is an $A' \in \Sigma^+_A$ such that if $B \in \Sigma^+_A$, then we can find a $B' \in \Sigma^+_B$ with the following properties:

a) $\left\| \frac{\chi_B}{\mu(B)} - \frac{\chi_{B'}}{\mu(B')} \right\| < \varepsilon$,

b) $\left\| \chi_{B'} \cdot T \left( \frac{\chi_{B'}}{\mu(B')} \right) \right\| < \varepsilon$;

(iii) $T \in \mathcal{M}$.

**Proof.** (i) implies (ii). We begin with the following observation.

Suppose $S : L_1(A) \mapsto L_1(\Omega)$ is a bounded linear operator. Then for any given $\varepsilon > 0$ there is a set $A_1 \in \Sigma^+_A$ with $\mu(A_1) < \infty$ such that for every non-negative function $f \in S(L_1(A_1))$ we have $\|Sf\| \geq \|S\| - \varepsilon$.

Indeed, we can assume that $\mu(A) < \infty$ and choose $g^* \in S(L_1(\Omega))$ so that $\|S^*g^*\| > \|S\| - \varepsilon$. Then, regarding $S^*g^*$ as an element of $L_\infty(A)$ we find a set $A_1 \in \Sigma^+_A$ with $\theta S^*g^*(A_1) \subset (\|S\| - \varepsilon, \|S\|]$, where $\theta$ is a sign. Now, if $f \in S(L_1(A_1))$, $f \geq 0$ and $\text{supp}(f) \subset A_1$, then $\|Sf\| > \theta g^*(f) = \theta S^*g^*(f) > \|S\| - \varepsilon$, from where the observation follows.

We know that $\|Id_A + T_A\| = 1 + \|T_A\|$. By scaling, without loss of generality we can and do assume that $\|T_A\| = 1$. So there is an $A_1 \in \Sigma^+_A$ with $\mu(A_1) < \infty$ such that

$$\left\| \frac{\chi_B}{\mu(B)} + T \left( \frac{\chi_B}{\mu(B)} \right) \right\| > 2 - \varepsilon,$$

whenever $B \in \Sigma^+_A$. We also know that $\|Id_{A_1} - T_{A_1}\| = 1 + \|T_{A_1}\| > 2 - \varepsilon$. Thus there exists an $A' \in \Sigma^+_A$ such that

$$\left\| \frac{\chi_B}{\mu(B)} - T \left( \frac{\chi_B}{\mu(B)} \right) \right\| > 2 - \varepsilon,$$

whenever $B \in \Sigma^+_A$. 
We prove that $A'$ is the desired set.

To this end, let us fix $B \in \Sigma_3^+$. It follows from (3), (4) and a theorem of Dor [3] that there are two disjoint measurable sets $\Omega_1$ and $\Omega_2$ in $\Omega$ such that

$$\int_{\Omega_1} \left| T \left( \frac{\chi_B}{\mu(B)} \right) \right| (t) d\mu(t) > (1 - \varepsilon)^2$$

and

$$\int_{\Omega_2} \frac{\chi_B}{\mu(B)} (t) d\mu(t) > (1 - \varepsilon)^2.$$  

The last inequality implies

$$\mu(B \cap \Omega_1) = \mu(B) \int_{B \cap \Omega_1} \frac{\chi_B}{\mu(B)} (t) d\mu(t) < \mu(B) \int_{\Omega_2} \frac{\chi_B}{\mu(B)} (t) d\mu(t)$$

$$< (1 - (1 - \varepsilon)^2) \mu(B) = (2\varepsilon - \varepsilon^2) \mu(B).$$

Let us put $B' = B \setminus \Omega_1$ and show that $B'$ meets conditions a) and b).

First,

$$\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| = \int_{\Omega} \left| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} + \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right| (t) d\mu(t)$$

$$\leq 1 - \frac{\mu(B')}{\mu(B)} + \frac{\mu(B \cap \Omega_1)}{\mu(B)} = 2 \frac{\mu(B \cap \Omega_1)}{\mu(B)},$$

and taking into account (6), we obtain

$$\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| < 2(2\varepsilon - \varepsilon^2).$$

Second, from (5), (7) and $\|T_A\| = 1$ it follows that

$$\left\| \chi_{B'} \cdot T \left( \frac{\chi_{B'}}{\mu(B')} \right) \right\| = \int_{B'} \left| T \left( \frac{\chi_{B'}}{\mu(B')} \right) \right| (t) d\mu(t)$$

$$< \int_{B'} \left| T \left( \frac{\chi_B}{\mu(B)} \right) \right| (t) d\mu(t) + 2(2\varepsilon - \varepsilon^2)$$

$$\leq \int_{\Omega \setminus \Omega_1} \left| T \left( \frac{\chi_B}{\mu(B)} \right) \right| (t) d\mu(t) + 2(2\varepsilon - \varepsilon^2)$$

$$\leq 3(2\varepsilon - \varepsilon^2).$$

In view of the arbitrariness of $\varepsilon$, this gives the desired result.

It is obvious that (iii) follows from (ii).

Let us finally prove that (iii) implies (i). Since $\mathcal{M}$ is stable under scalar multiplication, it is sufficient to prove (1) only for $T$.

To this end, we fix an arbitrary $A \in \Sigma^+$ and as above for any given $\varepsilon > 0$ we find an $A' \in \Sigma^+_A$ with $\mu(A') < \infty$ such that for every $B \in \Sigma^+_A$, $\| T \left( \frac{\chi_B}{\mu(B)} \right) \| > \| T_A \| - \varepsilon$.

By condition (2), there is a $B_0 \in \Sigma^+_A$ such that $\| \chi_{B_0} \cdot T \left( \frac{\chi_{B_0}}{\mu(B_0)} \right) \| < \varepsilon$. This means that $\chi_{B_0} / \mu(B_0)$ and $T \left( \frac{\chi_{B_0}}{\mu(B_0)} \right)$ are almost disjoint functions, and as a consequence we
have the following estimate:

\[
\|Id_A + T_A\| \geq \left\| \frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right\| \\
= \int_{B_0} \left\| \frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right\| (t) d\mu(t) + \int_{\Omega} T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) (t) d\mu(t) \\
- \int_{B_0} \left\| T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right\| (t) d\mu(t) \\
> 1 - \varepsilon + \|T_A\| - \varepsilon - \varepsilon = 1 + \|T_A\| - 3\varepsilon.
\]

This finishes the proof. □

Now we are in a position to prove our main result.

**Proof of Theorem 1.** Proposition 2 implies that \(\mathcal{M}\) consists of operators satisfying (1) for all \(A \in \Sigma^+\), and that every linear space of such operators is contained in \(\mathcal{M}\). \(\mathcal{M}\) is obviously closed and stable under scaling. So, the only thing we have to prove is that if operators \(U\) and \(V\) belong to \(\mathcal{M}\), then their sum belongs to \(\mathcal{M}\) too.

To show this, we check condition (ii) of Proposition 2 for \(U + V\). Further on, we assume that \(\|V\| \leq 1\).

Indeed, let \(A \in \Sigma^+\) and \(\varepsilon > 0\) be arbitrary. Applying Proposition 2 to the operator \(U\) we find a set \(A' \in \Sigma^+_A\) as in condition (ii). Then, by the same proposition applied to \(V\) we find a set \(A'' \in \Sigma^+_A\) with the corresponding properties. To show that \(A''\) is the required set, suppose \(B \in \Sigma^+_A\). By the choice of \(A''\) there is a \(B'' \in \Sigma^+_B\) such that

\[
\left\| \frac{\chi_{B''}}{\mu(B')} - \frac{\chi_{B}}{\mu(B)} \right\| < \frac{\varepsilon}{4} \tag{8}
\]

and

\[
\left\| \chi_{B''} \cdot V\left(\frac{\chi_{B''}}{\mu(B''')}\right) \right\| < \frac{\varepsilon}{4}. \tag{9}
\]

Since \(B' \subset A'\), by the analogous property of \(A'\), there is a \(B'' \in \Sigma^+_B\) with

\[
\left\| \frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_{B'}}{\mu(B')} \right\| < \frac{\varepsilon}{4} \tag{10}
\]

and

\[
\left\| \chi_{B''} \cdot U\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \frac{\varepsilon}{2}.
\]

From (8) and (10) we get

\[
\left\| \frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_{B'}}{\mu(B')} \right\| < \varepsilon. \quad \text{So, if we prove that } \left\| \chi_{B''} \cdot V\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \frac{\varepsilon}{2}, \quad \text{then } \left\| \chi_{B''} \cdot (V + U)\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \varepsilon, \quad \text{and we are done. But this easily follows from (9), (10) and the facts that } \|V\| \leq 1 \text{ and } B'' \subset B'.
\]

The proof is completed. □

**References**


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