THE UNSTABLE SPECTRUM OF THE SURFACE QUASI-GEOSTROPHIC EQUATION

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Dedicated to Victor Yudovich on the occasion of his 70th birthday

Abstract. We study the unstable spectrum of an equation that arises in geophysical fluid dynamics known as the surface quasi-geostrophic equation. In general the spectrum is the union of discrete eigenvalues and an essential spectrum. We demonstrate the existence of unstable eigenvalues in a particular example. We examine the spectra of the semigroup and the evolution operator. We exhibit the structure of these spectra for general flows and prove that a spectral mapping theorem holds. We observe that the spectral properties of the SQG equation are closely analogous to those of the two dimensional Euler equation .

1. Introduction

The so-called "quasi-geostrophic" or more precisely "surface quasi-geostrophic" (SQG) equation has received attention in the past few years for several reasons. It is an equation derived from the equations of motion for a 3-dimensional rapidly rotating, density stratified fluid that is a mathematical model for the ocean or atmosphere. Both the forces of rotation and stratification impose a tendency towards two dimensionality on a three dimensional fluid configuration. The SQG equation is obtained from the potential vorticity equation under certain approximations and it is the equation for the evolution of the temperature in a two dimensional layer. As such it has been used in geophysics as a model for frontogenesis. A derivation of the SQG equation from the full equations of geophysical fluid dynamics and discussion of its physical relevance can be found in Pedlosky [14], Salmon [16], and Held et al [11].

In addition to its physical relevance, the SQG equation has intrigued mathematicians as a 2 dimensional equation that, although analytically simpler than the full 3 dimensional Euler equation, retains several important features of this extremely challenging system. The mathematical analogues between the SQG equation and the 3 dimensional Euler equation are described in detail by Constantin et al [2] and mathematical properties of the SQG equation are studied by Cordoba [3, 4], Cordoba and Fefferman [5]. However, to date, the intriguing question of the possibility of finite time blow-up remains open for the SQG equation, as it famously does for the 3-dimensional Euler equation.

2000 *Mathematics Subject Classification.* 76E20,76E05,47A75.

Key words and phrases. surface quasi-geostrophic equation, Euler equation, eigenvalues, essential spectrum.

Susan Friedlander thanks IHES and ENS-Cachan for their hospitality and support during the writing of this paper. her work is partially supported by NSF grant DMS-0202767.

In this present paper we study the spectrum of the SQG equation linearized about a steady state. We exhibit the structure of the spectrum. We observe that the spectral properties of the SQG equation are closely analogous to those, not of 3-dimensional Euler, but the far less complex system of the 2-dimensional Euler equation.

2. The Surface Quasi-Geostrophic Equation

The SQG equation is an equation in 2 spatial dimensions for a scalar function $\theta(\mathbf{x},t)$ which represents the temperature which plays the role of a so called "active" scalar, i.e. a scalar that is coupled to the velocity field $\mathbf{q}(\mathbf{x},t)$ of the motion under which $\theta(\mathbf{x},t)$ evolves. The coupling is explicit via a stream function $\psi(\mathbf{x},t)$. The equations are

$$\left(\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right)\theta = 0, \tag{1}$$

where

$$\mathbf{q} = \nabla^{\perp}\psi \equiv (-\psi_{x_2}, \psi_{x_1}), \tag{2}$$

and

$$\theta = -(-\Delta)^{1/2}\psi. \tag{3}$$

The non-local operator $(-\Delta)^{1/2}$ is determined through the 2-dimensional Fourier transform

$$\psi(\mathbf{x},t) = \int_{\mathbb{R}^2} e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\hat{\psi}(\mathbf{k},t)\; d\mathbf{k} \tag{4}$$

by

$$(-\Delta)^{1/2}\psi = \int_{\mathbb{R}^2} e^{2\pi i\mathbf{k}\cdot\mathbf{x}}|\mathbf{k}|\hat{\psi}(\mathbf{k},t)\; d\mathbf{k}. \tag{5}$$

We note that (2), (3) and (5) imply that

$$\mathbf{q}(\mathbf{x}) = -i\int_{\mathbb{R}^2} \frac{\mathbf{k}^{\perp}}{|\mathbf{k}|}\hat{\theta}(\mathbf{k})e^{2\pi i\mathbf{k}\cdot\mathbf{x}}\; d\mathbf{k}. \tag{6}$$

or

$$\mathbf{q}(\mathbf{x}) = -\int_{\mathbb{R}^2} \frac{1}{|\mathbf{y}|}\nabla^{\perp}\theta(\mathbf{x}+\mathbf{y})\; d\mathbf{y}. \tag{7}$$

The SQG equation as defined through (1)-(3) has similarities with both the 2-dimensional and the 3-dimensional Euler equation for the motion of an incompressible inviscid fluid. In terms of the scalar vorticity $\omega(\mathbf{x},t)$ the 2D Euler equation becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right)\omega = 0 \tag{8}$$

with $\mathbf{q}$ given by (2) and

$$\omega = \Delta\psi. \tag{9}$$

Hence, again we have an equation for active scalar but a crucial difference between 2D Euler and SQG lies in the order of the operator that relates the active scalar to ψ through (9) or (3). We write (8) in the form

$$\left(\frac{\partial}{\partial t} + \operatorname{curl}^{-1} \omega \cdot \nabla\right) \omega = 0 \tag{10}$$

and observe that the operator curl^{-1} produces some valuable smoothing effects in the nonlinear PDE. However there is no such smoothing obtained in the SQG equation when $\mathbf{q}$ is given in terms of θ by the singular integral operator (7). The SQG analogue for (10) is

$$\left[\frac{\partial}{\partial t} - (\nabla^{\perp}(-\Delta)^{-1/2}\theta) \cdot \nabla\right] \theta = 0. \tag{11}$$

The operator $\nabla^{\perp}(-\Delta)^{-1/2}$ is of zero order and produces no smoothing in the sense of the operator curl^{-1}. It is well known that the properties of the existence and uniqueness hold for 2D Euler (with $\mathbf{q} \in H^s$, $s > 2$). However, at least at present such results are not known for the SQG equation. Constantin et al. [2] showed that the smooth solution of SQG is unique but it is known to exist only for finite time. Whether the smooth solution develops a singularity in finite time is an interesting question that has been addressed by several authors [1, 4, 5, 21] but remains open.

The vorticity equation for the 3-dimensional Euler equation is

$$\left(\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right) \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{q}, \tag{12}$$

where $\boldsymbol{\omega}$ is the vector $\operatorname{curl} \mathbf{q}$. In terms of operators on $\boldsymbol{\omega}$ we have

$$\left(\frac{\partial}{\partial t} + \operatorname{curl}^{-1} \boldsymbol{\omega} \cdot \nabla\right) \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla(\operatorname{curl}^{-1} \boldsymbol{\omega}). \tag{13}$$

The presence of the term on the RHS of (13) shows that in 3D, as opposed to 2D, there is no gain in smoothness in the vorticity equation. In this sense the SQG equation (11) may be "closer" to the behavior of 3D Euler than 2D Euler. Furthermore, applying $\nabla^{\perp}$ to equation (1) gives

$$\left(\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right) (\nabla^{\perp}\theta) = (\nabla^{\perp}\theta) \cdot \nabla \mathbf{q}. \tag{14}$$

Thus the quantity $\nabla^{\perp}\theta$ satisfies a 2D equation with the same structure as the 3D Euler equation (12) for the vorticity. In 3 dimensions the velocity is related to the vorticity via the singular integral known as the Biot-Savart law:

$$\mathbf{q}(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \times \boldsymbol{\omega}(\mathbf{y}) \, d\mathbf{y}, \tag{15}$$

which is the 3D analogue of the SQG 2D singular integral given by (7).

3. The spectrum of the SQG equation

In this section we discuss the spectrum of the operator obtained by linearizing the SQG equation about a steady state.

Let $\Psi_0(\mathbf{x})$ be the stream function for a steady 2D flow with velocity $\mathbf{q}_0(\mathbf{x})$ and temperature $\theta_0(\mathbf{x})$. The time independent form of (1)-(3) gives

$$\mathbf{q}_0 \cdot \nabla \theta_0 = 0, \tag{16}$$

with

$$\mathbf{q}_0 = \nabla^\perp \Psi_0 \tag{17}$$

and

$$\theta_0 = -(-\Delta)^{1/2}\Psi_0. \tag{18}$$

The linearization of (1) about the steady state gives

$$\frac{\partial \theta}{\partial t} = -\mathbf{q}_0 \cdot \nabla\theta - \nabla^\perp\psi \cdot \nabla\theta_0 \equiv L\theta \tag{19}$$

with

$$\psi = -(-\Delta)^{-1/2}\theta. \tag{20}$$

Let $\{G_t\}_{t\geq 0}$ be the semigroup generated by L. We will examine the discrete unstable spectrum (i.e. eigenvalues of L) and properties of the essential spectra of L and G_t for $\theta \in L^2(\mathbb{T}^2)$ and for $\theta \in H^m(\mathbb{T}^2)$.

3.1. **The discrete spectrum.** We consider the following class of solutions to (16)-(18), namely functions $\Psi_0 \in C^\infty(\mathbb{T}^2)$ that satisfy a pseudodifferential equation of the form

$$(-\Delta)^{1/2}\Psi_0 = F(\Psi_0). \tag{21}$$

We substitute (21) into (19) to obtain

$$\left(\frac{\partial}{\partial t} - D\right)\theta - D(F'(\Psi_0)\psi) = 0, \tag{22}$$

where D is the skew-symmetric operator

$$D \equiv \Psi_{0x_2}\frac{\partial}{\partial x_1} - \Psi_{0x_1}\frac{\partial}{\partial x_2} \tag{23}$$

and ψ and θ are related via (20). We note that the formal structure of equation (22) is analogous to the PDE governing the linearized 2D Euler equation in terms of the stream function (c.f. Friedlander et al. [10]). We investigate properties of the discrete spectrum of the SQG operator in $L^2(\mathbb{T}^2)$ by considering the eigenvalue equation

$$\lambda(-\Delta)^{1/2}\psi - D[(-\Delta)^{1/2}\psi - F'(\Psi_0)\psi] = 0. \tag{24}$$

we construct energy integrals by pairing (24) with ψ and $(-\Delta)^{1/2}\psi$ to obtain the condition

$$\operatorname{Re}\lambda \int_{\mathbb{T}^2} \left\{ |(-\Delta)^{1/4}\psi|^2 - \frac{|(-\Delta)^{1/2}\psi|^2}{F'(\Psi_0)} \right\} d\mathbf{x} = 0 \tag{25}$$

for functions F where $F' \neq 0$. Hence we obtain the same stability criterion as in the case of the 2D Euler equation, namely $F'(\Psi_0) < 0$ (or $F'(\Psi_0) > 0$ and sufficiently large) is a sufficient condition for spectral stability of the SQG equations.

Consider the case where $F(\Psi_0)$ is a linear function of Ψ_0, thus (21) becomes

$$(-\Delta)^{1/2}\Psi_0 = \lambda\Psi_0. \tag{26}$$

Applying the operator $(-\Delta)^{1/2}$ to both sides of (26) gives

$$-\Delta\Psi_0 = \lambda^2\Psi_0. \tag{27}$$

Hence any eigenfunction of the Laplacian gives a stream function for the SQG equation. In order to demonstrate the existence of an unstable eigenvalue for (24) we turn to a particular example, namely shear flows with an oscillating profile. Let

$$\mathbf{q}_0 = (\sin(mx_2), 0). \tag{28}$$

We note that this example was first used by Meshalkin and Sinai [12] to demonstrate the existence of an unstable eigenvalue for the Navier-Stokes operator and later by Friedlander et al. [8] to obtain unstable eigenvalues for the 2D Euler operator. We seek to construct a smooth eigenfunction in $L^2(\mathbb{T}^2)$ for (24) of the form

$$\psi(x_1, x_2) = e^{ikx_1} \sum_{n=-\infty}^{\infty} a_n e^{inx_2}. \tag{29}$$

In this example (with m taken to be positive)

$$\Psi_0 = \frac{1}{m}\cos(mx_2), \quad F(\Psi_0) = \cos(mx_2)$$

and hence (24) becomes

$$\lambda(-\Delta)^{1/2}\psi + \sin(mx_2)\frac{\partial}{\partial x_1}[(-\Delta)^{1/2}\psi - m\psi] = 0. \tag{30}$$

We substitute (29) into (30) to obtain

$$\lambda \sum_{n=-\infty}^{\infty} (k^2+n^2)^{1/2} a_n e^{inx_2} + (e^{imx_2} - e^{-imx_2})\frac{k}{2} \sum_{n=-\infty}^{\infty} [(k^2+n^2)^{1/2} - m] a_n e^{inx_2} = 0. \tag{31}$$

Using orthogonality of the terms in the Fourier series, we obtain a recurrence relation for the coefficients in (31) which yields the following triadiagonal infinite algebraic system:

$$\frac{2\lambda}{k}(k^2+n^2)^{1/2}a_n + ((k^2+(n-m)^2)^{1/2} - m)a_{n-m} - ((k^2+(n+m)^2)^{1/2} - m)a_{n+m} = 0. \tag{32}$$

We assume that $k \neq 0$ since otherwise (32) does not have nontrivial solutions. We may also assume that $k > 0$. We rewrite (32) in the form

$$\beta_n(z)d_n + d_{n-m} = d_{n+m} \tag{33}$$

where

$$\beta_n(z) = 2z(k^2+n^2)^{1/2}/((k^2+n^2)^{1/2} - m) \tag{34}$$

and

$$d_n = a_n((k^2+n^2)^{1/2} - m) \tag{35}$$

and

$$z = \lambda/k. \tag{36}$$

We analyze the system (33) with $-\infty < n < \infty$ and construct a solution for $0 < k < m$. The method of construction follows closely the method used for the 2D Euler problem in Friedlander et al. [8]. We define

$$l_n = \frac{d_n}{d_{n-m}}, \quad n > 0 \tag{37}$$

and

$$\tilde{l}_n = \frac{d_{n-m}}{d_n}, \quad n \leq 0. \tag{38}$$

Hence, from (33) we have

$$l_m = -\frac{1}{[\beta_m, \beta_{2m}, ...]} \tag{39}$$

$$\tilde{l}_0 = \frac{1}{[\beta_{-m}, \beta_{-2m}, ...]}, \tag{40}$$

where $[\beta_m, \beta_{2m}, ...]$ denotes the infinite continued fractions

$$\beta_m + \frac{1}{\beta_{2m} + \frac{1}{\beta_{3m} + ...}}. \tag{41}$$

We observe that

$$\beta_{pm} = \frac{2z}{\left(1 - \frac{m}{(k^2+(pm)^2)^{1/2}}\right)} \to 2z \tag{42}$$

as $p \to \infty$. Hence, for z real and positive the continued fractions in (39) and (41) are convergent. Furthermore, as $p \to \infty$,

$$l_{pm} \to z - \sqrt{1+z^2}, \quad \tilde{l}_{-pm} \to -z + \sqrt{1+z^2}. \tag{43}$$

Hence, $-1 < l_\infty < 0$ and $0 < \tilde{l}_{-\infty} < 1$. Thus the sequences d_{pm} and $d_{-pm} \to 0$ exponentially and hence the function ψ given by the Fourier series (29) is C^∞-smooth for such values of z.

The characteristic equation for z is obtained by setting $n = 0$ in (33) and using (39) and (40) to give

$$\beta_0(z) + \frac{1}{[\beta_{-m}, \beta_{-2m}, ...]} = -\frac{1}{[\beta_m, \beta_{2m}, ...]}. \tag{44}$$

Since by definition (34) $\beta_{pm} = \beta_{-pm}$, (34) yields the equation

$$\frac{zk}{m-k} = \frac{1}{[\beta_m, \beta_{2m}, ...]}. \tag{45}$$

This equation has a real positive root z provided k and m are positive integers with k sufficiently smaller than m. Thus, we have constructed an unstable eigenvalue of the SQG equation using techniques that are analogous the those employed for 2D Euler.

3.2. **The essential spectrum.** In this section we apply asymptotic methods to investigate the essential spectrum of L under periodic boundary conditions. We start with a highly oscillating initial data

$$\theta(t=0) = b_0(\mathbf{x})\ e^{i\boldsymbol{\xi}_0\cdot\mathbf{x}/\delta},$$

where b_0 is a scalar amplitude and $\boldsymbol{\xi}_0$ is a frequency vector. Assuming δ is small, we expand the solution $\theta(t,\mathbf{x})$ using WKB-type asymptotic series

$$\theta(t,\mathbf{x}) = b(t,\mathbf{x})\ e^{iS(t,\mathbf{x})/\delta} + O(\delta).$$

To obtain the corresponding asymptotic formula for the velocity, let us first assume that $t=0$. Then, in the frequency space, θ is localized near ξ_0/δ. According to the integral formula (6), the velocity is thus given by

$$\mathbf{q}(0,\mathbf{x}) = -i\frac{\xi_0}{|\xi_0|}\ b_0(\mathbf{x})\ e^{i\boldsymbol{\xi}_0\cdot\mathbf{x}/\delta} + O(\delta). \tag{46}$$

At time t, the frequency direction changes to the gradient of the phase, so we obtain analogously [1]

$$\mathbf{q}(t,\mathbf{x}) = -i\frac{\nabla^\perp S}{|\nabla S|}\ b(t,\mathbf{x})\ e^{iS(t,\mathbf{x})/\delta} + O(\delta). \tag{47}$$

After substituting into (19), and cancelling the low order terms along with the exponents, we obtain

$$b_t + \frac{i}{\delta}\ b\ S_t = -\mathbf{q}_0\cdot\nabla b - \frac{i}{\delta}\ b\ \mathbf{q}_0\cdot\nabla S + i\ b\ \frac{\nabla^\perp S}{|\nabla S|}\cdot\nabla\theta_0. \tag{48}$$

Balancing the terms on both sides of the equation, we get

$$S_t + \mathbf{q}_0\cdot\nabla S = 0 \tag{49}$$

$$b_t + \mathbf{q}_0\cdot\nabla b = i\ b\ \frac{\nabla^\perp S}{|\nabla S|}\cdot\nabla\theta_0. \tag{50}$$

To find the evolution equation for the frequency ∇S we take the gradient of (49),

$$\nabla S_t + \mathbf{q}_0\cdot\nabla S = -\partial\mathbf{q}_0^\top\ \nabla S.$$

We denote ∇S by the vector $\boldsymbol{\xi}$. Written in the Lagrangian coordinates these equations yield the following system of ODE's

$$\mathbf{x}_t = \mathbf{q}_0(\mathbf{x}), \tag{51a}$$

$$\boldsymbol{\xi}_t = -\partial\mathbf{q}_0^\top(\mathbf{x})\ \boldsymbol{\xi}, \tag{51b}$$

$$b_t = i\ b\ \frac{(\boldsymbol{\xi}^\perp\cdot\nabla\theta_0)}{|\boldsymbol{\xi}|}, \tag{51c}$$

subject to initial conditions $\mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{T}^2$, $\boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 \in \mathbb{R}^2\backslash\{0\}$, and $b(0) = b_0 \in \mathbb{C}$. The time derivative in the ODE system (51) is the derivative along the trajectories of $\mathbf{q}_0$.

As in the case with the Euler equation, solutions of the amplitude equation (51c) determine the asymptotics of the SQG equation in the following way.

[1] Rigorous proof of this formula can be obtained using the Method of Stationary Phase provided certain conditions on $S(t,x)$ are satisfied [6]. Since the phase is a priori unknown, we regard (47) as a formal expansion.

Theorem 3.1. *Let $\{G_t\}_{t\geq 0}$ be the semigroup generated by the SQG equation* (1)*. In the L^2-norm of the temperature, the essential spectrum of G_t is concentrated on the unit circle.*

Proof. Let $B_t(\mathbf{x}_0, \boldsymbol{\xi}_0)b_0 = b(t, \mathbf{x}_0, \boldsymbol{\xi}_0, b_0)$ be the solution of (51c) with initial data $(\mathbf{x}_0, \boldsymbol{\xi}_0, b_0)$. Explicitly, it is given by

$$B_t(\mathbf{x}_0, \boldsymbol{\xi}_0)b_0 = b_0 \exp\left\{ i \int_0^t \frac{\boldsymbol{\xi}^{\perp}(s, \mathbf{x}_0, \boldsymbol{\xi}_0) \cdot \nabla\theta_0(\mathbf{x}(s, \mathbf{x}_0))}{|\boldsymbol{\xi}(s, \mathbf{x}_0, \boldsymbol{\xi}_0)|} ds \right\}. \tag{52}$$

Let us denote by $\varphi_t(\mathbf{x})$ the integral flow of $\mathbf{q}_0$. Repeating the argument given in [17] we prove that, modulo compact error, $G_t\theta$ is equal to

$$P_t\theta(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^2\setminus\{0\}} B_t(\varphi_{-t}(\mathbf{x}), \mathbf{k})\hat{\theta}(\mathbf{k}) e^{i\varphi_{-t}(\mathbf{x})\cdot\mathbf{k}} \tag{53}$$

in any Sobolev space over the torus. Therefore, according to Nussbaum's formula [13], their respective essential spectral radii are equal. On the other hand, as in [17], the following identity is valid in $L^2(\mathbb{T}^2)$,

$$r_{\mathrm{ess}}(P_t) = e^{\mu t},$$

where μ is the maximal Lyapunov exponent of the b-equation (51c), i.e.

$$\mu = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{\mathbf{x}_0, \boldsymbol{\xi}_0, b_0 \\ |\boldsymbol{\xi}_0|=1, |b_0|=1}} |b(t, \mathbf{x}_0, \boldsymbol{\xi}_0, b_0)|. \tag{54}$$

It is clear from the explicit formula (52) that all solutions to (51c) are bounded in time. Hence, $\mu = 0$. This finishes the proof. □

In particular, since

$$e^{t\sigma_{\mathrm{ess}}(L)} \subset \sigma_{\mathrm{ess}}(G_t), \quad t \geq 0, \tag{55}$$

we conclude that the essential spectrum of the generator in the L^2-norm resides on the imaginary axis. Thus we see that the SQG equation is stable with respect to localized disturbances of the temperature (or velocity) in L^2. In the case of 2D Euler localized disturbances may be unstable for the velocity in L^2 (see, for example [9, 19]). However, such disturbances are stable with respect to growth of the vorticity in L^2. Thus the equivalence of the essential spectra in L^2 occurs between the velocity in the SQG equation and the vorticity in 2D Euler. Theorem 3.2, given below, shows the this equivalence carries through to the essential spectra in Sobolev space norms.

We point out that Theorem 3.1 is in contradiction to the result in Friedlander [7] which claimed that in L^2 the essential spectrum of G_t filled the complex plane. There is an error in [7] due to an erroneous factor of i in equation (2.32) in [7] which leads to spurious exponential growth in the amplitude term in the WKB asymptotics.

In our next result we detail the structure of the essential spectrum in the more interesting case of Sobolev spaces.

Theorem 3.2. *In the Sobolev space $H^m(\mathbb{T}^2)$, $m \in \mathbb{Z}$, the essential spectra of the linearized SQG equation and its semigroup are*

$$\sigma_{\text{ess}}(G_t) = \{e^{-t|m|\Lambda} \le |z| \le e^{t|m|\Lambda}\} \tag{56}$$

$$\sigma_{\text{ess}}(L) = \{-|m|\Lambda \le \operatorname{Re} z \le |m|\Lambda\}. \tag{57}$$

Proof. If m is a positive integer, we can differentiate P_t m times. The result is the operator of the same form as P_t, only with B_t replaced by the tensor product of B_t with m copies of the fundamental matrix of the $\boldsymbol{\xi}$-equation (51b). For the Euler equation this construction is explained in [18]. Repeating the same argument as in the proof of Theorem 3.1 we arrive at the following formula

$$r_{\text{ess}}(G_t) = e^{\mu_m t},$$

where μ_m is the corresponding Lyapunov exponent given by

$$\mu = \lim_{t\to\infty} \frac{1}{t} \log \sup |b(t, \mathbf{x}_0, \boldsymbol{\xi}_0, b_0)| \cdot |\xi(t, \mathbf{x}_0, \boldsymbol{\xi}_0)|^m, \tag{58}$$

in which the supremum is taken over all possible initial conditions. Since $|b(t)|$ is preserved in time, we see that μ is equal to m times the Lyapunov exponent of the basic fluid flow $\mathbf{q}_0$,

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\mathbf{x}\in\mathbb{T}^2} |\partial\varphi_t(\mathbf{x})|. \tag{59}$$

Thus, $\sigma_{\text{ess}}(G_t)$ is confined to the circle of radius $e^{tm\Lambda}$, and by the symmetry, it is confined to the annulus

$$\{e^{-tm\Lambda} \le |z| \le e^{tm\Lambda}\}. \tag{60}$$

By passing to adjoints, we draw the same conclusion for the Sobolev spaces of negative smoothness, in which case m has to be replaced by its absolute value.

In the rest of the proof we show that any point on the strip

$$\{-|m|\Lambda \le \operatorname{Re} z \le |m|\Lambda\}$$

belongs to the spectrum of the generator L. In view of inclusion (55), this completes the description of the essential spectrum.

It suffices to show that any point from the strip lies in the spectrum of L. Our attention will be focused on the case $m = -1$. All the other cases follow by duality or by a straightforward generalization of our argument (see [19] for details in a similar situation).

So, let us rewrite (19) as follows

$$L\theta = -\mathbf{q}_0 \cdot \nabla\theta + T\theta \cdot \nabla\theta_0,$$

where $T\theta = -\nabla^\perp(-\Delta)^{-1/2}\theta$. On the Fourier transform side,

$$T\theta(\mathbf{x}) = i \sum_{\mathbf{k}\in\mathbb{Z}^2} \begin{pmatrix} -k_2 \\ k_1 \end{pmatrix} \frac{1}{|\mathbf{k}|} \hat{\theta}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = i \begin{pmatrix} -R_2 \\ R_1 \end{pmatrix} \theta,$$

where R_j is the corresponding Riesz transform. The operator L is a bounded perturbation of the principal advection term

$$A\theta = -\mathbf{q}_0 \cdot \nabla\theta.$$

Indeed,

$$L\theta = A\theta + i\partial_{x_2}\theta_0 \cdot R_1\theta - i\partial_{x_1}\theta_0 \cdot R_2\theta.$$

For technical reasons, we work with the adjoint operator, which acts on H^1. Since the vertical strip is invariant with respect to complex conjugation, it suffices to prove (57) for L^*. In fact, we will show that every point on the strip belongs to the approximate point spectrum of L^*.

In view of the skew-symmetry of A with respect to the L^2-pairing, we have

$$L^*\theta = -A\theta - iR_1(\partial_{x_2}\theta_0 \cdot \theta) + iR_2(\partial_{x_1}\theta_0 \cdot \theta). \tag{61}$$

Let us examine the perturbation on the right hand side. Denote

$$\mathbf{v}_0 = (v^1, v^2) = (-\partial_{x_2}\theta_0, \partial_{x_1}\theta_0).$$

Let M be the multiplier with symbol $m(\mathbf{k}) = 1/|\mathbf{k}|$, $\mathbf{k} \neq (0,0)$ and $m(0,0) = 0$. Then we get

$$\begin{aligned} -iR_1(\partial_{x_2}\theta_0 \cdot \theta) + iR_2(\partial_{x_1}\theta_0 \cdot \theta) &= M(\partial_{x_1}(v^1\theta) + \partial_{x_2}(v^2\theta)) \\ &= M(\theta \operatorname{div} \mathbf{v}_0) + M(\mathbf{v}_0 \cdot \nabla\theta) = M(\mathbf{v}_0 \cdot \nabla\theta). \end{aligned}$$

Notice that M is a compact operator. So, if a sequence $\{\theta_n\}$ is such that $\{\mathbf{v}_0 \cdot \nabla\theta_n\}$ is bounded in H^1 and the measure of the supports of θ_n vanishes as $n \to \infty$, then $\{\mathbf{v}_0 \cdot \nabla\theta_n\}$ tends to zero weakly. This implies that the sequence $\{M(\mathbf{v}_0 \cdot \nabla\theta_n)\}$ tends to zero in the norm.

In order to prove that any $z \in \mathbb{C}$ with $-\Lambda \leq \operatorname{Re} z \leq \Lambda$ belongs to the spectrum of L^* we show that there is a normalized sequence $\{\theta_{\varepsilon,N}\}_{\varepsilon,N}$ such that

$$\limsup_{\varepsilon \to 0} \|L^*\theta_{\varepsilon,N} - z\theta_{\varepsilon,N}\| = \alpha(N), \tag{62}$$

for some vanishing function $\alpha(N)$ as $N \to \infty$.

We use the same sequence as constructed in [20]. In that construction, $\theta_{\varepsilon,N}$ is a function localized around the stable or unstable orbit of the flow induced by $\mathbf{q}_0$ near a hyperbolic point that produces Λ. As a consequence, for every fixed $N > 0$ we have

$$|\mathbf{q}_0(\mathbf{x})| > c_N > 0 \tag{63}$$

for all $\mathbf{x}$ in the support of $\theta_{\varepsilon,N}$. In addition, these functions possess the following two properties. First, their supports shrink towards the flow orbit as $\varepsilon \to 0$. Second, there exists an $\alpha(N)$ vanishing at infinity such that

$$\|A\theta_{\varepsilon,N} - z\theta_{\varepsilon,N}\| \to \alpha(N) \quad \text{as } \varepsilon \to 0. \tag{64}$$

From (64), by the triangle inequality, we obtain

$$\|A\theta_{\varepsilon,N}\| \lesssim 1 + \alpha(N). \tag{65}$$

With the help of (63) and (65) we can now obtain a bound on the H^1-norm of $\mathbf{v}_0 \cdot \nabla\theta_{\varepsilon,N}$. Indeed, notice that the vector fields $\mathbf{v}_0$ and $\mathbf{q}_0$ are parallel due to the definition of $\mathbf{v}_0$ and the stationary equation (16). Then,

$$\|\mathbf{v}_0 \cdot \nabla\theta_{\varepsilon,N}\|_{H^1} = \left\| \frac{|\mathbf{v}_0|}{|\mathbf{q}_0|} (\mathbf{q}_0 \cdot \nabla\theta_{\varepsilon,N}) \right\|_{H^1} \lesssim \frac{\|\partial\mathbf{v}_0\|_\infty \|\partial\mathbf{q}_0\|_\infty}{c_N^2} (1 + \alpha(N)).$$

So, for every fixed N the sequence $\{\mathbf{v}_0 \cdot \nabla\theta_{\varepsilon,N}\}$ is bounded in H^1 and has vanishing supports. So, the perturbation $M(\mathbf{v}_0 \cdot \nabla\theta_{\varepsilon,N})$ tends to zero with the ε. In view of (64) this proves (62). □

We note that the structure of essential spectrum described in Theorem 3.2 is identical to that of the 2D Euler equation in vorticity form (see [20]). This can be explained by the apparent similarity in dynamics of their bicharacteristic-amplitude systems. In both cases they remain bounded, so that in the higher Sobolev norms the WKB-asymptotics is determined solely by the basic fluid flow φ_t. This brings us to the conclusion that analogy between the linearized SQG and 2D Euler equations becomes more important – as opposed to 3D Euler – as far as the evolution of shortwave perturbations is concerned.

Theorem 3.2 proves the following spectral mapping property for the SQG semigroup:

$$\sigma(G_t) = e^{t\sigma(L)}. \tag{66}$$

We infer from (66) that the linear instabilities of the SQG equation due to the generator or the semigroup are equivalent. For general evolution semigroups this property may fail (see [15]).

4. Concluding remarks

The thrust of this paper is to examine the unstable spectra of the SQG equation and to point out the close similarities between the spectra of the SQG equation and the two dimensional Euler equation. We observe that the same stability criterion holds for both equations and that the existence of unstable eigenvalues can be demonstrated for oscillating shear flows in both settings. The equivalent structural properties hold for the unstable essential spectra in both spectral problems. Theorems 3.1 and 3.2, which detail the structure of the unstable SQG spectra and imply the spectral mapping property, also hold for the vorticity form of the 2D Euler equation with the scalar vorticity ω substituted for the temperature θ. A summary of the spectral properties of the 2D Euler equation and the equality of the spectra of the velocity and vorticity operators on the respective spaces of the Sobolev tower can be found in Shvydkoy and Friedlander [18]. We remark that to date we have far less information about the spectra of the 3D Euler equation which is much less amenable to the tools that have been used successfully to characterize the spectra of the 2D Euler and the SQG equations.

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