NARROW OPERATORS ON VECTOR-VALUED SUP-NORMED SPACES

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Abstract. We characterise narrow and strong Daugavet operators on $C(K,E)$-spaces; these are in a way the largest sensible classes of operators for which the norm equation $\|\mathrm{Id}+T\| = 1+\|T\|$ is valid. For certain separable range spaces E including all finite-dimensional ones and all locally uniformly convex ones we show that an unconditionally pointwise convergent sum of narrow operators on $C(K,E)$ is narrow, which implies for instance the known result that these spaces do not have unconditional FDDs. In a different vein, we construct two narrow operators on $C([0,1],\ell_1)$ whose sum is not narrow.

1. Introduction and preliminaries

This paper is a follow-up contribution to our paper [6] where we defined and investigated narrow operators on Banach spaces with the Daugavet property. We shall first review some definitions and results from [5] and [6] before we describe the contents of the present paper.

A Banach space X is said to have the *Daugavet property* if every rank-1 operator $T\colon X \to X$ satisfies

$$\|\mathrm{Id}+T\| = 1+\|T\|. \tag{1.1}$$

For instance, $C(K)$ and $L_1(\mu)$ have the Daugavet property provided that K is perfect, i.e., has no isolated points, and μ does not have any atoms. We shall have occasion to use the following characterisation of the Daugavet property from [5]; the equivalence of (ii) and (iii) results from the Hahn-Banach theorem.

Lemma 1.1. *The following assertions are equivalent:*

(i) *X has the Daugavet property.*

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(ii) *For every* $x \in S(X)$, $x^* \in S(X^*)$ *and* $\varepsilon > 0$ *there exists some* $y \in S(X)$ *such that* $x^*(y) > 1 - \varepsilon$ *and* $\|x + y\| > 2 - \varepsilon$.

(iii) *For all* $x \in S(X)$ *and* $\varepsilon > 0$, $B(X) = \overline{\text{co}}\{z \in B(X)\colon \|x + z\| > 2 - \varepsilon\}$.

It is shown in [5] and [9] that (1.1) automatically extends to wider classes of operators, e.g., weakly compact ones and, more generally, those that do not fix copies of ℓ_1 or strong Radon-Nikodým operators. (A strong Radon-Nikodým operator maps the unit ball into a set with the Radon-Nikodým property.) In [6] we found new proofs of these results based on the notions of a strong Daugavet operator and a narrow operator. An operator $T\colon X \to Z$ is said to be a *strong Daugavet operator* if for every two elements $x, y \in S(X)$, the unit sphere of X, and for every $\varepsilon > 0$ there is an element $u \in B(X)$, the unit ball of X, such that $\|x + u\| > 2 - \varepsilon$ and $\|T(y - u)\| < \varepsilon$. It is almost obvious that a strong Daugavet operator $T\colon X \to X$ satisfies (1.1), and the nontrivial task is now to find sufficient conditions on T to be strongly Daugavet. In this vein we could show that for instance strong Radon-Nikodým operators and operators not fixing copies of ℓ_1 are indeed strong Daugavet operators.

For some applications the concept of a strong Daugavet operator is somewhat too wide. Therefore we defined an operator $T\colon X \to Z$ to be *narrow* if for every two elements $x, y \in S(X)$, every $x^* \in X^*$ and every $\varepsilon > 0$ there is an element $u \in B(X)$ such that $\|x+u\| > 2-\varepsilon$ and $\|T(y-u)\| + |x^*(y-u)| < \varepsilon$. It follows that X has the Daugavet property if and only if all rank-1 operators are strong Daugavet operators if and only if there is at least one narrow operator on X. We denote the set of all strong Daugavet operators on X by $\mathscr{SD}(X)$ and the set of all narrow operators on X by $\mathscr{NAR}(X)$. Actually, in [6] we took a slightly different point of view in that we declared two operators $T_1\colon X \to Z_1$ and $T_2\colon X \to Z_2$ to be equivalent if $\|T_1 x\| = \|T_2 x\|$ for all $x \in X$; $\mathscr{SD}(X)$ and $\mathscr{NAR}(X)$ should really denote the sets of corresponding equivalence classes. However, in this paper we shall not make this point explicitly.

In this paper we shall continue our investigations of this type of operator, mostly in the setting of vector-valued function spaces $C(K, E)$. One of the drawbacks of the definition of a strong Daugavet operator is that the sum of two such operators need not be a strong Daugavet operator whereas the definition of a narrow operator has some built-in additivity quality. It remained open in [6] whether the sum of any two narrow operators is always narrow, although we could prove this to be true on $C(K)$, and in general we showed that the sum of a narrow operator and an operator not fixing ℓ_1 is narrow and that the sum of a narrow operator and a strong Radon-Nikodým operator is narrow. (Note that the sum of two strong Radon-Nikodým operators need not be a strong Radon-Nikodým operator [8].) Our work in Section 3, where we completely characterise strong Daugavet and narrow operators on $C(K, E)$, enables us to give counterexamples to the sum problem.

For this we employ a special feature of ℓ_1 explained in Section 2. This section introduces a class of Banach spaces called *USD-nonfriendly* spaces that are sort of remote from spaces with the Daugavet property; USD stands for uniformly strongly Daugavet. All finite-dimensional and all locally uniformly convex spaces fall within this category, but we haven't been able to decide whether a reflexive space must be USD-nonfriendly.

The class of USD-nonfriendly spaces is custom-made for our applications in Section 4 where we study pointwise unconditionally convergent series $\sum_{n=1}^{\infty} T_n$ of narrow operators on $C(K,E)$. If E is separable and USD-nonfriendly, we prove that the sum operator must be narrow again, which is new even in the case $E=\mathbb{R}$. To achieve this we take a detour investigating the related class of C-narrow operators following ideas from [4]. An obvious corollary is the result from [4] that the identity on $C(K)$ is not a pointwise unconditional sum of narrow operators, which implies that $C(K)$ does not admit an unconditional Schauder decomposition into spaces not containing $C[0,1]$.

We finish this introduction with a technical reformulation of the definition of a strong Daugavet operator. Let

$$D(x,y,\varepsilon) = \{z \in X\colon \|x+y+z\| > 2-\varepsilon,\ \|y+z\| < 1+\varepsilon\}$$

and

$$\begin{aligned} \mathscr{D}(X) &= \{D(x,y,\varepsilon)\colon x \in S(X),\ y \in S(X),\ \varepsilon > 0\}, \\ \mathscr{D}_0(X) &= \{D(x,y,\varepsilon)\colon x \in S(X),\ y \in B(X),\ \varepsilon > 0\}. \end{aligned}$$

It is easy to see that $T\colon X \to Z$ is a strong Daugavet operator if and only if T is not bounded from below on any $D \in \mathscr{D}(X)$ [6, Prop. 3.4]. In Section 3 it will be more convenient to work with $\mathscr{D}_0(X)$ instead; therefore we formulate a lemma saying that this doesn't make any difference.

Lemma 1.2. *An operator $T\colon X \to Z$ is a strong Daugavet operator if and only if T is not bounded from below on any $D \in \mathscr{D}_0(X)$.*

Proof. We have to show that $T \in \mathscr{S}\mathscr{D}(X)$ is not bounded from below on $D(x,y,\varepsilon)$ whenever $\|x\|=1$, $\|y\| \le 1$, $\varepsilon > 0$. By the above, T is not bounded from below on $D(x,-x,1)$; hence, given $\varepsilon' > 0$, for some $\zeta \in S(X)$ we have $\|T\zeta\| < \varepsilon'$. Now pick $\lambda \ge 0$ such that $y+\lambda\zeta \in S(X)$; then there is some $z' \in X$ such that

$$\|x+(y+\lambda\zeta)+z'\| > 2-\varepsilon,\ \|(y+\lambda\zeta)+z'\| < 1+\varepsilon,\ \|Tz'\| < \varepsilon';$$

i.e., $z := \lambda\zeta + z' \in D(x,y,\varepsilon)$ and $\|Tz\| < 3\varepsilon'$. □

2. USD-nonfriendly spaces

In this section we introduce a class of Banach spaces that are geometrically opposite to spaces with the Daugavet property. These spaces will arise naturally in Section 4.

Proposition 2.1. *The following conditions for a Banach space E are equivalent.*

1. $\mathscr{SD}(E) = \{0\}$.
2. *No nonzero linear functional on E is a strong Daugavet operator.*
3. *For every $x^* \in S(E^*)$ there exist some $\delta > 0$ and $D \in \mathscr{D}(E)$ such that $|x^*(z)| > \delta$ for all $z \in D$.*
4. *Every closed absolutely convex subset $A \subset E$ such that for every $\alpha > 0$ and every $D \in \mathscr{D}(E)$ the intersection $(\alpha A) \cap D$ is nonempty coincides with the whole space E.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are evident.

(3) $\Rightarrow$ (4): Assume there is some closed absolutely convex subset $A \subset E$ with the property from (4) that does not coincide with the whole space E. By the Hahn-Banach theorem there is a functional $x^* \in S(E^*)$ and a number $r > 0$ such that $|x^*(a)| \le r$ for every $a \in A$. If $\delta > 0$ and $D \in \mathscr{D}(E)$ are arbitrary, pick $z \in (\frac{\delta}{r}A) \cap D$; this intersection is nonempty by assumption on A. It follows that $|x^*(z)| \le \delta$, hence (3) fails.

(4) $\Rightarrow$ (1): Suppose $T \in \mathscr{SD}(E)$ and put $A = \{e \in E\colon \|Te\| \le 1\}$. By the definition of a strong Daugavet operator this A satisfies (4). So $A = E$ and hence $T = 0$. □

This proposition suggests the following definition.

Definition 2.2. A Banach space E is said to be an *SD-nonfriendly space* (i.e., strong Daugavet-nonfriendly) if $\mathscr{SD}(E) = \{0\}$. A space E is said to be a *USD-nonfriendly space* (i.e., uniformly strong Daugavet-nonfriendly) if there exists an $\alpha > 0$ such that every closed absolutely convex subset $A \subset E$ which intersects all the elements of $\mathscr{D}(E)$ contains $\alpha B(E)$. The largest admissible α is called the *USD-parameter* of E.

Proposition 2.1 shows that a USD-nonfriendly space is indeed SD-nonfriendly; but the converse is false as will be shown shortly. Also, SD-nonfriendliness is opposite to the Daugavet property in that the latter is equivalent to the condition that every functional is a strong Daugavet operator.

To further motivate the uniformity condition in the above definition, we supply a lemma.

Lemma 2.3. *A Banach space E is USD-nonfriendly if and only if*

(3*) *There exists some $\delta > 0$ such that for every $x^* \in S(E^*)$ there exists $D \in \mathscr{D}(E)$ such that $|x^*(z)| > \delta$ for all $z \in D$.*

Proof. It is enough to prove the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) for the following assertions about a fixed number $\delta > 0$:

(a) There exists a closed absolutely convex set $A \subset E$ not containing $\delta B(E)$ that intersects all $D \in \mathscr{D}(E)$.

(b) There exists a functional $x^* \in S(E^*)$ such that for all $D \in \mathscr{D}(E)$ there exists $z_D \in D$ satisfying $|x^*(z_D)| \le \delta$.

(c) There exists a closed absolutely convex set $A \subset E$ not containing $\delta' B(E)$ for any $\delta' > \delta$ that intersects all $D \in \mathscr{D}(E)$.

To see that (a) implies (b), pick $u \notin A$, $\|u\| \le \delta$. By the Hahn-Banach theorem we can separate u from A by means of a functional $x^* \in S(E^*)$; then we shall have for some number $r > 0$ that $|x^*(z)| \le r$ for all $z \in A$ and $x^*(u) > r$. On the other hand, $x^*(u) \le \|x^*\|\,\|u\| \le \delta$; hence (b) holds for x^*.

If we assume (b), we define A to be the closed absolutely convex hull of the elements z_D, $D \in \mathscr{D}(E)$, appearing in (b). Obviously A intersects each $D \in \mathscr{D}(E)$. If $\delta' B(E) \subset A$ for some $\delta' > 0$, then since $|x^*| \le \delta$ on A, we must have $|x^*| \le \delta$ on $\delta' B(E)$, i.e., $\delta' \le \delta$. Therefore, A works in (c). □

In Proposition 2.1 and Lemma 2.3 we may replace $\mathscr{D}(E)$ by $\mathscr{D}_0(E)$.

We now turn to some examples.

Proposition 2.4.

(a) *The space c_0 is SD-nonfriendly, but not USD-nonfriendly.*

(b) *The space ℓ_1 is not SD-nonfriendly and hence not USD-nonfriendly either.*

Proof. (a) Theorem 3.5 of [6] implies that $Te_k = 0$ for every unit basis vector e_k if $T \in \mathscr{SD}(c_0)$. [Actually, the theorem quoted is formulated for operators on $C(K)$ for compact K, but the theorem works likewise on $C_0(L)$ with L locally compact.] Hence $T = 0$ is the only strong Daugavet operator on c_0. (Another way to see this is to apply Corollary 3.6.)

To show that c_0 is not USD-nonfriendly we shall exhibit a closed absolutely convex set A intersecting each $D \in \mathscr{D}(c_0)$, yet containing no ball. Let $A = 2B(\ell_1) \subset c_0$, i.e.,

$$A = \left\{ (x(n)) \in c_0\colon \sum_{n=1}^{\infty} |x(n)| \le 2 \right\},$$

which is closed in c_0. Fix $x \in S(c_0)$ and $y \in S(c_0)$. If $|x(k)| = 1$, say $x(k) = 1$, pick $|\beta| \le 2$ such that $y(k) + \beta = 1$. Then $\beta e_k \in D(x, y, \varepsilon) \cap A$ for every $\varepsilon > 0$. Obviously, A does not contain a multiple of $B(c_0)$.

(b) We claim that $x^*_\sigma(x) = \sum_{n=1}^{\infty} \sigma_n x(n)$ defines a strong Daugavet functional on ℓ_1 whenever σ is a sequence of signs, i.e., if $|\sigma_n| = 1$ for all n. Indeed, let $x \in S(\ell_1)$, $y \in S(\ell_1)$ and $\varepsilon > 0$. Pick N such that $\sum_{n=1}^{N} |x(n)| > 1 - \varepsilon$ and define $u \in S(\ell_1)$ by $u(n) = 0$ for $n \le N$ and $u(n) = \sigma_{n-N} y(n-N)/\sigma_n$ for $n > N$. Then $x^*(u) = x^*(y)$ and $\|x + u\| > 2 - \varepsilon$; hence $z := u - y \in D(x, y, \varepsilon)$ and $x^*(z) = 0$. □

Next we wish to give some examples of USD-nonfriendly spaces. Recall that a point of local uniform rotundity of the unit sphere of a Banach space

E (a LUR-point) is a point $x_0 \in S(E)$ such that $x_n \to x_0$ whenever $\|x_n\| \le 1$ and $\|x_n + x_0\| \to 2$.

Proposition 2.5. *If the unit sphere of E contains a LUR-point, then E is a USD-nonfriendly space with USD-parameter ≥ 1.*

Proof. Let $x_0 \in S(E)$ be a LUR-point and $A \subset E$ be a closed absolutely convex subset which intersects all the elements of $\mathscr{D}(E)$. In particular for every fixed $y \in S(E)$ the set A intersects all the sets $D(x_0, y, \varepsilon) \subset E$, $\varepsilon > 0$. By definition of a LUR-point this means that all the points of the form $x_0 - y$, $y \in S(E)$, belong to A, i.e., $B(E) + x_0 \subset A$. But $-x_0$ is also a LUR-point, so $B(E) - x_0 \subset A$, and by convexity of A, $B(E) \subset A$. □

Corollary 2.6. *Every locally uniformly convex space is USD-nonfriendly with USD-parameter 2. In particular, the spaces $L_p(\mu)$ are USD-nonfriendly for $1 < p < \infty$.*

Proof. This follows from the previous proposition; that the USD-parameter is 2 is a consequence of $B(E) + x_0 \subset A$ for all $x_0 \in S(E)$; see the above proof. □

It is clear that no finite-dimensional space enjoys the Daugavet property, but more is true.

Proposition 2.7. *Every finite-dimensional Banach space E is a USD-nonfriendly space.*

Proof. Assume to the contrary that there is a finite-dimensional space E that is not USD-nonfriendly. By Lemma 2.3 we can find a sequence of functionals $(x_n^*) \subset S(E^*)$ such that $\inf_{z\in D} |x_n^*(z)| \le 1/n$ for each $D \in \mathscr{D}(E)$. By compactness of the ball we can pass to the limit and obtain a functional $x^* \in S(E^*)$ with the property that $\inf_{z\in D} |x^*(z)| = 0$ for each $D \in \mathscr{D}(E)$.

Denote $K = \{e \in B(E) \colon x^*(e) = 1\}$; this is a norm-compact convex set. Let $x_0 \in K$ be an arbitrary point. If we apply the above property to $D(x_0, -x_0, \varepsilon)$ for all $\varepsilon > 0$, we obtain, again by compactness, some z_0 such that $\|z_0 - x_0\| = 1$, $\|z_0\| = 2$ and $x^*(z_0) = 0$. We have $x^*(x_0 - z_0) = 1$, so $x_0 - z_0 \in K$. Therefore

$$2 \ge \operatorname{diam} K \ge \sup_{y\in K} \|x_0 - y\| \ge \|x_0 - (x_0 - z_0)\| = \|z_0\| = 2;$$

hence $\operatorname{diam} K = 2$ and x_0 is a diametral point of K, meaning

$$\sup_{y\in K} \|x_0 - y\| = \operatorname{diam} K.$$

But any compact convex set of positive diameter contains a nondiametral point [3, p. 38]; thus we have reached a contradiction. □

We shall later estimate the worst possible USD-parameter of an n-dimensional normed space.

We haven't been able to decide whether every reflexive space is USD-nonfriendly. Proposition 2.10 below presents a necessary condition a hypothetical reflexive USD-friendly (= not USD-nonfriendly) space must fulfill.

First an easy geometrical lemma.

Lemma 2.8. *Let $x, h \in E$, $\|x\| \le 1+\varepsilon$, $\|h\| \le 1+\varepsilon$, $\|x+h\| \ge 2-\varepsilon$. Let $f \in S(E^*)$ be a supporting functional of $(x+h)/\|x+h\|$. Then $f(x)$ as well as $f(h)$ are estimated from below by $1-2\varepsilon$.*

Proof. Denote $a = f(x)$, $b = f(h)$. Then $\max(a,b) \le 1+\varepsilon$ but $a+b \ge 2-\varepsilon$. So $\min(a,b) = a+b-\max(a,b) \ge 1-2\varepsilon$. □

Let E be a reflexive space, x_0^* be a strongly exposed point of $S(E^*)$ with strongly exposing evaluation functional x_0; i.e., the diameter of the slice $\{x^* \in S(E^*)\colon x^*(x_0) > 1-\varepsilon\}$ tends to 0 when ε tends to 0. Denote

$$S_{x_0^*} = \{x \in S(E)\colon x_0^*(x) = 1\}.$$

Proposition 2.9. *Let E, x_0^*, x_0 be as above, A be a closed convex set which intersects all the sets $D(x_0, 0, \varepsilon)$, $\varepsilon > 0$. Then A intersects $S_{x_0^*}$.*

Proof. For every $n \in \mathbb{N}$ select $h_n \in A \cap D(x_0, 0, \frac{1}{n})$. Then $\|h_n\| \le 1+\frac{1}{n}$, $\|x_0 + h_n\| \ge 2-\frac{1}{n}$. Denote by f_n a supporting functional of $(x_0+h_n)/\|x_0+h_n\|$. By the previous lemma $f_n(x_0)$ tends to 1 when n tends to infinity. So by the definition of an exposing functional, f_n tends to x_0^*. By the same lemma $f_n(h_n)$ tends to 1, so $x_0^*(h_n)$ also tends to 1. Hence every weak limit point of the sequence (h_n) belongs to the intersection of A and $S_{x_0^*}$, so this intersection is nonempty. □

Proposition 2.10. *Let E be a reflexive space.*

(a) *If E is USD-nonfriendly with USD-parameter $< \alpha$, then there exists a functional $x^* \in S(E^*)$ such that for every strongly exposed point x_0^* of $B(E^*)$ the numerical set $x^*(S_{x_0^*})$ contains the interval $[-1+\alpha, 1-\alpha]$.*

(b) *If E is not USD-nonfriendly, then for every strongly exposed point x_0^* of $B(E^*)$ the set $S_{x_0^*}$ has diameter 2. Moreover, for every $\delta > 0$ there exists a functional $x^* \in S(E^*)$ such that for every strongly exposed point x_0^* of $B(E^*)$ the numerical set $x^*(S_{x_0^*})$ contains the interval $[-1+\delta, 1-\delta]$.*

Proof. (a) Let A be a closed absolutely convex set which intersects all the sets $D \in \mathscr{D}(E)$, but does not contain $\alpha B(E)$. By the Hahn-Banach theorem there exists a functional $x^* \in S(E^*)$ such that $|x^*(a)| < \alpha$ for every $a \in A$. We fix $y \in S(E)$ with $x^*(y) = -1$.

Let $x_0^* \in S(E^*)$ be a strongly exposed point of $B(E^*)$. As before, we denote an exposing evaluation functional by x_0. Now $A \cap D(x_0, y, \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. By Proposition 2.9 and the evident equality $D(x_0, 0, \varepsilon) - y = D(x_0, y, \varepsilon)$ this implies that the set $A+y$ intersects $S_{x_0^*}$. If z_1 is an element of this intersection, we see that $x^*(z_1) < \alpha - 1$.

Likewise, since $D(-x_0, 0, \varepsilon) = -D(x_0, 0, \varepsilon)$, we find some $z_2 \in (-A - y) \cap S_{x_0^*}$; hence $x^*(z_2) > -\alpha + 1$. Therefore, $[-1+\alpha, 1-\alpha] \subset x^*(S_{x_0^*})$.

(b) The argument is the same as in (a). □

This proposition allows us to estimate the USD-parameter of finite-dimensional spaces.

Proposition 2.11. *If E is n-dimensional, then its USD-parameter is $\geq 2/n$.*

Proof. Assume that $\dim(E) = n$ and that its USD-parameter is $< 2/n$; then this parameter is strictly smaller than some $\alpha < 2/n$. Choose x^* as in Proposition 2.10 so that

$$[-1+\alpha, 1-\alpha] \subset x^*(S_{x_0^*}) \tag{2.1}$$

for every strongly exposed functional $x_0^* \in S(E^*)$.

We now claim that in any ε-neighbourhood of x^* there is some $y^* \in B(E^*)$ which can be represented as a convex combination of $\leq n$ strongly exposed functionals. First of all, the convex hull of the set $\operatorname{stexp} B(E^*)$ of strongly exposed functionals is norm-dense in $B(E^*)$; in fact, this is true of any bounded closed convex set in a separable dual space [1, p. 110]. Hence for some $\|y_1^* - x^*\| < \varepsilon$, $\lambda_1', \dots, \lambda_r' \geq 0$ with $\sum_{k=1}^r \lambda_k' = 1$ and $x_1^*, \dots, x_r^* \in \operatorname{stexp} B(E^*)$

$$y_1^* = \sum_{k=1}^r \lambda_k' x_k^*.$$

Let $C = \operatorname{co}\{x_1^*, \dots, x_r^*\}$ and let y^* be the point of intersection of the segment $[y_1^*, x^*]$ with the relative boundary of C, i.e., $y^* = \tau x^* + (1-\tau) y_1^*$ with $\tau = \sup\{t \in [0,1] \colon tx^* + (1-t)y_1^* \in C\}$. Let F be the face of C generated by y^*; then F is a convex set of dimension $< n$. Therefore an appeal to Carathéodory's theorem shows that y^* can be represented as a convex combination of no more than n extreme points of F. But $\operatorname{ex} F \subset \operatorname{ex} C \subset \{x_1^*, \dots, x_r^*\} \subset \operatorname{stexp} B(E^*)$, and our claim is established.

We apply the claim with some $\varepsilon < 2/n - \alpha$ to obtain some convex combination $y^* = \sum_{k=1}^n \lambda_k x_k^*$ of n strongly exposed functionals such that $\|y^* - x^*\| < \varepsilon$. One of the coefficients must be $\geq 1/n$, say $\lambda_n \geq 1/n$. Now if $x \in S_{x_n^*}$,

$$\begin{aligned} x^*(x) &\geq x^*(y) - \varepsilon = \sum_{k=1}^{n-1} \lambda_k x_k^*(x) + \lambda_n - \varepsilon \\ &\geq -\sum_{k=1}^{n-1} \lambda_k + \lambda_n = -1 + 2\lambda_n - \varepsilon \geq -1 + 2/n - \varepsilon. \end{aligned}$$

By (2.1) we have $-1+\alpha \geq -1+2/n-\varepsilon$ which contradicts our choice of ε. □

For ℓ_∞^n we can say more, namely, its USD-parameter is worst possible.

Proposition 2.12. *The USD-parameter of ℓ_∞^n is $2/n$.*

Proof. The argument of Proposition 2.4(a) implies in the setting of ℓ_∞^n rather than c_0 that the USD-parameter of ℓ_∞^n is $\le 2/n$, and the converse estimate follows from Proposition 2.11. □

3. Strong Daugavet and narrow operators in spaces of vector-valued functions

Let E be a Banach space and X be a subspace of the space of all bounded E-valued functions defined on a set K, equipped with the sup-norm. It will be convenient to use the following notation: A disjoint pair (U, V) of subsets of K is said to be *interpolating* for X if for every $f, g \in X$ with $\|f\| < 1$ and $\|g\chi_V\| < 1$ there exists $h \in B(X)$ such that $h = f$ on U and $h = g$ on V.

For arbitrary $V \subset K$ denote by X_V the subspace of all functions from X vanishing on V.

Proposition 3.1. *Let X be as above and let (U, V) be an interpolating pair for X. Then for every $f \in X$*

$$\operatorname{dist}(f, X_V) \le \sup_{t\in V} \|f(t)\|.$$

Proof. By the definition of an interpolating pair, for an arbitrary $\varepsilon > 0$ there exists an element $h \in X$, $\|h\| < \sup_{t\in V} \|f(t)\| + \varepsilon$, such that $h = 0$ on U and $h = f$ on V. Then the element $f - h$ belongs to X_V, so

$$\operatorname{dist}(f, X_V) \le \|f - (f - h)\| = \|h\| < \sup_{t\in V} \|f(t)\| + \varepsilon,$$

which completes the proof. □

Lemma 3.2. *Let $X \subset \ell_\infty(K, E)$, $U, V \subset K$, $f \in S(X_V)$ and $\varepsilon > 0$. Assume that $U \supset \{t \in K\colon \|f(t)\| > 1 - \varepsilon\}$ and that (U, V) is an interpolating pair for X. If T is a strong Daugavet operator on X and $g \in B(X)$, there is a function $h \in X_V$, $\|h\| \le 2 + \varepsilon$, satisfying*

$$\|Th\| < \varepsilon,\ \ \|(g + h)\chi_U\| < 1 + \varepsilon \text{ and } \|(f + g + h)\chi_U\| > 2 - \varepsilon.$$

Proof. Before we enter the proof proper, we formulate a number of technical assertions that are easy to verify and will be needed later.

Sublemma 3.3. *If T is a strong Daugavet operator on a Banach space X, if $1 - \eta < \|x\| < 1 + \eta$ and $\|y\| < 1 + \eta$, then there is some $z \in X$ such that*

$$\|x + y + z\| > 2 - 3\eta,\ \ \|y + z\| < 1 + 2\eta,\ \ \|Tz\| < \eta.$$

Proof. Choose $x_0 \in S(X)$ and $y_0 \in B(X)$ such that $\|x_0 - x\| < \eta$, $\|y_0 - y\| < \eta$ and pick by Lemma 1.2 $z \in D(x_0, y_0, \eta)$ such that $\|Tz\| < \eta$; this z clearly works. □

Sublemma 3.4. *If $\|x\| < 1+\eta$, $\|y\| < 1+\eta$ and $\|(x+y)/2\| > 1-\eta$ in a normed space, then $\|\lambda x + (1-\lambda)y\| > 1-3\eta$ whenever $0 \le \lambda \le 1$.*

Proof. Should $\|\lambda x + (1-\lambda)y\| \le 1-3\eta$ for some $0 \le \lambda \le 1/2$, then, since $\lambda_1 x + (1-\lambda_1)(\lambda x + (1-\lambda)y) = (x+y)/2$ for $\lambda_1 = (\frac{1}{2}-\lambda)/(1-\lambda) \in [0,1/2]$,

$$\left\|\frac{x+y}{2}\right\| \le \lambda_1(1+\eta) + (1-\lambda_1)(1-3\eta) = 1-(3-4\lambda_1)\eta \le 1-\eta.$$

(The case $\lambda > 1/2$ is analogous.) □

Sublemma 3.5. *If $\|y\| < 1+\eta$ and $\|x+Ny\|/(N+1) > 1-3\eta$ in a normed space, then $\|(x+y)/2\| > 1-(2N+1)\eta$.*

Proof. Should $\|(x+y)/2\| \le 1-(2N+1)\eta$, then

$$\begin{aligned}\left\|\frac{x+Ny}{1+N}\right\| &\le \frac{2}{1+N}\left\|\frac{x+y}{2}\right\| + \Big(1-\frac{2}{1+N}\Big)\|y\| \\ &\le \frac{2}{1+N}\big(1-(2N+1)\eta\big) + \Big(1-\frac{2}{1+N}\Big)(1+\eta) = 1-3\eta.\end{aligned}$$

□

To start the actual proof we may assume that $\|T\| = 1$. Fix $N > 6/\varepsilon$ and $\delta > 0$ such that $2(2N+1)9^N\delta < \varepsilon$; let $\delta_n = 9^n\delta$ so that $(2N+1)\delta_N < \varepsilon/2$. Put $f_1 = f$, $g_1 = g$ and pick $h_1 \in X$ such that

$$\|f_1 + g_1 + h_1\| > 2-\delta_1,\ \|g_1 + h_1\| < 1+2\delta_0,\ \|Th_1\| < \delta_0.$$

We are going to construct functions $f_n, g_n, h_n \in X$ by induction so as to satisfy

(a) $f_{n+1} = \frac{1}{n+1}(f_1 + \sum_{k=1}^n (g_k+h_k)) = \frac{n}{n+1}f_n + \frac{1}{n+1}(g_n+h_n)$, $1-3\delta_n < \|f_{n+1}\| < 1+\delta_n$,

(b) $g_{n+1} = g_1$ on U and $g_{n+1} = g_n + h_n$ $(= g_1 + h_1 + \cdots + h_n)$ on V, $\|g_{n+1}\| < 1+\delta_n$,

(c) $\|f_{n+1}+g_{n+1}+h_{n+1}\| > 2-\delta_{n+1}$, $1-2\delta_n < \|g_{n+1}+h_{n+1}\| < 1+6\delta_n < 1+\delta_{n+1}$, $\|Th_{n+1}\| < 3\delta_n$.

Suppose that these functions have already been constructed for the indices $1,\dots,n$. We then define f_{n+1} as in (a). Since by induction hypothesis $\|f_n\| < 1+\delta_{n-1}$ and $\|g_n+h_n\| < 1+\delta_n$, we clearly have $\|f_{n+1}\| < 1+\delta_n$. From $\|f_n+g_n+h_n\| > 2-\delta_n$ we conclude using Sublemma 3.4 (with $\eta = \delta_n$) that $\|f_{n+1}\| > 1-3\delta_n$. Thus (a) is achieved. To achieve (b) it is enough to use that (U,V) is interpolating along with the induction hypothesis that $\|g_n+h_n\| < 1+\delta_n$. Finally (c) follows from Sublemma 3.3 with $\eta = 3\delta_n$.

Next we argue that

$$\left\|f_1 + \frac{1}{N}\sum_{k=1}^N (g_k+h_k)\right\| > 2-\varepsilon/2.$$

This follows from Sublemma 3.5, (c) and (a) and our choice of δ. But for $t \notin U$ we can estimate

$$\left\| f_1(t) + \frac{1}{N}\sum_{k=1}^{N}(g_k(t) + h_k(t))\right\| \le 1 - \varepsilon + 1 - \delta_N \le 2 - 2\varepsilon,$$

therefore, letting $w = \frac{1}{N}\sum_{k=1}^{N} h_k$,

$$\|(f + g + w)\chi_U\| = \left\| \left(f_1 + \frac{1}{N}\sum_{k=1}^{N}(g_k + h_k)\chi_U \right) \right\| > 2 - \varepsilon/2.$$

Furthermore we have the estimates

$$\begin{aligned}
\|(g + w)\chi_U\| &= \left\| \frac{1}{N}\sum_{k=1}^{N}(g_k + h_k)\chi_U \right\| \le 1 + \delta_N < 1 + \varepsilon/2,\\
\|Tw\| &\le \frac{1}{N}\sum_{k=1}^{N}\|Th_k\| < 3\delta_{N-1} = \frac{1}{3}\delta_N < \varepsilon/2,\\
\|h_k\| &\le \|g_k + h_k\| + \|g_k\| \le 2 + 2\delta_k \le 2 + 2\delta_N \le 2 + \varepsilon/2,\\
\|w\| &\le \frac{1}{N}\sum_{k=1}^{N}\|h_k\| \le 2 + \varepsilon/2,
\end{aligned}$$

and for $t \in V$

$$\|w(t)\| = \frac{1}{N}\|g_{N+1}(t) - g_1(t)\| \le \frac{2 + \delta_N}{N} < \frac{3}{N} < \varepsilon/2.$$

By Proposition 3.1 and the above we see that $\operatorname{dist}(w, X_V) < \varepsilon/2$. Hence it is left to replace w by an element $h \in X_V$, $\|h - w\| \le \varepsilon/2$, to finish the proof. □

Let us remark that the conditions of Lemma 3.2 are fulfilled for an arbitrary compact Hausdorff space K, for a closed subset $V \subset K$ and for $X = C(K, E)$ as well as for $X = C_w(K, E)$. Here is another example.

Corollary 3.6. *If $X = X_1 \oplus_\infty X_2$ and $T \in \mathscr{SD}(X)$, then $T|_{X_1} \in \mathscr{SD}(X_1)$.*

To see this let $K = \operatorname{ex} B(X^*)$, $K_1 = \operatorname{ex} B(X_1^*)$, $K_2 = \operatorname{ex} B(X_2^*)$ so that $K = K_1 \cup K_2$ and $X \subset \ell_\infty(K)$ canonically. It is left to apply Lemma 3.2 with the interpolating pair (K_1, K_2). A direct proof of Corollary 3.6 is given in [2].

In the sequel, for some element $y \in E$ we also use the symbol y to denote the constant function in $C(K, E)$ taking that value.

Theorem 3.7. *Let K be a compact Hausdorff space, E a Banach space and T an operator on $X = C(K, E)$. Then the following conditions are equivalent:*

(1) *$T \in \mathscr{SD}(X)$.*

(2) *For every closed subset* $V \subset K$*, every* $x \in S(E)$*, every* $y \in B(E)$ *and every* $\varepsilon > 0$ *there exists an open subset* $W \subset K \setminus V$*, an element* $e \in E$ *with* $\|e + y\| < 1 + \varepsilon$*,* $\|e + y + x\| > 2 - \varepsilon$*, and a function* $h \in X_V$*,* $\|h\| \le 2 + \varepsilon$*, such that* $\|Th\| < \varepsilon$ *and* $\|e - h(t)\| < \varepsilon$ *for* $t \in W$*.*

(3) *For every closed subset* $V \subset K$*, every* $x \in S(E)$*, every* $y \in B(E)$ *and every* $\varepsilon > 0$ *there exists a function* $f \in X_V$ *such that* $\|Tf\| < \varepsilon$*,* $\|f + y\| < 1 + \varepsilon$*,* $\|f + y + x\| > 2 - \varepsilon$*.*

If K *has no isolated points, then these conditions are equivalent to*

(4) $T \in \mathscr{NAR}(X)$*.*

Proof. The implication (1) ⇒ (2) follows from Lemma 3.2, as follows. Let us apply Lemma 3.2 to $\varepsilon/4 > 0$, $g = \chi_K \otimes y$, $f = f_1 \otimes x \in S(X)$, where f_1 is a positive scalar function vanishing on V, and $U = \{t \in K\colon \|f(t)\| > 1 - \varepsilon/4\}$. Then for $h \in X_V$ which we get from Lemma 3.2 let us find a point $t_0 \in U$ such that $\|(f + g + h)(t_0)\| = \|(f + h)(t_0) + y\| > 2 - \varepsilon/4$. Because $\|h(t_0) + y\| < 1 + \varepsilon/4$ we have $\|f(t_0)\| > 1 - \varepsilon/2$, i.e., $\|f(t_0) - x\| < \varepsilon/2$. Now select an open neighbourhood $W \subset U$ of t_0 such that $\|f(\tau) - x\| < \varepsilon/2$ for all $\tau \in W$ and put $e = h(t_0)$.

To prove the implication (2) ⇒ (3) let us fix a positive $\varepsilon < 1/10$, $\delta < \varepsilon/4$ and $N > 6 + 2/\varepsilon$. Now apply inductively condition (2) to obtain elements x_k, y_k, e_k, $x_1 = x$, $y_k = y$, $k = 1, \dots, N$, open subsets $W_1 \supset W_2 \supset \dots$, closed subsets $V_{k+1} = K \setminus W_k$, $V_1 = V$ and functions $h_k \in X_{V_k}$ with the following properties:

(a) $x_{n+1} = \dfrac{x + \sum_{k=1}^{n}(y_k + e_k)}{\|x + \sum_{k=1}^{n}(y_k + e_k)\|} \in S(E)$,

(b) $\|e_k + y_k\| < 1 + \delta$, $\|e_k + y_k + x_k\| > 2 - \delta$,

(c) $h_k \in X_{V_k}$, $\|h_k(t) - e_k\| < \varepsilon/4$ for all $t \in W_k$, $\|h_k\| \le 2 + \varepsilon$, and $\|Th_k\| < \varepsilon$.

By an argument similar to the one in Lemma 3.2, we have for a proper choice of δ

$$\left\| x + y + \frac{1}{N}\sum_{k=1}^{N} e_k \right\| = \left\| x + \frac{1}{N}\sum_{k=1}^{N}(y_k + e_k) \right\| > 2 - \frac{\varepsilon}{2}.$$

Let us put $f = \frac{1}{N}\sum_{k=1}^{N} h_k$. Then the last inequality and (c) of our construction yield that $f \in X_V$, $\|f + y + x\| > 2 - \varepsilon$ and $\|Tf\| < \varepsilon$. The only thing left to do now is to estimate $\|f + y\|$ from above. If $t \in V$, then $\|f(t) + y\| = \|y\| \le 1$. If $t \in W_n \setminus W_{n+1}$ for some n then

$$\|f(t) + y\| = \left\| \frac{1}{N}\sum_{k=1}^{n} h_k(t) + y \right\| = \left\| \frac{1}{N}\sum_{k=1}^{n}(h_k(t) + y) \right\|$$

In this sum all the summands except for the last one satisfy the inequality $\|h_k(t) + y\| \le 1 + \varepsilon/2$ and the last summand $h_n(t) + y$ is bounded by $3 + \varepsilon$.

So

$$\|f(t)+y\| \le \frac{1}{N}\sum_{k=1}^{n-1}\Big(1+\frac{\varepsilon}{2}\Big)+\frac{1}{N}(3+\varepsilon) \le 1+\frac{\varepsilon}{2}+\frac{1}{N}(3+\varepsilon) \le 1+\varepsilon.$$

The same estimate holds for $t \in W_N$.

To prove the implication (3) $\Rightarrow$ (1) fix $f, g \in S(X)$ and $0 < \varepsilon < 1/10$. Pick a point $t \in K$ with $\|f(t)\| > 1-\varepsilon/4$ and a neighbourhood U of t such that

$$\|f(t)-f(\tau)\| + \|g(t)-g(\tau)\| < \frac{\varepsilon}{4} \qquad \forall \tau \in U.$$

Denote $x = f(t)/\|f(t)\|$ and $y = g(t)$ and apply condition (3) to obtain a function $h \in X_V$ such that $\|Th\| < \varepsilon$, $\|h+y\| < 1+\varepsilon/4$ and $\|h+y+x\| > 2-\varepsilon/4$. For this h we have $\|h+g\| < 1+\varepsilon$ and $\|h+g+f\| > 2-\varepsilon$, so $T \in \mathscr{SD}(X)$.

Let us now pass to the case of a perfect compact K. The implication (4) $\Rightarrow$ (1) is evident.

The proof of the remaining implication (3) $\Rightarrow$ (4) is similar to that of (3) $\Rightarrow$ (1). Namely, let $f, g \in S(X)$, $x^* \in X^*$ and let $\varepsilon > 0$ be small. We have to show that there is an element $h \in X$ such that

$$\|f+g+h\| > 2-\varepsilon, \quad \|g+h\| < 1+\varepsilon \tag{3.1}$$

and

$$\|Th\| + |x^*h| < \varepsilon. \tag{3.2}$$

To this end let us pick a closed subset $V \subset K$ (whose complement $K \setminus V$ we denote by U) and a point $t \in U$ in such a way that $\|f(t)\| > 1-\varepsilon/4$,

$$|x^*|_{X_V} < \frac{\varepsilon}{4}, \tag{3.3}$$

and for every $\tau \in U$

$$\|f(t)-f(\tau)\| + \|g(t)-g(\tau)\| < \frac{\varepsilon}{4}. \tag{3.4}$$

Denote $x = f(t)/\|f(t)\|$, $y = g(t)$ and apply condition (3) to obtain a function $h \in X_V$ such that $\|Th\| < \varepsilon/4$, $\|h+y\| < 1+\varepsilon/4$ and $\|h+y+x\| > 2-\varepsilon/4$. For this h (3.1) follows from (3.4) and (3.2) follows from (3.3). □

In [6] we have introduced the tilde-sum of two operators $T_1\colon X \to Y_1$, $T_2\colon X \to Y_2$ by

$$T_1 \tilde{+} T_2\colon X \to Y_1 \oplus_1 Y_2,\ x \mapsto (T_1x, T_2x).$$

There we proved that the $\tilde{+}$-sum and therefore also the ordinary sum of two narrow operators on $C(K)$ is narrow (another proof will be given in the next section), and we inquired whether this is so on any space with the Daugavet property. We are now in a position to provide a counterexample.

Let $T\colon E \to F$ be an operator on a Banach space. By T^K let us denote the corresponding "multiplication" or "diagonal" operator $T^K\colon C(K,E) \to C(K,F)$ defined by

$$(T^K f)(t) = T(f(t)).$$

Proposition 3.8. *$T^K \in \mathscr{SD}(C(K,E))$ if and only if $T \in \mathscr{SD}(E)$.*

Proof. Criterion (3) of Theorem 3.7 immediately provides the proof. □

Here is the announced counterexample.

Theorem 3.9. *There exists a Banach space X for which $\mathscr{NAR}(X)$ does not form a semigroup under the operation $\tilde{+}$; in fact, $C([0,1],\ell_1)$ is such a space.*

Proof. The key feature of ℓ_1 is that $\mathscr{SD}(\ell_1)$ is not a $\tilde{+}$-semigroup; for we have shown in Proposition 2.4(b) that $x_1^*(x) = \sum_{n=1}^\infty x(n)$ and $x_2^*(x) = x(1) - \sum_{n=2}^\infty x(n)$ define strong Daugavet functionals on ℓ_1, but $x_1^* + x_2^*\colon x \mapsto 2x(1)$ is not in $\mathscr{SD}(\ell_1)$ and hence $x_1^* \,\tilde{+}\, x_2^*$ is not, either.

Now if $\mathscr{SD}(E)$ fails to be a $\tilde{+}$-semigroup, pick $T_1, T_2 \in \mathscr{SD}(E)$ with $T_1 \,\tilde{+}\, T_2 \notin \mathscr{SD}(E)$. Put $X = C(K,E)$ for a perfect compact Hausdorff space K; then by Proposition 3.8 and Theorem 3.7 $T_1^K, T_2^K \in \mathscr{NAR}(X))$, but $T_1^K \,\tilde{+}\, T_2^K \notin \mathscr{NAR}(X)$. □

Another example of a space for which $\mathscr{SD}(E)$ is no $\tilde{+}$-semigroup is $E = L_1[0,1]$. This is much subtler than for ℓ_1; the proof is presented in [6, Th. 6.3]. This example has the additional benefit of involving a space with the Daugavet property; by Theorem 3.9, however, $E = C([0,1],\ell_1)$ is another example of this kind.

4. Narrow and C-narrow operators on $C(K,E)$

The following definition extends the notion of a C-narrow operator studied in [4] and [6] to the vector-valued setting.

Definition 4.1. An operator $T \in L(C(K,E),W)$ is called *C-narrow* if there is a constant λ such that given any $\varepsilon > 0$, $x \in S(E)$ and open set $U \subset K$ there is a function $f \in C(K,E)$, $\|f\| \le \lambda$, satisfying the following conditions:

(a) $\operatorname{supp}(f) \subset U$,
(b) $f^{-1}(B(x,\varepsilon)) \ne \emptyset$, where $B(x,\varepsilon) = \{z \in E\colon \|z - x\| < \varepsilon\}$,
(c) $\|Tf\| < \varepsilon$.

As the following proposition shows, condition (b) of the previous definition can be substantially strengthened. In particular, the size of the constant λ is immaterial; but introducing this constant in the definition allows for more flexibility in applications. Also, Proposition 4.2 shows that for $E = \mathbb{R}$ the new notion of C-narrowness coincides with the one from [6].

Proposition 4.2. *If T is a C-narrow operator, then for every $\varepsilon > 0$, $x \in S(E)$ and open set $U \subset K$ there is a function f of the form $g \otimes x$, where $g \in C(K)$, $\operatorname{supp}(g) \subset U$, $\|g\| = 1$ and g is nonnegative, such that $\|Tf\| < \varepsilon$.*

Proof. Let us fix $\varepsilon > 0$, an open set U in K and $x \in S(E)$. By Definition 4.1 we find a function $f_1 \in C(K,E)$ as described there corresponding to ε, U and x. Put $U_1 = U$ and $U_2 = f_1^{-1}(B(x, \frac{1}{2}))$. As above, there is a function f_2 corresponding to ε, U_2 and x. We denote $U_3 = f_2^{-1}(B(x, \frac{1}{4}))$ and continue the process. In the r^{th} step we get the set $U_r = f_{r-1}^{-1}(B(x, \frac{1}{2^{r-1}}))$ and apply Definition 4.1 to obtain a function f_r corresponding to U_r.

Choose $n \in \mathbb{N}$ so that $(\lambda + 2)/n < \varepsilon$ and put $f = \frac{1}{n}(f_1 + f_2 + \cdots + f_n)$. Now using the Urysohn Lemma we find a continuous function g satisfying $\frac{k-1}{n} \le g(t) \le \frac{k}{n}$ for all $t \in U_k$, $k = 1, \ldots, n$, $\|g\| = 1$ and vanishing outside U_1. We claim that $\|f - g \otimes x\| < \varepsilon$. Indeed, by our construction, if $t \in K \setminus U_1$, then $\|(f - g \otimes x)(t)\| = 0$, and if $t \in U_k \setminus U_{k+1}$ (with the understanding that U_{n+1} stands for $\emptyset$), then

$$\begin{aligned}
\|(f - g \otimes x)(t)\| &= \left\| \frac{1}{n}(f_1 + \cdots + f_k)(t) - g(t) \cdot x \right\| \\
&\le \left\| \frac{1}{n}\big((f_1(t) - x) + \cdots + (f_{k-1}(t) - x) + f_k(t)\big) \right\| + \frac{1}{n} \\
&\le \frac{1}{n}\Big(\frac{1}{2} + \cdots + \frac{1}{2^{k-1}} + \lambda\Big) + \frac{1}{n} < \frac{\lambda + 2}{n} < \varepsilon.
\end{aligned}$$

Moreover,

$$\|Tf\| \le \frac{1}{n}\left(\|Tf_1\| + \|Tf_2\| + \cdots + \|Tf_n\|\right) < \varepsilon.$$

Thus $\|T(g \otimes x)\| < \varepsilon + \varepsilon\|T\|$, and since ε was chosen arbitrarily, we are done. ☐

Another way to express this proposition is to say that $T\colon C(K,E) \to W$ is C-narrow if and only if, for each $x \in E$, the restriction $T_x\colon C(K) \to W$, $T_x(g) = T(g \otimes x)$, is C-narrow.

Proposition 4.3.

(a) *Every C-narrow operator on $C(K,E)$ is a strong Daugavet operator. Hence, in the case of a perfect compact K every C-narrow operator on $C(K,E)$ is narrow.*

(b) *If E is a separable USD-nonfriendly space, then every strong Daugavet operator on $C(K,E)$ is C-narrow.*

(c) *If every strong Daugavet operator on $C(K,E)$ is C-narrow, then E is SD-nonfriendly.*

Proof. (a) Let T be C-narrow. We will use criterion (3) of Theorem 3.7. Let $F \subset K$ be a closed subset, $x \in S(E)$, $y \in B(E)$ and $\varepsilon > 0$. According to Proposition 4.2 there exists a function f vanishing on F of the form $g \otimes (x - y)$, where $g \in C(K)$, $\|g\| = 1$ and g is nonnegative, such that $\|Tf\| < \varepsilon$. Evidently this f satisfies all the demands of criterion (3) in Theorem 3.7.

(b) Let T be a strong Daugavet operator, and suppose E is separable. Let $U \subset K$ be a nonvoid open subset. Given $x, y \in S(E)$ and $\varepsilon' > 0$ we define

$$\begin{aligned} O(x, y, \varepsilon') = \{t \in U\colon\ \exists f \in C(K, E)\colon\ \operatorname{supp} f \subset U,\ \|f + y\| < 1 + \varepsilon', \\ \|f(t) + y + x\| > 2 - \varepsilon',\ \|Tf\| < \varepsilon'\}. \end{aligned}$$

This is an open subset of K, and by Theorem 3.7(3) it is dense in U. Now pick a countable dense subset $\{(x_n, y_n)\colon n \in \mathbb{N}\}$ of $S(E) \times S(E)$ and a null sequence (ε_n). Then by Baire's theorem, $G := \bigcap_n O(x_n, y_n, \varepsilon_n)$ is nonempty.

Let $\varepsilon > 0$, and fix $t_0 \in G$. We denote by $A(U, \varepsilon)$ the closure of

$$\{f(t_0)\colon f \in C(K, E),\ \|f\| < 2 + \varepsilon,\ \|Tf\| < \varepsilon,\ \operatorname{supp} f \subset U\};$$

this is an absolutely convex set. We claim that $A(U, \varepsilon)$ intersects each set $D(x, y, \varepsilon') \in \mathscr{D}(E)$. Indeed, if $\|x_n - x\| < \varepsilon'/4$, $\|y_n - y\| < \varepsilon'/4$, $\varepsilon_n < \varepsilon'/2$ and $\varepsilon_n < \varepsilon$, then for a function f_n as appearing in the definition of $O(x_n, y_n, \varepsilon_n)$ we have $f_n(t_0) \in A(U, \varepsilon) \cap D(x_n, y_n, \varepsilon_n) \subset A(U, \varepsilon) \cap D(x, y, \varepsilon')$.

Since E is USD-nonfriendly, say with parameter α, the set $A(U, \varepsilon)$ contains $\alpha B(E)$. This implies that T satisfies the definition of a C-narrow operator with constant $\lambda = 3/\alpha$.

(c) Let $T \in \mathscr{SD}(E)$; then by Proposition 3.8 T^K is a strong Daugavet operator on $C(K, E)$. But

$$\big(T^K(g \otimes e)\big)(t) = T\big((g \otimes e)(t)\big) = g(t)Te,$$

hence T^K is not C-narrow unless $T = 0$. $\square$

The example $E = c_0$ shows that the converse of (b) is false. We have already pointed out in Proposition 2.4(a) that c_0 fails to be USD-nonfriendly; yet every strong Daugavet operator on $C(K, c_0)$ is C-narrow. To see this we first remark that it is enough to check the condition spelt out in Proposition 4.2 for x in a dense subset of $S(E)$. In our context we may therefore assume that the sequence x vanishes eventually, say $x(n) = 0$ for $n > N$. If we write $c_0 = \ell_\infty^N \oplus_\infty Z$, with Z the space of null sequences supported on $\{N+1, N+2, \dots\}$, we also have $C(K, c_0) = C(K, \ell_\infty^N) \oplus_\infty C(K, Z)$. By Corollary 3.6 the restriction of any strong Daugavet operator T on $C(K, c_0)$ to $C(K, \ell_\infty^N)$ is again a strong Daugavet operator, and hence it is C-narrow, for ℓ_∞^N is USD-nonfriendly (Proposition 2.7). This implies that T is C-narrow.

We do not know whether (c) is actually an equivalence.

One of the fundamental properties of C-narrow operators is stated in our next theorem.

Theorem 4.4. *Suppose that operators T, $T_n \in L(C(K, E), W)$ are such that the series $\sum_{n=1}^\infty w^*(T_n f)$ converges absolutely to $w^*(Tf)$, for every $w^* \in W^*$*

and $f \in C(K,E)$. If all the T_n are C-narrow, then so is T. In particular, the sum of two C-narrow operators is a C-narrow operator.

Corollary 4.5. *A pointwise unconditionally convergent sum of narrow operators on $C(K,E)$ is a narrow operator itself if E is separable and USD-nonfriendly.*

Indeed, this follows from Theorem 4.4 and Proposition 4.3; note that K is perfect if there exists a narrow operator defined on $C(K,E)$ in case E fails the Daugavet property. To see the latter assume that $K = \{k\} \cup K'$ for some isolated point k. If there exists a narrow operator on $C(K,E) \cong E \oplus_\infty C(K',E)$, then this space has the Daugavet property and so has E [5, Lemma 2.15].

We remark that the case of a sum of two narrow operators on $C(K)$ was treated earlier in [4] and [6], but the assertion about infinite sums is new even there. It was proved in [5] for a pointwise unconditionally convergent sum $T = \sum_{n=1}^\infty T_n$ on a space with the Daugavet property that

$$\|\mathrm{Id} + T\| \geq 1$$

whenever $\|\mathrm{Id} + S\| = 1 + \|S\|$ for every S in the linear span of the T_n. In the context of Theorem 4.4 we even obtain

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{4.1}$$

when all the T_n are narrow on $C(K)$. In particular, the identity on $C(K)$ cannot be represented as an unconditional sum of narrow operators, since obviously (4.1) fails for $T = -\mathrm{Id}$. This last consequence shows for an unconditional Schauder decomposition $C(K) = X_1 \oplus X_2 \oplus \dots$ with corresponding projections $P_1, P_2, \dots$ that one of the P_n must be non-narrow, since $\mathrm{Id} = \sum_{n=1}^\infty P_n$ pointwise unconditionally. Hence one of the X_n must be infinite-dimensional if K is a perfect compact Hausdorff space. In fact, one of the X_n must contain a copy of $C[0,1]$ and therefore be isomorphic to $C[0,1]$ by a theorem due to Pełczyński [7] if K is in addition metrisable; see [4] and [5] for more results along these lines.

We now turn to the proof of Theorem 4.4 for which we need an auxiliary concept. A similar idea has appeared in [4].

Definition 4.6. Let G be a closed G_δ-set in K and $T \in L(C(K), W)$. We say that G is a *vanishing set* of T if there is a sequence of open sets $(U_i)_{i\in\mathbb{N}}$ in K and a sequence of functions $(f_i)_{i\in\mathbb{N}}$ in $S(C(K))$ such that

(a) $G = \bigcap_{i=1}^\infty U_i$;
(b) $\mathrm{supp}(f_i) \subset U_i$;
(c) $\lim_{i\to\infty} f_i = \chi_G$ pointwise;
(d) $\lim_{i\to\infty} \|Tf_i\| = 0$.

The collection of all vanishing sets of T is denoted by $\mathrm{van}\, T$.

Let $T \in L(C(K), W)$. By the Riesz Representation Theorem, T^*w^* can be viewed as a regular measure on the Borel subsets of K whenever $w^* \in W^*$. For convenience, we denote it by T^*w^* as well.

Lemma 4.7. *Suppose G is a closed G_δ-set in K and $T \in L(C(K), W)$. Then $G \in \operatorname{van} T$ if and only if $T^*w^*(G) = 0$ for all $w^* \in W^*$.*

Proof. Let $G \in \operatorname{van} T$, and pick functions $(f_i)_{i\in\mathbb{N}}$ as in Definition 4.6. Then by the Lebesgue Dominated Convergence Theorem, for any given $w^* \in W^*$ we have

$$T^*w^*(G) = \int_K \chi_G \, dT^*w^* = \lim_{i\to\infty} \int_K f_i \, dT^*w^* = \lim_{i\to\infty} w^*(Tf_i) = 0.$$

Conversely, let $(U_i)_{i\in\mathbb{N}}$ be a sequence of open sets in K such that $\overline{U}_{i+1} \subset U_i$ and $G = \bigcap_{i=1}^\infty U_i$. By the Urysohn Lemma there exist functions $(f_i)_{i\in\mathbb{N}}$ having the following properties: $0 \le f_i(t) \le 1$ for all $t \in K$, $\operatorname{supp}(f_i) \subset U_i$, and $f_i(t) = 1$ if $t \in \overline{U}_{i+1}$. Clearly, $\lim_{i\to\infty} f_i = \chi_G$ pointwise and

$$\lim_{i\to\infty} w^*(Tf_i) = \lim_{i\to\infty} T^*w^*(f_i) = T^*w^*(G) = 0$$

whenever $w^* \in W^*$. This means that the sequence $(Tf_i)_{i\in\mathbb{N}}$ is weakly null. Applying the Mazur Theorem we finally obtain a sequence of convex combinations of the functions $(f_i)_{i\in\mathbb{N}}$ which satisfies all the conditions of Definition 4.6.

This completes the proof. □

Lemma 4.8. *An operator $T \in L(C(K), W)$ is C-narrow if and only if every nonvoid open set $U \subset K$ contains a nonvoid vanishing set of T. Moreover, if $(T_n)_{n\in\mathbb{N}} \subset L(C(K), W)$ is a sequence of C-narrow operators, every open set $U \neq \emptyset$ contains a set $G \neq \emptyset$ that is simultaneously a vanishing set for all T_n.*

Proof. We first prove the more general "moreover" part. Put $U_{1,1} = U$. By the definition of a C-narrow operator and Proposition 4.2 there is a function $f_{1,1} \subset S(C(K))$ with $\operatorname{supp}(f_{1,1}) \subset U_{1,1}$, $U_{1,2} := f_{1,1}^{-1}(\frac{1}{2}, 1] \neq \emptyset$ and $\|T_1 f_{1,1}\| < \frac{1}{2}$. Obviously, $\overline{U}_{1,2} \subset f_{1,1}^{-1}[\frac{1}{2}, 1] \subset U_{1,1}$. Again applying the definition we find $f_{1,2} \in S(C(K))$ with $\operatorname{supp}(f_{1,2}) \subset U_{1,2}$, $U_{2,1} = f_{1,2}^{-1}(\frac{2}{3}, 1] \neq \emptyset$ and $\|T_1 f_{1,2}\| < \frac{1}{3}$. As above $\overline{U}_{2,1} \subset U_{1,2}$.

In view of the C-narrowness of T_2 there exists a function $f_{2,1} \in S(C(K))$ with $\operatorname{supp}(f_{2,1}) \subset U_{2,1}$, $U_{1,3} = f_{2,1}^{-1}(\frac{2}{3}, 1] \neq \emptyset$ and $\|T_2 f_{2,1}\| < \frac{1}{3}$. In the next step we construct $f_{1,3} \in S(C(K))$ such that $U_{2,2} = f_{1,3}^{-1}(\frac{3}{4}, 1] \neq \emptyset$ and $\|T_1 f_{1,3}\| < \frac{1}{4}$.

Proceeding in the same way, in the n^{th} step we find a set of functions $(f_{k,l})_{k+l=n} \subset S(C(K))$ and nonempty open sets $(U_{k,l})_{k+l=n}$ in K such that $\operatorname{supp}(f_{k,l}) \subset U_{k,l}$, $\|T_k f_{k,n-k}\| < \frac{1}{n}$ and $U_{k,l} = f_{k-1,l+1}^{-1}(\frac{n-1}{n}, 1]$, if $k \neq 1$. Then we put $U_{1,n} = f_{n-1,1}^{-1}(\frac{n-1}{n}, 1]$ to start the next step.

It remains to show that the set $G = \bigcap_{k,l\in\mathbb{N}} U_{k,l} = \bigcap_{k,l\in\mathbb{N}} \overline{U}_{k,l}$ is as desired. Indeed, G is clearly a nonempty closed G_δ-set and $G = \bigcap_{i=1}^{\infty} U_{n,i}$ for every $n \in \mathbb{N}$. It is easily seen that the sequences $(f_{n,i})_{i\in\mathbb{N}}$ and $(U_{n,i})_{i\in\mathbb{N}}$ meet the conditions of Definition 4.6 for the operator T_n. So, $G \in \operatorname{van} T_n$ for every $n \in \mathbb{N}$.

To prove the converse, let $U \neq \emptyset$ be any open set in K and let $\varepsilon > 0$. By assumption on $\operatorname{van} T$ we can find a closed G_δ-set $\emptyset \neq G \subset U$, $G \in \operatorname{van} T$. Consider the open sets $(U_i)_{i\in\mathbb{N}}$ and functions $(f_i)_{i\in\mathbb{N}}$ provided by Definition 4.6. For sufficiently large $i \in \mathbb{N}$ we have $U_i \subset U$ and $\|Tf_i\| < \varepsilon$ so that f_i may serve as a function as required in Definition 4.1.

This finishes the proof. □

Now we are in a position to prove Theorem 4.4.

Proof of Theorem 4.4. By virtue of Proposition 4.2, we may assume that $E = \mathbb{R}$. By Lemma 4.8 it suffices to show that $\bigcap_{n=1}^{\infty} \operatorname{van} T_n \subset \operatorname{van} T$.

Suppose $G \in \bigcap_{n=1}^{\infty} \operatorname{van} T_n$. According to Lemma 4.7 we need to prove that $T^*w^*(G) = 0$ for all $w^* \in W^*$. By the condition of the theorem, the series $\sum_{n=1}^{\infty} T_n^* w^*$ is weak*-unconditionally Cauchy and hence weakly unconditionally Cauchy. Since $C(K)^*$ does not contain a copy of c_0, it is actually unconditionally norm convergent by the Bessaga-Pełczyński Theorem. This implies that for the bounded sequence of functions $(f_i)_{i\in\mathbb{N}}$ satisfying $f_i \to \chi_G$ pointwise constructed in the proof of Lemma 4.7, we have

$$\begin{aligned} T^*w^*(G) &= \lim_{i\to\infty} T^*w^*(f_i) = \lim_{i\to\infty} \sum_{n=1}^{\infty} T_n^* w^*(f_i) \\ &= \sum_{n=1}^{\infty} T_n^* w^*(\chi_G) = \sum_{n=1}^{\infty} T_n^* w^*(G) = 0. \end{aligned}$$

The proof is complete. □

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