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Nonlinear instability for the Navier-Stokes equations

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Abstract: It is proved, using a bootstrap argument, that linear instability implies nonlinear instability for the incompressible Navier-Stokes equations in L^p for all $p \in (1, \infty)$ and any finite or infinite domain in any dimension n.

Key words. Navier-Stokes equations – (non)linear instability – analytic semigroup

1. Introduction

The stability/instability of a flow of viscous incompressible fluid governed by the Navier-Stokes equations is a classical subject with a very extensive literature over more than 100 years. Much of the classical literature has concerned the stability of relatively simple specific flows (e.g. Couette flows and Poiseuille flows), the spectrum of the Navier-Stokes equations linearized about such flows and the role of the critical Reynolds number delineating the linearly stable and unstable regimes. An elegant result for general bounded flows was proved by Serrin [17] who used energy methods to show that all flows are nonlinearly stable in L^2 norm when the Reynolds number is less than a specific constant ($\pi\sqrt{3}$). Hence all steady flows that are sufficiently slow or sufficiently viscous are stable. However for many physical situations the Reynolds number is much larger than $\pi\sqrt{3}$, often by many orders of magnitude and observations indicate that such flows are unstable.

Linear instability has been confirmed in some specific examples by demonstrating existence of an nonempty unstable spectrum for the linearized Navier-Stokes operator. For example, Meshalkin and Sinai [14] used Fourier series and continued fractions to show the existence of unstable eigenvalues in the case of so called Kolmogorov flows (i.e. plane parallel shear flow with a sinusoidal profile). In a book published in Russian in 1984 (and in English in 1989) Yudovich [20] obtained an important result relating linear stability/instability for the Navier-Stokes equations with nonlinear stability/instability (see also Henry [8]). These results were proved in the function space $L^q(\Omega)$ with $q \geq n$ in n-spatial dimensions. A fairly general abstract theorem of Friedlander et al [5] can be applied to the Navier-Stokes equations in a finite domain to prove nonlinear instability in H^s, $s > \frac{n}{2}+1$ when the linearized operator has an unstable eigenvalue in L^2.

In this present paper we extend the result that linear instability implies nonlinear instability for the Navier-Stokes equations to all L^p spaces with $1 < p < \infty$ and both finite domains and $\mathbb{R}^n$. We note that our result includes nonlinear instability in the L^2 energy norm which we claim is the natural norm in which to consider issues of stability and instability.

The technique we employ to prove our main result is a bootstrap argument. Such arguments have been previously employed by several authors to prove under certain restrictions that linear instability implies nonlinear instability for the 2 dimensional Euler equation (Bardos et al [1], Friedlander and Vishik [19], Lin [13]). Because in general the spectrum of the Euler operator has a continuous component, unlike the Navier-Stokes operator in a finite domain whose spectrum is purely discrete, these nonlinear instability results for the Euler equation are much more limited than those presented here for the Navier-Stokes equations.

2. Notation and Formulation

We consider solutions to the Navier-Stokes equations

$$\frac{\partial q}{\partial t} = -(q \cdot \nabla)q - \nabla p + R^{-1}\Delta q + f, \tag{2.1a}$$

$$\nabla \cdot q = 0, \tag{2.1b}$$

where $q(x,t)$ denotes the n-dimensional velocity vector, $p(x,t)$ denotes the pressure and $f(x)$ is an external force vector. The dimensionless parameter R is the Reynolds number defined as $R = \frac{VL}{\nu}$ where V and L are characteristic velocity and length scales of the system and ν is the viscosity of the fluid. In Section 3 we consider the system on the n-dimensional torus $\mathbb{T}^n$ and in a bounded domain $\Omega \subset \mathbb{R}^n$. In Section 4 we consider the system in $\mathbb{R}^n$. The results are valid in all dimensions n although the most relevant physical cases are $n = 2$ and 3. We impose the standard boundary conditions on solutions of (2.1) for each type of domain: the no-slip condition $q|_{\partial\Omega} = 0$, in the case of Ω; vanishing velocity $q(x) \to 0$, as $x \to \infty$, in the case of $\mathbb{R}^n$; and periodic boundary condition in case of the torus. The results in Sections 3 and 4 prove that spectral instability for the linearized Navier-Stokes equations implies nonlinear instability in L^p for $1 < p < \infty$. In Section 5 we prove a result relating spectral stability with nonlinear stability in L^p for $p > n$.

Here and thereafter, for any $p \in [1,\infty)$, L^p denotes the usual Lebesgue space, with norm denoted $\|\cdot\|_p$, intersected with the space of divergence free functions. We let $W^{s,p}$ stand for the Sobolev space in the same context with norm denoted $\|\cdot\|_{s,p}$.

We consider an arbitrary steady solution of (2.1)

$$0 = -(U_0 \cdot \nabla)U_0 - \nabla P_0 + R^{-1}\Delta U_0 + f, \tag{2.2a}$$

$$\nabla \cdot U_0 = 0. \tag{2.2b}$$

We assume $U_0(x) \in C^\infty$ and $f(x) \in C^\infty$. To discuss stability of U_0 we rewrite the Navier-Stokes equations (2.1) in perturbation form with $q(x,t) = U_0(x) + v(x,t)$

$$\frac{\partial v}{\partial t} = -(U_0 \cdot \nabla)v - (v \cdot \nabla)U_0 + R^{-1}\Delta v - \nabla \cdot (v \otimes v) - \nabla p \tag{2.3a}$$

$$\nabla \cdot v = 0 \tag{2.3b}$$

$$v|_{t=0} = v_0 \tag{2.3c}$$

Applying the Leray projector $\mathbb{P}$ onto the space of divergence free functions, we write (2.3a) in the operator form:

$$\frac{\partial v}{\partial t} = Av + N(v,v) \tag{2.4}$$

where

$$Av = \mathbb{P}[-(U_0 \cdot \nabla)v - (v \cdot \nabla)U_0 + R^{-1}\Delta v] \tag{2.5}$$

$$N(v,v) = \mathbb{P}[-\nabla \cdot (v \otimes v)] \tag{2.6}$$

We note that the linear operator A is a bounded perturbation, to lower order, of the Stokes operator $R^{-1}\mathbb{P}\Delta$. The operator A generates a strongly continuous semigroup in every Sobolev space $W^{s,p}$ which we denote by e^{At}:

$$v(t) = e^{At}v_0, \quad v_0 \in W^{s,p} \tag{2.7}$$

(the case of a bounded domain is treated in [20], and in the case of $\mathbb{R}^n$ the statement can be proved with the use of the Fourier transform and the Hörmander-Mikhlin multiplier theorem).

We now define a suitable version of Lyapunov (nonlinear) stability for the Navier-Stokes equations.

Definition 2.1. *Let (X, Z) be a pair of Banach spaces. An equilibrium U_0 which is the solution of (2.2) is called (X, Z) nonlinearly stable if, no matter how small $\rho > 0$, there exists $\delta > 0$ so that $v_0 \in X$ and*

$$\|v_0\|_Z < \delta \tag{2.8}$$

imply the following two assertions

(i) there exists a global in time solution to (2.3) such that $v(t) \in C([0,\infty); X)$;
(ii) $\|v(t)\|_Z < \rho$ for a.e. $t \in [0,\infty)$.

An equilibrium U_0 that is not stable in the above sense is called Lyapunov unstable.

We will drop the reference to (X, Z) where it does not lead to confusion.

We note that under this strong definition of stability, loss of existence of a solution to (2.4) is a particular case of instability. We remark that in literature there are many definitions of a solution to the Navier-Stokes equations. These include "classical" solutions that are continuous functions of each argument (and very few such solutions are known), "weak" solutions defined via test functions by Leray [12] and "mild" solutions introduced by Kato-Fujita [10]. It is this last concept of existence that we will invoke because we utilize a "mild" integral representation of the solution to (2.4) via Duhamel's formula. We remark that to date local in time existence of mild solutions for the Navier-Stokes equations is proved only in L^p, $p \geq n$, (for $p > n$ by Fabes-Jones-Riviere [4] and for $p = n$ by Kato [9]). The existence of weak solutions has been proved in L^2 by Leray [12], in L^p for all $2 \leq p < \infty$ by C. Calderon [2], and for uniformly locally square integrable initial data by Lemarié [11]. For a survey of existence results see for example, Temam [18] and Cannone [3].

We now state the main result of this paper:

Theorem 2.2. *Let $1 < p < \infty$ be arbitrary. Suppose that the operator A over L^p has spectrum in the right half of the complex plane. Then the flow U_0 is (L^q, L^p) nonlinearly unstable for any $q > \max\{p, n\}$.*

The proof of this theorem essentially uses properties of the operator A which are stated in Lemmas 3.1 and 3.2. The instability result is proved using a bootstrap argument which is presented in Section 3 in the case of finite domains Ω and $\mathbb{T}^n$ and in Section 4 in the case of $\mathbb{R}^n$.

Here we state a version of the Sobolev embedding theorem that we shall invoke in the proof of Theorem 2.2.

Proposition 2.3. *Let $s > 0$, $1 < r_1 < \infty$, and $1 < r_2 < \infty$ satisfy*

$$\frac{1}{r_1} < 1 - \frac{s}{n}, \quad r_2 \leq r_1, \quad \frac{1}{r_2} \leq \frac{1}{r_1} + \frac{s}{n}. \tag{2.9}$$

Then

$$\|f\|_{-s,r_1} \lesssim \|f\|_{r_2} \tag{2.10}$$

Proof. Recall that for $s > 0$ and $1 < r < \infty$, $W^{-s,r}$ is defined as the dual space to $W_0^{s,r'}$, where $1/r + 1/r' = 1$. The inequalities (2.9) can be rewritten as

$$sr_1' < n, \quad r_1' \leq r_2' \leq \frac{nr_1'}{n - sr_1'}. \tag{2.11}$$

Thus, the standard Sobolev embedding theorem implies that

$$\|f\|_{r_2'} \lesssim \|f\|_{s,r_1'}. \tag{2.12}$$

Applying (2.12) we obtain

$$\|f\|_{-s,r_1} = \sup_{\|g\|_{s,r_1'} \leq 1} \langle f, g \rangle \lesssim \sup_{\|g\|_{r_2'} \leq 1} \langle f, g \rangle = \|f\|_{r_2},$$

which proves the proposition.

3. Finite domain

In this section we present a proof of Theorem 2.2 in the case of finite domains $\mathbb{T}^n$ and $\Omega \subset \mathbb{R}^n$.

Let μ be the eigenvalue of A with maximal positive real part, which we denote by λ, and let $\phi \in L^p$, with $\|\phi\|_p = 1$, be the corresponding eigenfunction. We note that in the case of a finite domain all eigenfunctions of A are infinitely smooth.

For a fixed $0 < \delta < \lambda$ we denote by A_δ the following operator:

$$A_\delta = A - \lambda - \delta. \tag{3.1}$$

Now we state two auxiliary lemmas which hold both in the case of a finite and in the case of an infinite domain.

Lemma 3.1. *For every $0 < \alpha < 1$ and $p > 1$ there exists a constant $M > 0$ such that for all $t > 0$ one has*

$$\|A_\delta^\alpha \, e^{A_\delta t}\|_{L^p \to L^p} \leq \frac{M}{t^\alpha}. \tag{3.2}$$

This lemma holds generally for any bounded analytic semigroup (see [15]). The rescaling of A given by (3.1) ensures that the semigroup $e^{A_\delta t}$ is bounded. The fact that it is analytic is proved by Yudovich [20] and Giga [6].

Lemma 3.2. *For every $1/2 < \alpha < 1$ and $p > 1$ there exists a constant $C > 0$ such that*

$$\|A_\delta^{-\alpha} \, f\|_p \leq C\|f\|_{-2\alpha,p}. \tag{3.3}$$

In the case of a bounded domain the lemma follows by duality from the papers of Giga [7] and Seeley [16]. On the torus and $\mathbb{R}^n$ one can check (3.3) directly using the Fourier transform and integral representation for fractional power of a generator [15].

We are now in a position to prove Theorem 2.2.

Let us fix an arbitrary small $\epsilon > 0$, and solve the Cauchy problem (2.3) with initial condition $v_0 = \epsilon\phi$. We note that for such initial condition, with ϵ small enough, there exists unique global in time classical solution to (2.3) (see, for example, [18]). Using Duhamel's formula we write the solution in the form

$$v(t) = \epsilon e^{t\mu}\phi + B(t), \tag{3.4}$$

where

$$B(t) = \int_0^t e^{A(t-\tau)} N(v,v)(\tau) \, d\tau.$$

The main idea of the proof is to show that the bilinear term $B(t)$ grows at most like the square of the norm of $v(t)$ for as long as the latter is bounded by a constant multiple of $\epsilon e^{\lambda t}$. The L^q-metric in which such control is possible has to satisfy the assumption $q > n$. Since this condition is not assumed for p we will use L^q as an auxiliary space, while our final instability result will be proved in L^p as stated.

Lemma 3.3. *Let $q > n$. Then there exists a constant $C > 0$ such that the following estimate holds*

$$\|B(t)\|_q \le C \int_0^t e^{(\lambda+\delta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} \|v(\tau)\|_q^2 \, d\tau, \tag{3.5}$$

for some $1/2 < \alpha < 1$.

Proof. Indeed, for any $0 < \alpha < 1$, we can write

$$B(t) = \int_0^t e^{(\lambda+\delta)(t-\tau)} \; A_\delta^\alpha \; e^{A_\delta(t-\tau)} A_\delta^{-\alpha} N(v,v)(\tau)\, d\tau.$$

Hence, by Lemma 3.1,

$$\|B(t)\|_q \lesssim \int_0^t e^{(\lambda+\delta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} \|A_\delta^{-\alpha} N(v,v)(\tau)\|_q \, d\tau.$$

By Lemma 3.2, we have

$$\|A_\delta^{-\alpha} N(v,v)\|_q \lesssim \|N(v,v)\|_{-2\alpha,q} \lesssim \|v \otimes v\|_{1-2\alpha,q},$$

where the last inequality follows from the continuity of the Leray projection. We now choose α sufficiently close to 1 so that $q > n/(2\alpha - 1)$. This would fulfill the conditions of Proposition 2.3 with $s = 2\alpha - 1$, $r_1 = q$ and $r_2 = q/2$. Thus,

$$\|A_\delta^{-\alpha} N(v,v)\|_q \lesssim \|v \otimes v\|_{q/2} \lesssim \|v\|_q^2. \tag{3.6}$$

Inserting this in the last estimate for $\|B(t)\|_q$ we finally obtain (3.5).

Let us fix $q > \max\{n,p\}$. So, in particular, (3.5) holds. For any $Q > \|\phi\|_q$ let $T = T(Q)$ be the maximal time such that

$$\|v(t)\|_q \le Q\epsilon e^{\lambda t}, \quad \forall\, t \le T. \tag{3.7}$$

Notice that (3.7) holds for $t = 0$. Hence, $T > 0$ by continuity. In fact, we show that this critical time T is sufficiently large for any choice of Q. First, let us observe that for any $t \le T$, by Lemma 3.3,

$$\|B(t)\|_q \le CQ^2\epsilon^2 \int_0^t e^{(\lambda+\delta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} e^{2\lambda\tau}\, d\tau.$$

Splitting the integral into two integrals over $[0, t-1]$ and $[t-1, t]$, one can show that it behaves asymptotically as $e^{2\lambda t}$. Hence, perhaps with a different $C > 0$ independent of Q or t, we obtain the following estimate

$$\|B(t)\|_q \le C(Q\epsilon e^{\lambda t})^2, \quad \forall\, t \le T. \tag{3.8}$$

Using (3.8) we now prove an estimate on the size of T.

Lemma 3.4. *For any $Q > \|\phi\|_q$ one has the following inequality*

$$\epsilon e^{\lambda T} \ge \frac{Q - \|\phi\|_q}{CQ^2}. \tag{3.9}$$

Proof. If $T = \infty$, the inequality is trivial. If $T < \infty$, then at time $t = T$ the inequality (3.7) turns into equality and we obtain using (3.4) and (3.8)

$$Q\epsilon e^{\lambda T} = \|v(T)\|_q \le \epsilon e^{\lambda T}\|\phi\|_q + C(Q\epsilon e^{\lambda T})^2.$$

The lemma now easily follows.

Let $\mathfrak{X}_*$ denote the constant on the right hand side of (3.9), i.e.

$$\mathfrak{X}_* = \frac{Q - \|\phi\|_q}{CQ^2}. \tag{3.10}$$

In view of (3.9) there exists time $t_* \le T$ such that $\mathfrak{X}_* = \epsilon e^{\lambda t_*}$. Since $q > p$ we trivially have

$$\|B(t)\|_p \le C'\|B(t)\|_q, \tag{3.11}$$

for some $C' > 0$. So, by the triangle inequality applied to (3.4) we obtain using (3.8), (3.11), and our assumption $\|\phi\|_p = 1$

$$\|v(t_*)\|_p \ge \mathfrak{X}_* - C'C\mathfrak{X}_*^2 = \mathfrak{X}_*(1 - C'C\mathfrak{X}_*). \tag{3.12}$$

Since C and C' are independent of Q, we could choose $Q = Q_0$ in the beginning of the argument so close to $\|\phi\|_q$ that $\mathfrak{X}_* < 1/(2C'C)$. Then

$$\|v(t_*)\|_p \ge \mathfrak{X}_*/2 = c_0.$$

This finishes the proof of Theorem 2.2 in the case of a finite domain.

We remark that in the case of a finite domain our method proves a stronger result. Since the eigenfunction ϕ belongs to C^∞, the size of initial perturbation can be measured in the stronger metric of C^∞ so that $\|v_0\|_{C^\infty} \le \epsilon$, whereas instability at the critical time t_* is measured in the weak L^p-metric.

4. Infinite domain

The case of $\mathbb{R}^n$ brings two main difficulties to the proof. First, we no longer have the inclusion $L_q \subset L_p$ to satisfy (3.11). Second, there may not be an exact smooth eigenfunction ϕ corresponding to $\mu \in \sigma(A)$, because the operator A has a non-compact resolvent over $\mathbb{R}^n$.

4.1. Estimates for $B(t)$. In the case of $\mathbb{R}^n$ we replace the single estimate (3.11) with a sequence of recursive estimates improving integrability exponent on each step.

Let L be the first integer such that $2^L p > n$. By Lemma 3.3, which is valid on $\mathbb{R}^n$ too, we have

$$\|B(t)\|_{2^L p} \le C\int_0^t e^{(\lambda+\delta)(t-\tau)}\frac{1}{(t-\tau)^\alpha}\|v(\tau)\|_{2^L p}^2\,d\tau, \tag{4.1}$$

for some $1/2 < \alpha < 1$. On the other hand, for every $l = 0, \ldots, L-1$ one has, in place of (3.6),

$$\|A_\delta^{-\alpha}N(v,v)\|_{2^l p} \lesssim \|v \otimes v\|_{1-2\alpha,2^l p} \lesssim \|v\otimes v\|_{2^l p} \lesssim \|v\|_{2^{l+1}p}^2.$$

Thus, we obtain

$$\|B(t)\|_{2^l p} \leq C \int_0^t e^{(\lambda+\delta)(t-\tau)} \frac{1}{(t-\tau)^\alpha} \|v(\tau)\|^2_{2^{l+1}p} \, d\tau. \tag{4.2}$$

We postpone the use of (4.1) and (4.2) till Lemma 4.3, where we show the analogue of (3.12) for the case of $\mathbb{R}^n$.

4.2. Construction of approximate eigenfunctions. Suppose now that $\mu \in \sigma(A)$ lies on the boundary of the spectrum and has the greatest positive real part λ. In this case there exists a sequence of functions $\{f_m\}_{m=1}^\infty \subset L^p(\mathbb{R}^n)$ such that

$$\|f_m\|_p = 1,$$
$$\lim_{m\to\infty} \|Af_m - \mu f_m\|_p = 0,$$

and as a consequence, for every $t > 0$,

$$\lim_{m\to\infty} \|e^{tA} f_m - e^{t\mu} f_m\|_p = 0.$$

Lemma 4.1. *There exists a sequence $\{\phi_m\}_{m=1}^\infty \subset L^p(\mathbb{R}^n)$ such that the following is true*

(i) $\|\phi_m\|_p = 1$, $m \in \mathbb{N}$;
(ii) For every $q > p$ there is a constant M_q such that

$$\|\phi_m\|_q \leq M_q$$

holds for all $m \in \mathbb{N}$;
(iii) $\|e^{tA}\phi_m\|_p \geq \frac{1}{2} e^{t\lambda}$, for all $0 \leq t \leq m$;
(iv) $\|e^{tA}\phi_m\|_q \leq 2\|\phi_m\|_q e^{t\lambda}$, for all $0 \leq t \leq m$ and $p \leq q \leq 2^L p$.

Proof. Let $\tilde{\phi}_m = e^A f_m$. Since

$$\|e^A f_m - e^\mu f_m\|_p \to 0$$

we conclude that

$$c \leq \|\tilde{\phi}_m\|_p \leq C, \tag{4.3}$$

for all $m \in \mathbb{N}$. Denote $\phi_m = \tilde{\phi}_m \cdot \|\tilde{\phi}_m\|_p^{-1}$. Clearly, (i) is satisfied. To prove the other three statements we fix $s > 0$ such that $n = sp$. By the end-point Sobolev embedding theorem and (4.3) we have, for any $q > p$,

$$\begin{aligned} \|\phi_m\|_q &\lesssim \|\tilde{\phi}_m\|_q = \|e^A f_m\|_q \lesssim \|e^A f_m\|_{s,p} \\ &\lesssim \|A_\delta^s e^A f_m\|_p \lesssim \|f_m\|_p = 1. \end{aligned}$$

This proves (ii).

Furthermore, we have

$$\begin{aligned} \|e^{tA}\phi_m - e^{t\mu}\phi_m\|_q &\lesssim \|e^A(e^{tA} f_m - e^{t\mu} f_m)\|_{s,p} \lesssim \|A_\delta^s e^A (e^{tA} f_m - e^{t\mu} f_m)\|_p \\ &\lesssim \|e^{tA} f_m - e^{t\mu} f_m\|_p \to 0, \end{aligned}$$

as $m \to \infty$ for each fixed $t > 0$ and $p \leq q$. So, by choosing an appropriate subsequence, we achieve (iii) and (iv).

4.3. Bootstrap argument. Let us fix an arbitrary $\epsilon > 0$ and find $m \in \mathbb{N}$ such that

$$\epsilon e^{\lambda m} > 1. \tag{4.4}$$

This m will be fixed through the rest of the argument. We solve the Cauchy problem (2.3) with initial condition $v_0 = \epsilon\phi_m$. Lemma 4.1 shows that $\phi_m \in L^q$ uniformly in m for all $q > p$. In particular, for any fixed $q > \max\{p, n\}$ there exists a mild solution in $Z = L^q$ for which the Duhamel formulation holds:

$$v(t) = \epsilon e^{At}\phi_m + B(t). \tag{4.5}$$

We note that failure for $v(t)$ to satisfy (4.5) for all $t > 0$ or being in $C([0,\infty), X)$ is regarded as instability by definition. We thus can assume in the rest of the argument that (4.5) holds for all $t > 0$ and $v \in C([0,\infty), X)$. In addition, since $\phi_m \in L^{2^L p}$ and $2^L p > n$, the solution $v(t)$ belongs to $L^{2^L p}$ at least for a certain initial period of time. Our subsequent estimates will show that, in fact, $v(t) \in L^{2^L p}$ over a time interval of the order $\log 1/\epsilon$.

Let $Q > 2\|\phi_m\|_{2^L p}$ be arbitrary, and define $T = T(Q)$ to be the maximal time such that

$$\|v(t)\|_{2^L p} \le Q\epsilon e^{\lambda t}, \text{ for all } t \le T. \tag{4.6}$$

Like in the previous section the following inequality holds

$$\|B(t)\|_{2^L p} \le C(Q\epsilon e^{\lambda t})^2, \quad \forall t \le T. \tag{4.7}$$

Lemma 4.2. *For any $Q > 2\|\phi_m\|_{2^L p}$ we have*

$$\epsilon e^{\lambda T} \ge \min\left\{1; \frac{Q - 2\|\phi_m\|_{2^L p}}{CQ^2}\right\}, \tag{4.8}$$

where $C > 0$ is independent of Q.

Proof. If $T \ge m$, we appeal to (4.4). If $T < m$, then at $t = T$ the inequality (4.6) must turn into equality. Thus, in view of (4.7) we have

$$Q\epsilon e^{\lambda T} = \|v(T(Q))\|_{2^L p} \le 2\|\phi_m\|_{2^L p}\epsilon e^{\lambda T} + C(Q\epsilon e^{\lambda T})^2,$$

which implies (4.8).

We will choose Q appropriately after the following key lemma.

Lemma 4.3. *There are constants $C_2, \ldots, C_{2^{L+1}}$ and $2 \le K \le 2^{L+1}$ independent of Q and m such that for any $t \le \min\{T, m\}$ one has the following inequality*

$$\|v(t)\|_p \ge \frac{1}{2}\mathfrak{X} - C_2\mathfrak{X}^2 - \ldots - C_{K-1}\mathfrak{X}^{K-1} - \\ - C_K Q^K \mathfrak{X}^K - \ldots - C_{2^{L+1}} Q^{2^{L+1}} \mathfrak{X}^{2^{L+1}}, \tag{4.9}$$

where $\mathfrak{X} = \epsilon e^{\lambda t}$.

Proof. First we bound all the norms $\|v(t)\|_{2^l p}$, $l = 1, \dots, L$ from above using the estimates on the nonlinear term (4.2), (4.7). We start with $l = L$ and invoke (4.7) to obtain

$$\|v(t)\|_{2^L p} \leq 2\epsilon e^{\lambda t}\|\phi_m\|_{2^L p} + C(\epsilon e^{\lambda t})^2 Q^2 = C_1 \mathfrak{X} + C_2 Q^2 \mathfrak{X}^2,$$

for all $t \leq T$. We note that our constants may change during the proof.

By the previous inequality and (4.2) with $l = L-1$ we obtain

$$\begin{aligned}\|v(t)\|_{2^{L-1}p} &\leq 2\mathfrak{X}\|\phi_m\|_{2^{L-1}p} + \\ &+ \int_0^t \frac{e^{(t-\tau)(\lambda+\delta)}}{(t-\tau)^\alpha}(C_1 \epsilon e^{\lambda\tau} + C_2 Q^2 \epsilon^2 e^{2\lambda\tau})^2 d\tau \\ &\leq C_1 \mathfrak{X} + C_2 \mathfrak{X}^2 + C_3 Q^3 \mathfrak{X}^3 + C_4 Q^4 \mathfrak{X}^4.\end{aligned}$$

Here and thereafter we use the fact that for any $k \geq 2$ one has

$$\int_0^t e^{(t-\tau)(\lambda+\delta)}(t-\tau)^{-\alpha} e^{k\lambda\tau} d\tau \lesssim e^{k\lambda t}.$$

By induction on l we arrive at

$$\|v(t)\|_{2p} \leq C_1 \mathfrak{X} + \ldots + C_{\tilde{K}-1}\mathfrak{X}^{\tilde{K}-1} + C_{\tilde{K}} Q^{\tilde{K}} \mathfrak{X}^{\tilde{K}} + \ldots + C_{2^L} Q^{2^L} \mathfrak{X}^{2^L},$$

and hence,

$$\|B(t)\|_p \leq C_2 \mathfrak{X}^2 + \ldots + C_{K-1}\mathfrak{X}^{K-1} + C_K Q^K \mathfrak{X}^K + \ldots + C_{2^{L+1}} Q^{2^{L+1}} \mathfrak{X}^{2^{L+1}}.$$

Finally, using (iii) of Lemma (4.1) and the triangle inequality on (4.5) in the opposite direction to get (4.9).

We will choose a Q so that the RHS of (4.9) is bigger than an absolute constant at

$$\mathfrak{X} = \mathfrak{X}_* = \min\left\{1; \frac{Q - 2\|\phi_m\|_{2^L p}}{CQ^2}\right\}.$$

For this $\mathfrak{X}_*$, due to Lemma 4.2 and our initial assumption (4.4), there exists a $t_* \leq \min\{T, m\}$ such that $\mathfrak{X}_* = \epsilon e^{t_* \lambda}$. Hence, Lemma 4.3 applies to obtain instability at time $t = t_*$.

It is convenient to seek Q in the form

$$Q = (2 + a\|\phi_m\|_{2^L p})\|\phi_m\|_{2^L p},$$

where $0 < a < 1$. Then

$$\frac{Q - 2\|\phi_m\|_{2^L p}}{CQ^2} = \frac{a}{C(2 + a\|\phi_m\|_{2^L p})^2} \leq \frac{a}{4C}.$$

Choosing $a < 4C$ we ensure that

$$\frac{Q - 2\|\phi_m\|_{2^L p}}{CQ^2} < 1$$

and hence,

$$\mathfrak{X}_* = \frac{Q - 2\|\phi_m\|_{2^L p}}{CQ^2}.$$

By the above estimate and (ii) of Lemma 4.1, we have

$$\frac{a}{C(2 + M_{2^L p})^2} \le \mathfrak{X}_* \le \frac{a}{4C},$$

or

$$\frac{a}{C'} \le \mathfrak{X}_* \le \frac{a}{c'}. \tag{4.10}$$

We notice that since Q is bounded by a constant independent of m and a, we can bound the minimum

$$\min\left\{1; \min_{2\le k\le K-1}(C_k 4^k)^{-1/k}; \min_{K\le k\le 2^{L+1}}(C_k Q^k 4^k)^{-1/k}\right\}$$

from below by some constant c_0 independent of Q. Let $a = \min\{4C, c'c_0/2\}$. Then from (4.10), we obtain

$$\tilde{c}_0 \le \mathfrak{X}_* \le c_0.$$

Thus, by (4.9),

$$\|v(t_*)\|_p \ge \tilde{c}_0\left(\frac{1}{2} - \frac{1}{16} - \ldots\right) = c.$$

This finishes the proof.

We remark again that like in the case of a finite domain our method yields a slightly stronger result. Since $\phi_m \in W^{s,p}$ uniformly, we can measure the size of initial perturbation in the metric of any Sobolev space $W^{s,p}$ for all $s > 0$.

5. Stability result

Bootstrap techniques can also be used to prove that linear stability implies nonlinear stability for the Navier-Stokes equations in L^q for $q > n$. In particular this reproves the classical stability theorem of Yudovich [20].

Theorem 5.1. *Let $q > n$ be arbitrary. Assume the operator A in L^q has spectrum confined to the left half of the complex plane. Then the flow U_0 is (L^q, L^q) nonlinearly stable. The result holds in $\mathbb{T}^n$ and Ω, and in any spatial dimension n.*

Proof. We recall that that any analytic semigroup possesses the spectral mapping property. From the assumption that the spectrum of A is confined to the left half plane we thus conclude that the exponential type of the semigroup e^{At} is negative. Hence, there exists $\lambda > 0$ such that

$$\|e^{At}v_0\|_q \le Me^{-\lambda t}\|v_0\|_q, \tag{5.1}$$

for all $t > 0$ and $v_0 \in L^q$. From Duhamel's formula (3.4) with the initial condition replaced by v_0, and by argument similar to that used in the proof of Lemma 3.3, we have

$$\|v(t)\|_q \le Me^{-\lambda t}\|v_0\|_q + C\int_0^t e^{-\lambda(t-\tau)}(t-\tau)^{-\alpha}\|v(\tau)\|_q^2 d\tau. \tag{5.2}$$

Again let T be the maximal time for which

$$\|v(t)\|_q \leq 2M\|v_0\|_q e^{-\lambda t}, \quad t \leq T. \tag{5.3}$$

Combining (5.2) and (5.3) gives

$$\begin{aligned}\|v(t)\|_q &\leq Me^{-\lambda t}\|v_0\|_q + 4M^2Ce^{-2\lambda t}\|v_0\|_q^2 \\ &\leq Me^{-\lambda t}\|v_0\|_q(1 + 4MC\|v_0\|_q),\end{aligned}$$

for $t \leq T$. We choose $\|v_0\|_q < (8MC)^{-1}$. Then the previous inequality implies that

$$\|v(t)\|_q \leq \frac{3}{2}M\|v_0\|_q e^{-\lambda t}, \tag{5.4}$$

for $t \leq T$. Hence, the assumption of (5.3) implies the smaller bound of (5.4), which gives a contradiction with a maximal finite T. Thus, $T = \infty$ and the bound (5.3) holds for all $t \geq 0$. This bound implies the global existence of the solution to (2.1) and condition (ii) of Definition 2.1 for a sufficiently small choice of $\|v_0\|_q$.

Remark 5.2. The instability/stability results in this paper can be generalized to all the equations of motion that are augmented versions of the equations for incompressible, dissipative fluids described in operator form by an appropriate version of (2.4). This includes the magnetohydrodynamic equations for a dissipative electrically conducting fluid, the equations for an incompressible, stratified fluid with viscous and thermal dissipation and the so called modified Navier-Stokes equations with $(-\Delta)$ replaced by $(-\Delta)^\beta$ where $\beta > 1/2$.

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