Geometric aspects of the Daugavet property.

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Abstract

Let X be a closed subspace of a Banach space Y and J be the inclusion map. We say that the pair (X, Y) has the Daugavet property if for every rank one bounded linear operator T from X to Y the following equality

$$\|J + T\| = 1 + \|T\| \tag{1}$$

holds. A new characterization of the Daugavet property in terms of weak open sets is given. It is shown that the operators not fixing copies of ℓ_1 on a Daugavet pair satisfy (1).

Some hereditary properties are found: if X is a Daugavet space and Y is its subspace, then Y is also a Daugavet space provided X/Y has the Radon-Nikodým property; if Y is reflexive, then X/Y is a Daugavet space. Becides, we prove that if (X, Y) has the Daugavet property and $Y \subset Z$, then Z can be renormed so that (X, Z) possesses the Daugavet property and the equivalent norm coincides with the original one on Y.

1 Introduction.

Let X be a closed subspace of a Banach space Y and $J : X \to Y$ be the inclusion map. We say that the pair (X, Y) has the Daugavet property (or is a Daugavet pair) if for every rank one bounded linear operator T from X to Y the following identity

$$\|J+T\| = 1 + \|T\|, \tag{2}$$

which is called the Daugavet equation, holds. If (2) is satisfied by operators from some class $\mathcal{M}$ we say that (X, Y) has the Daugavet property with respect to this class.

Investigation of (2) was originated with the work of Daugavet ([5]), in which he establishes the equality for compact operators on $C[0,1]$. This result was then generalized by Foias and Singer to weakly compact operators acting on arbitrary atomless $C(K)$ (see [9]). Shortly after that in [19] Lozanovsky proved Daugavet's theorem for $L_1[0,1]$. There was a series of works following the aforementioned ones (see [2], [3], [4], [7], [11], [12] and references therein). The authors established (2) for several classes of operators (including weakly compact ones) on atomless $C(K)$ and $L_1(\mu)$-spaces, and utilized it to various problems of classical Banach space theory.

For some time it had not been known whether $C(K)$ and $L_1(\mu)$-spaces are the only ones possessing the Daugavet property. In his work [2], Abramovich gave the negative answer to this question. He showed that, for example, $L_\infty(\mu)\oplus_1 L_\infty(\nu)$ and $L_1(\mu)\oplus_\infty L_1(\nu)$ are Daugavet spaces provided μ and ν are atomless measures. Afterwards, this result was generalized by Wojtaszczyk ([24]) as follows: the ℓ_1 and ℓ_∞ sums of any number of Daugavet spaces is a Daugavet space. Many interesting classes of Daugavet spaces were also discovered by Kadets, Popov, Werner and others (see, for example, [13], [14], [23]). It is worth mentioning that first the Daugavet property as such was introduced in [3]. The authors provided a unified approach for studying the Daugavet equation and obtained several new results in terms of Banach lattice theory (see also [1] and [21]).

Until recently there have been known little about properties possessed by *every* Daugavet space. It turned out, however, that the Daugavet equation, being of isometric nature, could strongly influence isomorphic structure of a Banach space. For instance,

(i) The unit sphere of a Daugavet space does not have a strongly exposed point. Thus, a Daugavet space cannot have the Radon-Nikodým property (see [23] and [24]).

(ii) A Daugavet space does not have an unconditional basis (see [13]).

A more systematic study of this type of questions was initiated in [16]. The authors gave a geometric characterization of the Daugavet property in terms of slices of the unit ball. It helped to find a lot of isomorphic properties of the Daugavet spaces.

The present paper is a natural continuation of [16]. We give affirmative answers for many questions posed there and provide alternative proofs of some known earlier results.

In Section 2 another characterization of the Daugavet property in terms of weak open sets intersecting the unit ball is given. Using this tool we prove that all operators not fixing a copy of ℓ_1 on a Daugavet pair satisfy the Daugavet equation (Theorem 2.3). Note that the analogous result for strong Radon-Nikodým operators was already obtained in [16]. We also present some new hereditary properties (Theorem 2.5). In particular, a pair (X, Y) has the Daugavet property, provided Y is a Daugavet space and Y/X has the Radon-Nikodým property.

Section 3 is entirely devoted to pairs of the form $(X, C(K))$, where K is a compact Hausdorff space. It is shown that in some natural cases, e.g., when K is the unit ball of X^*, such a pair possesses the Daugavet property whenever X does. We will see that this is also the case for some bigger $C(K)$-spaces containing X. In Section 4 one of them is shown to be, in a sense, universal: a Banach space Y can be isomorphically embedded into it, whenever $X \subset Y$ and Y/X is separable (Corollary 4.3).

At the end of Section 4 we prove the following renorming theorem: let (X, Y) have the Daugavet property and Z be a Banach space containing Y, then Z can be renormed so that (X, Z) possesses the Daugavet property and the equivalent norm remains unchanged on Y. A consequence of this result and the aforementioned Theorem 2.3 is that a Daugavet space does not embed into an unconditional sum of Banach spaces without a copy of ℓ_1 (Corollary 4.4). It is a cosiderable generalization of the well known Theorem of Pelczyński about impossibility of embedding $C[0, 1]$ and $L_1[0, 1]$ into a space with unconditional basis, and its extention (ii).

Throughout the text $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X into Y; $B(X)$ $(S(X))$ stands for the unit ball (unit sphere) of a Banach space X; by $\overline{ext} B(X^*)$ we denote the weak* closure of the set of all extreme points of the dual unit ball $B(X^*)$. For a subset A of a Banach space, $\overline{A}$ denotes the norm-closure of A.

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2 Some characterizations and direct consequences.

The central role in this section plays the notion of a slice.

Definition 2.1 *Let X be a Banach space. A* slice *of $B(X)$ is called the following set*

$$S(x^*, \varepsilon) = \{x \in B(X) : x^*(x) > 1 - \varepsilon\},$$

where $x^* \in X^*$ and $\varepsilon > 0$. We always assume that $x^* \in S(X^*)$. If X is a dual space and x^* is taken from the predual, then $S(x^*, \varepsilon)$ is called a weak slice.*

In paper [16] the following characterization of the Daugavet property in terms of slices was obtained.

Lemma 2.1 *The following are equivalent:*

(a) The pair (X, Y) has the Daugavet property;

(b) For every $y_0 \in S(Y)$ and for every slice $S(x_0^*, \varepsilon_0)$ of $B(X)$ there is another slice $S(x_1^*, \varepsilon_1) \subset S(x_0^*, \varepsilon_0)$ of $B(X)$ such that for every $x \in S(x_1^*, \varepsilon_1)$ the inequality $\|x + y_0\| \geq 2 - \varepsilon_0$ holds;

(c) For every $x_0^* \in S(X^*)$ and for every weak slice $S(y_0, \varepsilon_0)$ of $B(Y^*)$ there is another weak* slice $S(y_1, \varepsilon_1) \subset S(y_0, \varepsilon_0)$ of $B(Y^*)$ such that for every $y^* \in S(y_1, \varepsilon_1)$ the inequality $\|x_0^* + y^*_{|X}\| \geq 2 - \varepsilon_0$ holds.*

For the sake of completeness we present the proof here.

Proof.

(a)⇒(b). Define T: $X \to Y$ by $Tx = x_0^*(x)y_0$. Then $\|J^* + T^*\| = \|J + T\| = 2$, so there is a functional $y^* \in S(Y^*)$ such that $\|J^*y^* + T^*y^*\| \geq 2 - \varepsilon_0$ and $y^*(y_0) \geq 0$. Put

$$x_1^* = \frac{J^*y^* + T^*y^*}{\|J^*y^* + T^*y^*\|}, \quad \varepsilon_1 = 1 - \frac{2 - \varepsilon_0}{\|J^*y^* + T^*y^*\|}.$$

Then for all $x \in S(x_1^*, \varepsilon_1)$ we have

$$\langle (J^* + T^*)y^*, x\rangle \geq (1 - \varepsilon_1)\|J^*y^* + T^*y^*\| = 2 - \varepsilon_0,$$

therefore

$$y^*(x) + y^*(y_0)x_0^*(x) \geq 2 - \varepsilon_0, \tag{3}$$

which implies that $x_0^*(x) \geq 1 - \varepsilon_0$, i.e., $x \in S(x_0^*, \varepsilon_0)$. Moreover, by (3) we have $y^*(x) + y^*(y_0) \geq 2 - \varepsilon_0$ and hence $\|x + y_0\| \geq 2 - \varepsilon_0$.

(b)⇒(a). Let $T \in \mathcal{L}(X, Y)$, $Tx = x_0^*(x)y_0$ be a rank one operator. We can assume that $\|T\| = 1$ (see, for example, [3]) and $\|x_0^*\| = \|y_0\| = 1$. Fix any $\varepsilon > 0$. Then there is an $x \in S(x_0^*, \frac{\varepsilon}{2})$ such that $\|x + y_0\| > 2 - \frac{\varepsilon}{2}$. So,

$$\|J + T\| \geq \|x + x_0^*(x)y_0\| \geq \|x + y_0\| - |1 - x_0^*(x)| > 2 - \varepsilon.$$

Let ε go to zero.

The proof of equivalence (a)⇔(c) is analogous. □

One can see that the slices $S(x_1^*, \varepsilon_1)$ and $S(y_1, \varepsilon_1)$ in the statement of the previous lemma can be replaced by vectors x and y^*. We will often refer to Lemma 2.1 in this form.

Lemma 2.1 leads to several remarkable consequences known before (the proofs can be found in [16]). First, if X has the Daugavet property then X (and X^*) contains an isomorphic copy of ℓ_1, and moreover, vectors equivalent to the canonical basis of ℓ_1 can be chosen in arbitrary slices of $B(X)$ (and weak* slices of $B(X^*)$). Hence, neither X nor X^* possess the Radon-Nikodým property provided X has the Daugavet property (see also [23] and [24]). Second, all strong Radon-Nikodým operators and, in particular, all weakly compact operators on a Daugavet pair satisfy the Daugavet equation. Below we isolate another such a class of operators, namely those not fixing copies of ℓ_1, but first we need the following modification of Lemma 2.1, which shows that we can operate with weak open sets as well as with slices.

Lemma 2.2 *The following are equivalent:*

(a) The pair (X, Y) has the Daugavet property;

(b) For any given $\varepsilon > 0$, $y \in S(Y)$ and weak open set U in X with $U \cap B(X) \neq \emptyset$ there is a weak open set V in X with $V \cap B(X) \neq \emptyset$ and $V \cap B(X) \subset U \cap B(X)$ such that $\|v + y\| > 2 - \varepsilon$, whenever $v \in V \cap B(X)$;

(c) For any given $\varepsilon > 0$, $x^* \in S(X^*)$ and weak* open set U in Y^* with $U \cap B(Y^*) \neq \emptyset$ there is a weak* open set V in Y^* with $V \cap B(Y^*) \neq \emptyset$ and $V \cap B(Y^*) \subset U \cap B(Y^*)$ such that $\|v_{|X} + x^*\| > 2 - \varepsilon$, whenever $v \in V \cap B(Y^*)$.

Proof. Let us prove (a)⇒(b).

First we consider the weak* open set U^{**} in X^{**} that induces U on X, i.e. $U^{**} \cap X = U$. By the Krein-Milman Theorem, there is a convex combination of extreme points of $B(X^{**})$, $\sum_{i=1}^{n} \lambda_i x_i^{**}$, such that $\sum_{i=1}^{n} \lambda_i x_i^{**} \in U^{**}$. Clearly, we can find weak* open neighborhoods $\{U_i^{**}\}_{i=1}^{n}$ of the points $\{x_i^{**}\}_{i=1}^{n}$ respectively, for which the following inclusion holds:

$$\sum_{i=1}^{n} \lambda_i (U_i^{**} \cap B(X^{**})) \subset U^{**}. \tag{4}$$

Now by the Choquet Lemma (weak* slices containing an extreme point form a basis of its weak* neighborhoods, [10, p.49]), we can assume that the sets $\{U_i^{**} \cap B(X^{**})\}_{i=1}^{n}$ are weak* slices. Thus, inclusion (4) restricted on X looks as follows: $\sum_{i=1}^{n} \lambda_i S_i \subset U$, where $S_i = U_i^{**} \cap B(X^{**}) \cap X$ are slices for all $i = 1, 2, \ldots, n$.

Employing Lemma 2.1(b) we find a vector $x_1 \in S_1$ with $\|\lambda_1 x_1 + y\| > (\lambda_1 + 1 - \varepsilon)$. Analogously, there is an $x_2 \in S_2$ with $\|\lambda_2 x_2 + \lambda_1 x_1 + y\| > (\lambda_2 + \lambda_1 + 1 - \varepsilon)$. Continuing in the same way we finally find $x_n \in S_n$ with $\|\lambda_n x_n + \lambda_{n-1} x_{n-1} + \ldots + \lambda_1 x_1 + y\| > (\lambda_n + \lambda_{n-1} + \ldots + \lambda_1 + 1 - \varepsilon) = 2 - \varepsilon$, and $\sum_{i=1}^{n} \lambda_i x_i \in U$. It remains only to use the lower weak semicontinuity of a norm to get the required weak open set V.

This completes the proof of implication (a)⇒(b).

The implication (a)⇐(b) follows from Lemma 2.1 and the equivalence (a)⇔(c) is proved in the same way. □

Theorem 2.3 *If the pair (X, Y) has the Daugavet property, then every operator from $\mathcal{L}(X, Y)$ not fixing a copy of ℓ_1 satisfies the Daugavet equation.*

Proof. Let $T \in \mathcal{L}(X, Y)$, $\|T\| = 1$, be such an operator and $\varepsilon > 0$ be arbitrary.

Our considerations will rely on the following "releasing principle": suppose for some finite set of vectors $\{x_i\}_{i=1}^{n} \subset B(X)$ and some $\varepsilon > 0$ the inequalities

$$\left\| \sum_{i=1}^{n} \theta_i x_i \right\| > n - \varepsilon, \tag{5}$$

and

$$\left\| \sum_{i \in I_1} a_i x_i + \sum_{i \in I_2} a_i T x_i \right\| > \left(\sum_{i \in I_1 \cup I_2} a_i \right) (1 - \varepsilon) \tag{6}$$

hold for all non-negative reals a_i, signs θ_i, and some disjoint sets $I_1,\ I_2 \subset \{1, 2, \dots, n\}$. Then there is a weak open set $U \subset X$ such that (5) and (6) remain true for all $x_n \in U \cap B(X)$.

Let us prove it. By the compactness argument, there is a $\delta > 0$ such that

$$\left\| \sum_{i \in I_1} a_i x_i + \sum_{i \in I_2} a_i T x_i \right\| > 1 - \varepsilon + \delta, \tag{7}$$

whenever $\sum_{i \in I_1 \cup I_2} a_i = 1$ and $I_1,\ I_2$ as above. Fix a finite $\frac{\delta}{2}$-net $\{(a_{k,1}, a_{k,2}, \dots, a_{k,n})\}_{k=1}^{K}$ in the set $\{(a_1, a_2, \dots, a_n) : \ \sum_{i=1}^{n} a_i = 1,\ a_i \geq 0\}$ equipped with the ℓ_1-metric. Using the lower weak semicontinuity of a norm and weak continuity of a bounded linear operator we conclude that there is a weak open set U such that both (5) and (7) hold for $a_i = a_{k,i}$, $i = 1, 2, \dots, n$, $k = 1, 2, \dots, K$ and all $z_n \in U \cap B(X)$. It is not hard to see that U is desired.

Now we construct a sequence $\{x_i\}_{i=1}^{\infty} \subset B(X)$ which satisfies (5) and (6) for all non-negative reals a_i, signs θ_i and all disjoint finite sets $I_1,\ I_2 \subset \mathbb{N}$.

Assume that we have constructed such a sequence $\{x_i\}_{i=1}^{n}$ of length n. We want to prove now that altering only the last term x_n one can find another vector x_{n+1} such that the resulting sequence of length $n+1$ satisfies (5) and (6). Arguing in such a way, we produce the desired infinite sequence if only take $x_1 \in S(X)$ with $\|Tx_1\| > 1 - \varepsilon$ on the first step.

Let us put $x'_{n+1} = x_n$ for a moment. Clearly, (6) remains true for the sequence $x_1, x_2, \dots, x_n, x'_{n+1}$ and all I_1, I_2 with additional restriction: if one of them contains n, then the other does not contain $n+1$. We get rid of this restriction by alteration of x_n and x'_{n+1}. To this end, we use the "releasing principle" for x'_{n+1} and find the corresponding weak open set $U \subset X$. Application of Lemma 2.2(b) several times yields a vector $x_{n+1} \in U \cap B(X)$ such that (5) is valid for the sequence $x_1, x_2, \dots, x_n, x_{n+1}$ and (6) holds without the restriction: if I_1 contains $n+1$, then I_2 does not contain n. Then we use the "releasing principle" to release x_n so that both (5) and (6) remain true. Appealing to Lemma 2.2(b) we finally get an x'_n such that (6) holds for the sequence $x_1, x_2, \dots, x'_n, x_{n+1}$ without any restrictions on I_1 and I_2. Inequality (5) is satisfied automatically.

The constructed sequence is $(1-\varepsilon)$-equivalent to the canonical basis of ℓ_1, for if $\sum_{i=1}^n |\lambda_i| = 1$, then by (5) we have

$$\begin{aligned}\left\|\sum_{i=1}^n \lambda_i x_i\right\| &= \left\|\sum_{i=1}^n sign\lambda_i \cdot x_i + \sum_{i=1}^n (\lambda_i - sign\lambda_i)\cdot x_i\right\| \\ &> n-\varepsilon-\sum_{i=1}^n |\lambda_i - sign\lambda_i| = n-\varepsilon-\sum_{i=1}^n |1-|\lambda_i|| \\ &= n-\varepsilon-n+1 = 1-\varepsilon.\end{aligned}$$

Since T fixes no copies of ℓ_1, by Rosenthal's Lemma we may assume that the sequence $(Tx_n)_{n=1}^\infty$ is weakly Cauchy. Thus, $(Tx_{2n+1}-Tx_{2n})_{n=1}^\infty$ is weakly null. By Mazur's Theorem there are two finite disjoint sets I_1, $I_2 \subset N$ such that for some $p \in conv\{x_i : i \in I_1\}$ and $q \in conv\{x_i : i \in I_2\}$ we have $\|Tp - Tq\| < \varepsilon$. From this and (6) we finally obtain

$$\|p + Tp\| > \|p + Tq\| - \varepsilon > 2(1-\varepsilon) - \varepsilon = 2 - 3\varepsilon,$$

which implies $\|J + T\| = 2$ in view of arbitrariness of ε.

This finishes the proof. □

As was remarked in the introduction, $C(K)$ has the Daugavet property, provided K is a compact Hausdorff space without isolated points. Besides, due to a result of Rosenthal [20] and by Lemma 2.4 from [22] it follows that operators on $C(K)$ not fixing copies of $C[0,1]$ are precisely those not fixing copies of ℓ_1. So, from the previous theorem we obtain that all such operators satisfy the Daugavet equation. This result was first established by Weis and Werner in their paper [22]. By Theorem 2.3 we also solve a problem posed in [16].

Corollary 2.4 *Suppose X is a Daugavet space and Y is a complemented subspace in X such that X/Y contains no copies of ℓ_1, then the norm of every projection from X onto Y is at least 2.*

Proof. Let $P : X \to X$ be any projection onto Y. Then $-Id + P$ fixes no copies of ℓ_1 and hence, by Theorem 2.3, satisfies the Daugavet equation. So, we have $\|P\| = \|Id + (-Id + P)\| = 1 + \|P - Id\| \geq 2$. □

Problem 1. It remains open whether every Dunford-Pettis operator on a Daugavet pair satisfies the Daugavet equation.

Problem 2. One of the remarkable characterizations of Banach spaces not containing isomorphic copies of ℓ_1 is that the duals of such spaces possess the weak Radon-Nikodým property. Thus, no dual to a Daugavet space has this property. It is not known, however, if the same is true for a Daugavet space itself.

Now we discuss the following question: suppose X has the Daugavet property; what subspaces of X possess the same property?

It was shown in [16] that all subspaces with separable annihilator do. Such an effect could be attributed to extreme "spreadness" of a Daugavet unit ball (see Lemmas 2.1 and 2.2). We will repeatedly use this idea later on.

Theorem 2.5 *Let X have the Daugavet property and Y be a subspace of X.*

(a) If X/Y has the Radon-Nikodým property, then the pair (Y, X) has the Daugavet property;

(b) If Y is reflexive, then X/Y has the Daugavet property.

In the particular case when $X = L_1[0,1]$ part (b) of Theorem 2.5 was proved in [16].

Proof. Part (a). According to Lemma 2.1(b) it is sufficient to prove that given any $\delta > 0$, $S(y^*, \varepsilon)$ and $x \in B_X$ there is a $y \in S(y^*, \varepsilon)$ such that $\|x + y\| > 2 - \delta$.

Denote by j the quotient map : $X \mapsto X/Y$. Saving the notation for the functional y^*, we extend it to all of X by the Hahn-Banach Theorem. The set $A = j(S(y^*, \varepsilon))$ is convex and contains the origin. Since X/Y has the Radon-Nikodým property, the Phelps Theorem (see for example [6]) yields a convex combination $\sum_{i=1}^{n} \lambda_i a_i$ of strongly exposed points $\{a_i\}_{i=1}^{n}$ of the set $\overline{A}$ for which

$$\left\| \sum_{i=1}^{n} \lambda_i a_i \right\| < \frac{\delta}{2}. \tag{8}$$

Let $\{a_i^*\}_{i=1}^{n} \subset (X/Y)^*$ be functionals exposing $\{a_i\}_{i=1}^{n}$ respectively and let positive numbers $\{\varepsilon_i\}_{i=1}^{n}$ be such that

$$diam\left\{S(a_i^*, \varepsilon_i) \cap \overline{A}\right\} < \frac{\delta}{4},\ i = 1, 2, ..., n. \tag{9}$$

Since $S(a_i^*, \varepsilon_i) \cap A \neq \emptyset$, we have $S(j^* a_i^*, \varepsilon_i) \cap S(y^*, \varepsilon) \neq \emptyset$. Applying Lemma 2.2(b) we find $x_i \in S(j^* a_i^*, \varepsilon_i) \cap S(y^*, \varepsilon)$ such that

$$\left\| \sum_{i=1}^{n} \lambda_i x_i + x \right\| > 2 - \frac{\delta}{4}$$

Now taking into account (8) and (9) we obtain the following estimate:

$$\left\| j(\sum_{i=1}^{n} \lambda_i x_i) \right\| < \left\| \sum_{i=1}^{n} \lambda_i a_i \right\| + \frac{\delta}{4} < \frac{\delta}{2}.$$

It means that there is a $y \in B_Y$ for which

$$\left\| \sum_{i=1}^{n} \lambda_i x_i - y \right\| < \delta.$$

Then by (9) we finally get

$$\|x + y\| > 2 - \frac{3}{2}\delta.$$

Clearly, $y \in S(y^*, \varepsilon + \delta)$.

Because of arbitrariness of ε and δ, part (a) is proved.

The proof of part (b) is analogous (we have only to use the weak* topology and apply Lemma 2.2(c)). □

Problem 3. Under the conditions of Theorem 2.5,

(a) does Y have the Daugavet property if X/Y is an Asplund space (equivalently, $(X/Y)^*$ has the Radon-Nikodým property) or, more generally, if X/Y fails to contain isomorphic copies of ℓ_1?

(b) does X/Y have the Daugavet property if either Y or Y^* (or both) has the Radon-Nikodým property or fails to contain isomorphic copies of ℓ_1?

3 Subspaces of C(K)-spaces.

Now we study the case when in a pair (X, Y) the space Y is a $C(K)$-space for some compact Hausdorff space K. As was shown in various works (see, for example [3]) and as also follows from our Lemma 2.1, $C(K)$ has the Daugavet property if and only if K has no isolated points. Moreover, we can assert that if for some $X \subset C(K)$ the pair $(X, C(K))$ has the Daugavet property, then K does not have such a point k_0, for otherwise the rank one operator $Tx = -\chi_{\{k_0\}} \cdot x(k_0)$, where $\chi_{\{k_0\}}$ denotes the characteristic function of the singleton $\{k_0\}$, does not satisfy the Daugavet equation. So, investigating pairs of the form $(X, C(K))$ it is natural to require that K have no isolated points.

We begin with a characterization of those Banach spaces X, $X \subset C(K)$ that the pair $(X, C(K))$ has the Daugavet property. In the sequel, δ_k^*, $k \in K$ stands for the functional on $C(K)$ acting by the rule $\delta_k^*(f) = f(k)$, $f \in C(K)$.

Lemma 3.1 *Let X be a subspace of $C(K)$, where K is a compact Hausdorff space without isolated points. The following conditions are equivalent:*

(a) The pair $(X, C(K))$ has the Daugavet property;

(b) For every $\varepsilon > 0$, $x^* \in S(X^*)$ and open set U in K there exists a point $u \in U$ such that $\|x^* + \delta^*_{u|X}\| > 2 - \varepsilon$;

(c) For every $x^* \in S(X^*)$ and open set U in K there exists a (closed) G_δ-set G in U such that $\|x^* + \delta^*_{u|X}\| = 2$, whenever $u \in G$.

Proof. (a)⇒(b). Let $f \in S(C(K))$ be a function vanishing outside U. By Lemma 2.1(c), there is a slice $S \subset S(f, \frac{1}{2})$ such that $\|x^* + \mu\| > 2 - \varepsilon$, for all $\mu \in S$. Pick any $\delta_u^* \in S$. Clearly, $\delta_u^*(f) = f(u) > \frac{1}{2}$ and hence, $u \in U$. So, u is the required point.

(b)⇒(c). Apply part (b) countably many times and use the weak* lower semicontinuity of a dual norm and the regularity of a Hausdorff compact set.

(c)⇒(a). We apply Lemma 2.1 again. Pick arbitrary $x^* \in S(X^*)$ and weak* slice $S(f, \varepsilon)$ in $B(C^*(K))$. Let $U = \{k \in K : f(k) > 1 - \varepsilon\}$. By condition (c), we can find a point $u \in U$ such that $\|x^* + \delta^*_{u|X}\| = 2$. Moreover, we have $\delta_u^*(f) = f(u) > 1 - \varepsilon$ and hence, $\delta_u^* \in S(f, \varepsilon)$. This completes the proof. □

Of course, not every pair $(X, C(K))$ has the Daugavet property provided X does, e.g., this one $(C[0,1], C([0,1] \cup [2,3]))$. However, as the following theorem shows, in some natural and useful cases this is true.

Proposition 3.2 *If the pair (X, Y) has the Daugavet property and K is either $B(Y^*)$ or $\overline{ext}B(Y^*)$, then the pair $(X, C(K))$ also has the Daugavet property.*

Proof. In both cases we use condition (b) of Lemma 3.1.

First, consider $K = B(Y^*)$. Fix arbitrary $\varepsilon > 0$, open set $U \subset K$ and $x^* \in S(X^*)$. By Lemma 2.2(c) there is $y^* \in U$ such that $\|x^* + y^*_{|X}\| > 2 - \varepsilon$. We denote by u the functional y^* regarding it as a point of topological space K. It remains to notice that $\delta^*_{u|X} = y^*_{|X}$.

Let $K = \overline{ext}B(X^*)$. Fix ε, U and x^* as above. By the Choquet Lemma we may assume that U is induced by a slice S. By Lemma 2.1(c) there is a slice $S_1 \subset S$, and hence, there is a $y^* \in S \cap K$ such that $\|x^* + y^*_{|X}\| > 2 - \varepsilon$. So, as above the point $u = y^*$ is required. □

In the case $K = B(Y^*)$ this proposition solves a problem posed in [16]. The result was proved there for $K = \overline{ext}B(Y^*)$. However, we include both cases to emphasize their common origin.

Let K be a compact Hausdorff space without isolated points. We introduce the following spaces:

$$\begin{aligned} l_\infty(K) &= \{f : K \mapsto R, \quad \|f\|_\infty = \sup(|f(s)|,\, s \in K) < \infty\}, \\ m(K) &= \{f \in l_\infty(K) : supp(f) \text{ is a first category set}\}, \\ m_0(K) &= l_\infty(K)/m(K). \end{aligned}$$

In what follows we investigate Daugavet properties of the space $m_0(K)$. In the next section we use them to prove some general results on renormings.

$m_0(K)$ equipped with the factor-norm is a real C^*-algebra, and hence, is a $C(Q)$-space. The appropriate compact set $Q = Q_K$ can be defined as the set of all real homomorphisms on $m_0(K)$ endowed with the induced weak* topology. This is precisely limits by ultrafilters on K, which do not contain first category sets. Let $\mathfrak{U}$ be such an ultrafilter. We denote by $\lim \mathfrak{U}$ the point in K to which it converges and by $\mathfrak{U}\_\lim$ the real homomorphism on $m_0(K)$ it generates ($\mathfrak{U}\_\lim \in Q_K$).

Lemma 3.3 *Suppose U is an open set in Q_K, then there is an open set V in K such that for every $v \in V$ one can find an ultrafilter $\mathfrak{U}_v$ on K with $\lim \mathfrak{U}_v = v$ and $\mathfrak{U}_v\_\lim \in U$.*

Proof. By the construction of Q_K we may assume there are a finite set $(f_i)_{i=1}^n \subset m_0(K)$, $\varepsilon > 0$ and ultrafilter $\mathfrak{U}_0$ on K such that $U = \{\varphi \in Q_K : |\varphi(f_i) - \mathfrak{U}_0\text{-}\lim(f_i)| < \varepsilon\}$. Denote $a_i = \mathfrak{U}_0\text{-}\lim(f_i)$. We fix a second category set $A \in \mathfrak{U}_0$ with the following property:

$$f_i(A) \subset (a_i - \varepsilon\ ,\ a_i + \varepsilon), \quad i = 1, 2, \ldots, n. \tag{10}$$

Then we find an open set V in K such that for any open $W \subset V$, $W \cap A$ is a second category set (see [18]). It remains to show that V is required.

Indeed, let $v \in V$. Consider an ultrafilter $\mathfrak{U}_v$ containing $\{W \cap A : W$ is an open neighborhood of $v\}$. Plainly, $\lim \mathfrak{U}_v = v$. On the other hand, in view of (10) we have $\mathfrak{U}_v\text{-}\lim(f_i) \in (a_i - \varepsilon\ ,\ a_i + \varepsilon)$, $i = 1, 2, \ldots, n$. This means that $\mathfrak{U}_v\text{-}\lim \in U$. This finishes the proof. □

It is easy to see that $C(K)$ is isometrically embedded into $m_0(K)$ by the quotient map.

Proposition 3.4 *If the pair $(X, C(K))$ has the Daugavet property, then the pair $(X, m_0(K))$ also has the Daugavet property.*

Proof. We apply Lemma 3.1 again using the interpretation of $m_0(K)$ as a $C(Q)$-space. To this end, we fix $\varepsilon > 0$, open set $U \subset Q_K$ and $x^* \in S(X^*)$. Applying Lemma 3.3 to U we find the corresponding open set $V \subset K$. Lemma 3.1 applied to the pair $(X, C(K))$ yields $v \in V$ such that $\|x^* + \delta^*_{v|X}\| > 2 - \varepsilon$. Consider the ultrafilter $\mathfrak{U}_v$ with $\lim \mathfrak{U}_v = v$ and $\mathfrak{U}_v\text{-}\lim \in U$, and denote $u = \mathfrak{U}_v\text{-}\lim$. So, $\delta^*_{v|X} = \delta^*_{u|X}$ and $u \in U$. Hence, the point u is desired. □

Corollary 3.5 *The pair $(C(K), m_0(K))$ has the Daugavet property.* □

Corollary 3.6 *Let the pair (X, Y) have the Daugavet property and K be either $B(Y^*)$ or $\overline{ext} B(X^*)$, then the pair $(X, m_0(K))$ has the Daugavet property too.*

Proof. Combine Propositions 3.2 and 3.4. □

4 Renorming theorem.

The main goal of this section is to prove the following result.

Theorem 4.1 *Let X, Y, Z be Banach spaces such that $X \subset Y \subset Z$. If the pair (X, Y) has the Daugavet property, then Z can be renormed so that (X, Z) possesses the Daugavet property and the equivalent norm coincides with the original one on Y.*

In separable case this theorem was proved in [16]. The general case, however, requires more detailed consideration. Therefore we present the complete proof here.

First we prove a theorem which establishes, in some sense, a property of universality of $m_0(K)$-spaces, where K is the unit ball of a dual space. Since in the sequel we often deal with density character of a Banach space X (the minimal cardinality of a dense set in X), we denote it by $dens(X)$.

Theorem 4.2 *Let Y be a closed subspace of Banach spaces Z and W. Let also $dens(Z/Y) = \beta$, where β is an ordinal. Suppose $B(W^*)$ contains a family $\{B_\alpha\}_{\alpha<\beta}$ of disjoint second category sets such that if $B' = \cup_{\alpha<\beta} B_\alpha$, then $B' \cap -B' = \emptyset$. Then there is an isomorphic embedding $E : Z \to m_0(B(W^*))$, which coincides with the natural one on Y.*

Proof. Let us fix a dense set $([z_\alpha])_{\alpha<\beta} \subset B(Z/W)$ with $\|z_\alpha\| \le 1$, and for every $\alpha < \beta$ find a functional $\varphi_\alpha \in S(Y^\perp)$ so that $\varphi_\alpha(z_\alpha) = \|[z_\alpha]\|$. Also to every w^* we assign a functional $\widetilde{w}^*$ obtained by restriction of w^* on Y and then extension to all of Z by the Hahn-Banach Theorem.

Now we want to embed Z into $\ell_\infty(B(W^*))$ so that every element from the image of $B(Z)$ takes values greater than $\frac{1}{8}$ on a second category set. To this end, for each $z \in Z$ we define a function $f_z \in \ell_\infty(B(W^*))$ as follows:

$$f_z(w^*) = \begin{cases} \widetilde{w}^*(z), & w^* \in B(W^*) \backslash B_0 \\ \widetilde{w}^*(z) + 8\varphi_\alpha(z), & w^* \in B_\alpha \end{cases} .$$

Clearly the mapping $F : z \to f_z$ is linear and bounded. Moreover, $f_z(w^*) = w^*(z)$, if $z \in Y$. So, $F_{|X}$ is the natural embedding of Y into $\ell_\infty(B(W^*))$ (even into $C(B(W^*))$).

Suppose now $\|z\| = 1$. Then either $\|[z]\| \le \frac{1}{4}$ or $\|[z]\| > \frac{1}{4}$. In the former case there is a $y_0 \in Y$ such that $\|z - y_0\| < \frac{3}{8}$. Because of the condition imposed on B', the set $\{w^* \in B(W^*) \backslash B' : w^*(y_0) > \|y_0\| - \frac{1}{8}\}$ is of second category, and for every its element we have

$$|f_z(w^*)| = |\widetilde{w}^*(z)| > |\widetilde{w}^*(y_0)| - \frac{3}{8} = |w^*(y_0)| - \frac{3}{8} = \|y_0\| - \frac{1}{2} > \frac{1}{8}.$$

So, $|f_z(w^*)| > \frac{1}{8}$, for w^* from some second category set.

In the case $\|[z]\| > \frac{1}{4}$, there is an ordinal α, $\alpha < \beta$, and $y \in Y$ such that $\|[z_\alpha]\| > \frac{1}{4}$ and $\|z - z_\alpha - y\| < \frac{1}{16}$. From this we get for all $w^* \in B_\alpha$

$$\begin{aligned} |f_z(w^*)| &= |\widetilde{w}^*(z) + 8\varphi_\alpha^*(z)| \\ &> |8\varphi_\alpha^*(z_\alpha - y)| - \frac{1}{2} - |\widetilde{w}^*(z)| = 8\|[z_\alpha]\| - \frac{3}{2} \\ &> \frac{8}{4} - \frac{3}{2} = \frac{1}{2}. \end{aligned}$$

To define the desired isomorphic embedding $E : Z \to m_0(B(W^*))$ we just put $Ez = [Fz]$, $z \in Z$. □

It is not hard to construct countable number of second category sets satisfying the condition of the previous theorem. So, in the special case when Z/Y is separable, we obtain the following corollary.

Corollary 4.3 *Let Y be a closed subspace of Z such that Z/Y is separable. Then there exists an isomorphic embedding of Z into $m_0(B(Y^*))$, which coincides with the natural one on Y.*

Proof of Theorem 4.1.

Suppose (X, Y) is a Daugavet pair and Z is some Banach space containing Y. If $B(Y^*)$ were very "reach" of disjoint second category sets, i.e. enough to satisfy the condition of Theorem 4.2 (in this case $Y = W$), there would exist an isomorphic embedding E of Z into $m_0(B(Y^*))$. Appealing to Corollary 3.6, the equivalent norm $|||z||| = \|Ez\|$ would be desired.

That, however, may not be the case, for example, when $dens(Z) > dens(m_0(B(Y^*)))$. So, we should replace Y by a bigger space, say W, which meets the condition of Theorem 4.2 and at the same time possesses the Daugavet property in pair with X. If we can do this, the norm introduced in the previous case satisfies our requirements, and we are done.

Let β be as in Theorem 4.2. We define W to be the ℓ_∞-sum of β copies of $C(B(Y^*))$, i.e. $W = \{(f_\alpha)_{\alpha<\beta} : f_\alpha \in C(B(Y^*)) \text{ and } \|(f_\alpha)\| = \sup_{\alpha<\beta}\|f_\alpha\| < \infty\}$. Y embeds into W as follows:

$$\begin{aligned} y &\to (y_\alpha)_{\alpha<\beta}, \quad y \in Y; \\ y_\alpha(s) &= s(y), \quad s \in B(Y^*). \end{aligned}$$

So, Y can be regarded as a subspace of W. Using Proposition 3.2, it is not difficult to prove that the pair (X, W) has the Daugavet property.

Now fix $f \in C(B(Y^*))$, $\|f\| = 1$, and for every α, $\alpha < \beta$, define the vector $w_\alpha = (f_{\alpha'})_{\alpha' < \beta}$ so that $f_{\alpha'} = f$, if $\alpha' = \alpha$, and $f_{\alpha'} = 0$ otherwise. Put $B_\alpha = S(w_\alpha, \frac{1}{3})$. Since every B_α is weak* open, it is a second category set. Next, $B_{\alpha'} \cap B_{\alpha''} = \emptyset$, $\alpha' \neq \alpha''$, for otherwise every $w^* \in B_{\alpha'} \cap B_{\alpha''}$ would have norm bigger than 1. For the same reason, $B' = \cup_{\alpha < \beta} B_\alpha$ is disjoint with $-B'$.

So, we have constructed the space satisfying all our requirements. This finishes the proof. □

Corollary 4.4 *A Daugavet space does not isomorphically embed into an unconditional sum of Banach spaces without a copy of ℓ_1.*

The proof is the same as that of Corollary 2.7 in [16]. We only have to use our Theorem 2.3 and the fact that the sum of finite number operators not fixing a copy of ℓ_1 is an operator not fixing a copy of ℓ_1.

It is worthwhile to remark that the previous result is a direct generalization of the known Theorem of Pelczyński for $C[0, 1]$ and $L_1[0, 1]$ spaces (for more about that see [8], [17]) and its recent extensions (see [13], [15], [16])

Problem 4. It would be interesting to find answer to the following question: if (X, Y) is a Daugavet pair, can Y be renormed to have the Daugavet property. We can assert that such a renorming cannot be accomplish leaving the norm on X unchanged. In fact, look at the space $L_\infty[0, 1]$. It is 1-complemented in every containing Banach space. Since every 1-codimensional subspace of a Daugavet space is at least 2-complemented, $L_\infty[0, 1] \oplus \mathbb{R}$ cannot be renormed to have the Daugavet property so that the equivalent norm remains the same on $L_\infty[0, 1]$.

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