

# Some Remarks on Vector-Valued Integration

V. Kadets

Department of Mechanics and Mathematics

Kharkov State University

4 Svobody sq., Kharkov 310077

Ukraine

B. Shumyatskiy

R. Shvidkoy

L. Tseytlin

K. Zheltukhin

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## Abstract

The article presents a new method of integration of functions with values in Banach spaces. This integral and related notions prove to be a useful tool in the study of Banach space geometry.

## 1 Introduction

There are many definitions of integral for Banach-space-valued functions and their corresponding classes of integrable functions. The easiest one is the Riemann integral. This definition looks exactly like the one for real-valued functions.

**Definition 1.1** *Let  $f : [0; 1] \rightarrow X$  be a bounded function. This function is said to be Riemann integrable if there exists an  $x \in X$  (called  $f$ 's Riemann integral) such that for any  $\epsilon > 0$  there is a  $\delta > 0$  such that for any partition of the segment  $[0; 1]$  into a finite number of intervals  $\{\Delta_i\}_{i=1}^N$  with  $\max_i |\Delta_i| < \delta$  and any choice of sampling points  $t_i \in \Delta_i$  the corresponding Riemann integral sum  $S_R(f, \{\Delta_i\}, \{t_i\}) = \sum_{i=1}^N f(t_i) |\Delta_i|$  is close to  $x$ , that is,  $\|S_R(f, \{\Delta_i\}, \{t_i\}) - x\| < \epsilon$ .*

The most commonly used definition of a vector-valued integral is, however, that of Bochner integral.

**Definition 1.2** *Let  $(\Omega, \Sigma, \mu)$  be a measure space and  $f : \Omega \rightarrow X$  be a function. The function  $f$  is called Bochner-integrable if there exists a sequence of simple (measurable finite-valued) functions  $f_n$  such that  $\int_{\Omega} \|f - f_n\| d\mu \rightarrow 0$ , as  $n \rightarrow \infty$ . The Bochner integral of  $f$  is then defined as  $\int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu$ , where the integral for simple functions is defined in the obvious way: if  $f_n = \sum x_i \chi_{A_i}$ , then  $\int_{\Omega} f_n d\mu = \sum x_i \mu(A_i)$ . It is easy to see that such a limit indeed exists and does not depend on the choice of a sequence  $f_n$  approximating the given function  $f$ .*

Although the Bochner integrability is a direct generalization of the Lebesgue one and has many properties of the latter, there are Riemann integrable functions that are not Bochner integrable.

*Example.* Consider the space  $l_2([0; 1])$ . It consists of all functions  $f : [0; 1] \rightarrow \mathbb{R}$  such that  $\sum_{t \in [0; 1]} |f(t)|^2 < \infty$  (it follows that these functions take non-zero values on countable subsets of  $[0; 1]$ ). The norm on  $l_2([0; 1])$  is given by  $\|f\| = (\sum_{t \in [0; 1]} |f(t)|^2)^{1/2}$ . It is easy to see that  $l_2([0; 1])$  is a non-separable Hilbert space. Its uncountable basis is given by  $e_t = \chi_{\{t\}}$ .

Now consider a function  $f : [0; 1] \rightarrow l_2([0; 1])$ :  $f(t) = e_t$ . This function is not measurable, since it is not even “almost separable-valued”. Thus it is not Bochner-integrable. However, it is Riemann-integrable. To see this, note that for any partition  $\Pi = \{\Delta_i\}$  with any sample points  $T = \{t_i\}$ ,

$$\begin{aligned} \|S_R(f, \Pi, T)\| &= \left\| \sum f(t_i) |\Delta_i| \right\| = \left\| \sum e_{t_i} |\Delta_i| \right\| = \\ &= \left( \sum |\Delta_i|^2 \right)^{1/2} \leq \left( \sum d(\Pi) \cdot |\Delta_i| \right)^{1/2} = \\ &= \sqrt{d(\Pi)} \left( \sum_i |\Delta_i| \right)^{1/2} = \sqrt{d(\Pi)}, \end{aligned}$$

where  $d(\Pi)$  denotes  $\max\{|\Delta_i|\}$ . Now take an arbitrary  $\epsilon > 0$ . Fix a partition  $\Pi$  with  $d(\Pi) < \epsilon$ . Then for any  $\Pi' \succ \Pi$  and any  $T'$  we have  $\|S_R(f, \Pi', T')\| \leq \sqrt{d(\Pi')} \leq \sqrt{d(\Pi)} \leq \epsilon$ . Thus  $f$  is Riemann-integrable and the integral equals 0.

This gap may be covered by the idea of weak integration.

**Definition 1.3** Let  $(\Omega, \Sigma, \mu)$  be a measure space and  $f : \Omega \rightarrow X$  be a function. This function is called weakly integrable if for any functional  $F \in X^*$  the scalar function  $F \circ f$  is Lebesgue-integrable.

The function  $f$  is called Pettis-integrable if for any  $A \in \Sigma$  there exists an  $x \in X$  such that  $F(x) = \int_A F \circ f d\mu$ , whenever  $F \in X^*$ . The point  $x$  is then called  $f$ 's Pettis integral over  $A$ .

Pettis integrability is useful enough and covers both Riemann and Bochner types of integration, but it, in turn, has its own drawbacks. The definition of the Pettis integral does not use any “simple” approximations, that is why some of the usual Lebesgue-integration theorems do not hold for the Pettis integral. For example, there exists a Pettis-integrable function  $f : [0; 1] \rightarrow X$ , for which the “antiderivative”  $F(t) = \int_0^t f(\tau) d\mu$  is not differentiable ([3]); the space of all Pettis-integrable functions is not complete, etc. We refer the reader to classical texts [4] and [1] for the detailed treatment of this theory.

In the present paper we study a new definition of integrability, introduced by two of the authors in [2] and named the RL (Riemann-Lebesgue) integrability, which covers both Riemann and Bochner integrals (theorems 2.6 and 2.7), but is less general than the Pettis one. As will be shown below, this notion is a very natural and convenient one. The properties of the RL integration depend on the properties of the space  $X$ , which makes this concept a valuable tool for the study of Banach space structure (especially in the non-separable case).

## 2 RL-Integral Sums and RL-Integral

Let  $(\Omega, \Sigma, \mu)$  be a space with finite measure  $\mu$  and  $X$  be a Banach space.

**Definition 2.1** Let  $f : \Omega \rightarrow \mathbb{R}^+$  be a function. We call the value  $\overline{\int} f d\mu = \inf\{\int g d\mu : g(t) \geq f(t) \forall t; g \text{ is Lebesgue integrable}\}$   $f$ 's upper Lebesgue integral. In particular,  $\overline{\int} f d\mu = +\infty$ , if  $f$  has no a Lebesgue-integrable majorant.

Using the notion of the upper Lebesgue integral we introduce the *upper- $L_1$*  ( $\overline{L}_1$ ) space.

**Definition 2.2** The space  $\overline{L}_1(\Omega, \Sigma, \mu, X)$  is defined as the space of all functions  $f : \Omega \rightarrow X$  such that  $\overline{\int} \|f\| d\mu < +\infty$ . The norm on this space is given by  $\|f\| = \overline{\int} \|f(t)\| d\mu(t)$ , and is called the RL-norm ( $\|\cdot\|_{RL}$ ).

It is easy to see that any bounded function as well as any Bochner-integrable function belongs to  $\overline{L}_1$ . Using usual classical methods, one can also prove that  $\overline{L}_1$  is a Banach space.

Let us now introduce the notion of RL integral sums, first defined in [2].

**Definition 2.3** *Let  $f : \Omega \rightarrow X$  be a function. Let  $\Pi = \{\Delta_i\}_{i=1}^{\infty}$  be a partition of  $\Omega$  into a countable number of measurable subsets. Let  $T = \{t_i\}_{i=1}^{\infty}$  be the set of sampling points for  $\Pi$ , i.e.  $t_i \in \Delta_i$ . We can construct a formal series  $S(f, \Pi, T) = \sum_{i=1}^{\infty} f(t_i)\mu(\Delta_i)$ . This series is called the (absolute) RL integral sum of  $f$  with respect to  $\Pi$  and  $T$ , provided it is absolutely convergent.*

It seems natural to order the set of partitions in the following way. We will say that the partition  $\Pi_1 = \{D_i\}$  follows partition  $\Pi_2 = \{E_k\}$ , or  $\Pi_1$  is inscribed into  $\Pi_2$  ( $\Pi_1 \succ \Pi_2$ ), whenever  $\Pi_1$  is a finer partition, that is, the set of indices  $\mathbb{N}$  can be broken into disjoint subsets  $I_k, k \in \mathbb{N}, \bigcup_{k=1}^{\infty} I_k = \mathbb{N}$  such that  $E_k = \bigcup_{i \in I_k} D_i$ .

**Lma 2.1** *Let  $f \in \overline{L}_1(\Omega, \Sigma, \mu, X)$ . Then there exists a partition  $\Pi$  of  $\Omega$  so that for any finer partition  $\Gamma \succ \Pi$  and any set of sampling points  $T$  the series  $S(f, \Gamma, T)$  is absolutely convergent.*

*Proof.* Since  $f \in \overline{L}_1(\Omega, \Sigma, \mu, X)$ , there exists a Lebesgue-integrable function  $g : \Omega \rightarrow \mathbb{R}^+$  that dominates the norm  $f(t)$  pointwise. Since  $g$  is Lebesgue-integrable, we can choose an  $\epsilon > 0$  so small that the upper Lebesgue integral sum  $\sum_{i=1}^{\infty} i\epsilon \cdot \mu(g^{-1}([(i-1)\epsilon; i\epsilon]))$  is absolutely convergent. Denoting  $\Delta_i = g^{-1}([(i-1)\epsilon; i\epsilon])$  we obtain a partition  $\Pi$  for which the series  $\sum_i \sup_{t \in \Delta_i} g(t) \cdot \mu(\Delta_i)$  is absolutely convergent.

Now if  $\Gamma \succ \Pi$ , then we can write  $\Gamma = \{\Delta_{ij}\}$ , where  $\bigcup_j \Delta_{ij} = \Delta_i$ . Let  $T = \{t_{ij}\}$  be a set of sampling points for  $\Gamma$ . Then the integral sum  $S(f, \Gamma, T) = \sum_{ij} f(t_{ij})\mu(\Delta_{ij})$  is dominated in norm by the series  $\sum_{ij} \sup_{t \in \Delta_i} g(t)\mu(\Delta_{ij})$ , which is convergent. Thus  $S(f, \Gamma, T)$  is absolutely convergent.  $\square$

Now we can introduce the definition of the RL integral.

**Definition 2.4** *A function  $f : \Omega \rightarrow X$  is called Riemann-Lebesgue (RL) integrable over a measurable set  $A \subset \Omega$  if there exists a point  $x \in X$  such that for any  $\epsilon > 0$  there is a partition  $\Pi$  of  $A$  such that for any finer partition  $\Gamma \succ \Pi$  with any set of sampling points  $T$  we have  $\|S(f, \Gamma, T) - x\| < \epsilon$  and the sum  $S(f, \Gamma, T)$  converges absolutely. This point  $x$  is called then the Riemann-Lebesgue integral of  $f$  and denoted, as usual, by  $\int_A f(t)dt$ .*

Let us point out several simple properties of the RL integral:

1. RL integral is additive: if functions  $f$  and  $g$  are RL-integrable, then  $f + g$  is RL-integrable too and  $\int (f + g)d\mu = \int fd\mu + \int gd\mu$ .
2. If a function  $f$  is RL-integrable over a set  $A$ , it is also RL-integrable over any measurable subset of  $A$ .
3. If  $f : \Omega \rightarrow X$  is an RL-integrable function, then  $f \in \overline{L_1}(\Omega, \Sigma, \mu, X)$ .
4. As a function of set, RL integral is a countably-additive measure of bounded variation.
5. If  $f : \Omega \rightarrow X$  is an RL-integrable function and  $T : X \rightarrow Y$  is a continuous linear operator, then the composition  $Tf : \Omega \rightarrow Y$  is an RL-integrable function and we have  $T \int fd\mu = \int Tf d\mu$ .

**Definition 2.5** *The  $RL_1(\Omega, \Sigma, \mu, X)$  space is the linear subspace of  $\overline{L_1}(\Omega, \Sigma, \mu, X)$  consisting of all those functions that are RL-integrable.*

$RL_1$  is a closed subspace of  $\overline{L_1}$  and as such is a Banach space.

The following theorem is a useful sufficient condition for a function to be RL-integrable. We will use it later to prove the RL-integrability of the Bochner and Riemann integrable functions.

**Theorem 2.2** *Let  $f \in \overline{L_1}(\Omega, \Sigma, \mu, X)$  be a function and  $g \in L_1(\Omega, \Sigma, \mu)$  be its integrable majorant. Let also  $\{\Pi_n\}$  be a sequence of partitions, for each of which the upper integral sum of  $g$  is absolutely convergent. Assume that for any choice of sampling points  $T_n$  for  $\Pi_n$ , the sequence  $S(f, \Pi_n, T_n)$  converges to a certain element  $x \in X$ . Then  $f$  is RL-integrable and  $x = \int fd\mu$ .*

Let us prove some auxiliary lemmas first.

**Lma 2.3** *Let  $K_1, \dots, K_N$  be finite sets in a Banach space  $X$ . Then*

$$\text{conv}(K_1) + \dots + \text{conv}(K_N) \subset \text{conv}(K_1 + \dots + K_N) \quad (1)$$

*Proof.* Since the sets  $K_i$  are finite and their number is finite, both sides of (1) are convex compacts. Due to the Krein–Milman theorem, it is sufficient to show that the extreme points of the left-hand side of (1) belong to the right-hand one.

Denote the left-hand side of (1) by  $A$ . Let  $x$  be an extreme point of  $A$ . Obviously,  $x = x_1 + \dots + x_N$ , where  $x_i \in \text{conv}(K_i)$ ,  $1 \leq i \leq N$ . Note that each  $x_i$  must be an extreme point of  $\text{conv}(K_i)$ . Indeed, assume that one of the  $x_i$ 's is not extreme. Without loss of generality we can assume that  $i = 1$ . So  $x_1 = (y_1 + z_1)/2$ , where  $y_1, z_1 \in \text{conv}(K_1)$ . But then  $x = (y + z)/2$ , where  $y = y_1 + x_2 + \dots + x_n$  and  $z = z_1 + x_2 + \dots + x_n$ ,  $y, z \in A$ , which is impossible, since  $x$  is an extreme point of  $A$ .

Now note that an extreme point of  $\text{conv}(K_i)$  must belong to  $K_i$  itself. Thus  $x_i \in K_i$ , and therefore  $x = x_1 + \dots + x_N \in K_1 + \dots + K_N \subset \text{conv}(K_1 + \dots + K_N)$ , which proves the lemma.  $\blacksquare$

**Lma 2.4** *Let  $f : \Omega \rightarrow X$  be a function. Let  $\Pi$  be a finite partition of  $\Omega$ . Then for any finer finite partition  $\Gamma \succ \Pi$  the following inclusion holds:*

$$\begin{aligned} \{S(f, \Gamma, T) : T \text{ any choice of sampling points}\} &\subset \\ &\subset \text{conv}\{S(f, \Pi, T) : T \text{ any choice of sampling points}\}. \end{aligned}$$

*Proof.* Let  $\Pi = \{\Delta_i\}_{i=1}^N$  and let  $\Gamma = \{\Delta_{ij}\}_{i=1}^N \{j=1}^{M_i}$  where  $\bigcup_{j=1}^{M_i} \Delta_{ij} = \Delta_i$ . Consider arbitrary sampling points  $T = \{t_{ij}\}$ ,  $t_{ij} \in \Delta_{ij}$  and the integral sum  $S(f, \Gamma, T) = \sum_{ij} f(t_{ij})\mu(\Delta_{ij})$ . Denote by  $S_i$  the part of this sum where  $\Delta_{ij} \subset \Delta_i$ , i.e.  $S_i = \sum_j f(t_{ij})\mu(\Delta_{ij})$ . Then  $S(f, \Gamma, T) = S_1 + \dots + S_N$ . Denote also  $K_i = \{f(t_{ij})\mu(\Delta_i) : j = 1, \dots, M_i\}$  for each  $i = 1, \dots, N$ .

Note that

$$S_i = \sum_{j=1}^{M_i} f(t_{ij})\mu(\Delta_{ij}) \cdot \frac{\mu(\Delta_{ij})}{\mu(\Delta_i)},$$

where  $\sum_j \frac{\mu(\Delta_{ij})}{\mu(\Delta_i)} = 1$ . Thus  $S_i$  is a convex combination of elements of the form  $f(t_{ij})\mu(\Delta_{ij})$ , i.e.  $S_i \in \text{conv}(K_i)$ . Due to lemma 2.3,  $S(f, \Gamma, T) \in \text{conv}(K_1 + \dots + K_N)$ . But each element of  $K_1 + \dots + K_N$  has the form  $f(t_{1j_1})\mu(\Delta_1) + \dots + f(t_{Nj_N})\mu(\Delta_N)$ , where  $t_{ij_i} \in \Delta_{ij_i} \subset \Delta_i$ . Thus each element of  $K_1 + \dots + K_N$  is an integral sum of the form  $S(f, \Pi, T')$  for certain choice of sampling points  $T'$ . This proves the lemma.  $\square$

**Lma 2.5** *Let  $f \in \bar{L}_1(\Omega, \Sigma, \mu, X)$  be a function and  $g \in L_1(\Omega, \Sigma, \mu)$  be its integrable majorant. Let  $\Pi$  be a partition of  $\Omega$  for which the upper integral sum of  $g$  is absolutely convergent. Then for any finer partition  $\Gamma \succ \Pi$  the following inclusion holds:*

$$\begin{aligned} \{S(f, \Gamma, T) : T \text{ any choice of sampling points}\} &\subset \\ &\subset \overline{\text{conv}}\{S(f, \Pi, T) : T \text{ any choice of sampling points}\}. \end{aligned}$$

*Proof.* The lemma can be reduced to the previous one by approximating infinite series with their partial sums.  $\square$

Now we are in a position to prove theorem 2.2.

*Proof of theorem 2.2.*

Note that under the conditions of the theorem, the following fact is true: for any  $\epsilon > 0$  there exists a partition  $\Pi_n$  such that for any choice of sampling points  $T_n$  for it,  $\|S(f, \Pi_n, T_n) - x\| < \epsilon$ . Indeed, if this were not true, then there would exist an  $\epsilon > 0$  for which we could find a subsequence  $\{\Pi_{n_k}\}_{k=1}^{\infty}$  with sampling points  $T_{n_k}$  so that  $\|S(f, \Pi_{n_k}, T_{n_k}) - x\| \geq \epsilon$ . Then choosing arbitrary sets of sampling points  $T_n$  for partitions  $\Pi_n$ , where  $n \notin \{n_k\}$ , we would have a sequence  $(\Pi_n, T_n)$  for which  $S(f, \Pi_n, T_n)$  does not converge to  $x$ .

Now fix an  $\epsilon > 0$ . Take a partition  $\Pi_n$  for which  $\|S(f, \Pi_n, T_n) - x\| < \epsilon$  for any choice of sampling points  $T_n$ . Then, by lemma 2.5, for any  $\Gamma \succ \Pi_n$  and any choice of sampling points  $T$  for  $\Gamma$  we have  $\|S(f, \Gamma, T) - x\| < \epsilon$ , which proves that  $f$  is RL-integrable and the RL-integral of  $f$  equals  $x$ .  $\square$

This theorem in hand we can show that the notion of RL-integrability of functions is more general than some known ones.

**Theorem 2.6** *Let a bounded function  $f : [0; 1] \rightarrow X$  be Riemann integrable. Then it is RL-integrable and its RL integral equals its Riemann integral.*

*Proof.* Indeed, since  $f$  is bounded, it belongs to  $\overline{L}_1$ . The constant function that bounds  $f$  can serve as the dominant  $g$  from the theorem 2.2. Its upper Lebesgue integral sum for any partition is obviously convergent. Let  $\Pi_n$  be a partition of the segment  $[0; 1]$  into  $2^n$  equal subsegments. By the definition of the Riemann integral,  $\{\Pi_n\}$  is a sequence of partitions that satisfies the conditions of theorem 2.2. Therefore it is RL-integrable.  $\square$

**Theorem 2.7** *Let a function  $f : \Omega \rightarrow X$  be Bochner-integrable. Then it is RL-integrable and its RL-integral equals its Bochner integral.*

*Proof.* We use theorem 2.2. Since  $f$  is Bochner integrable, the function  $g(t) = \|f(t)\|$  is its integrable majorant and hence  $f$  belongs to  $\overline{L}_1(\Omega, \Sigma, \mu, X)$ . Put  $x = (\text{Bochner}) \int f d\mu$ . Let us fix a sequence of positive numbers  $\epsilon_n \rightarrow 0$  and construct the partition  $\Pi_n$  required in theorem 2.2.

Due to  $f$ 's Bochner integrability, it is measurable and therefore almost separable-valued. Thus there exists a subset  $\Omega' \subset \Omega$ ,  $\mu(\Omega') = \mu(\Omega)$ , such that  $f(\Omega')$  is separable. Let us cover the separable set  $f(\Omega')$  with countable number of disjoint sets  $A_i$  so that  $\text{diam}(A_i) < \epsilon_n$ . To do this we first fix a countable dense set  $\{x_i\} \subset f(\Omega')$ , then consider balls  $B_i = \{x \in X : \|x - x_i\| < \epsilon_n\}$ , and then define  $A_i$ 's as follows:  $A_1 = B_1$ ,  $A_2 = B_2 \setminus A_1$ ,  $A_3 = B_3 \setminus (A_1 \cup A_2)$ , and so on. It is clear from the construction that the sets  $f^{-1}(A_i)$  are measurable. Therefore  $\{f^{-1}(A_i)\}$  is a partition of  $\Omega'$ . By adding a negligible set  $\Omega \setminus \Omega'$  (which will have no effect on the integral sums) to the partition we obtain a partition of the entire  $\Omega$ . Denote this partition by  $\Pi_n = \{\Delta_i\}$ . Obviously, it has the following property:

$$\text{if } \mu(\Delta_i) \neq 0, \text{ then } \text{diam}(f(\Delta_i)) < \epsilon_n. \quad (2)$$

Let us prove that  $\Pi_n$  is required. To this end, suppose  $T = \{t_i\}$  is a set of sampling points for  $\Pi_n$ . Then, taking into account (2), we estimate:

$$\begin{aligned} \left\| f(t_i)\mu(\Delta_i) - \int_{\Delta_i} f d\mu \right\| &= \left\| \int_{\Delta_i} f(t_i) d\mu - \int_{\Delta_i} f d\mu \right\| \leq \\ &\leq \left\| \int_{\Delta_i} f(t_i) - f(t) d\mu \right\| \leq \epsilon_n \mu(\Delta_i). \end{aligned}$$

Therefore

$$\begin{aligned} \|S(f, \Pi_n, T) - x\| &= \left\| \sum_{i=1}^{\infty} f(t_i)\mu(\Delta_i) - \sum_{i=1}^{\infty} \int_{\Delta_i} f d\mu \right\| \leq \\ &\leq \left\| \sum_{i=1}^{\infty} \left( f(t_i)\mu(\Delta_i) - \int_{\Delta_i} f d\mu \right) \right\| \leq \epsilon_n \sum_{i=1}^{\infty} \mu(\Delta_i) = \epsilon_n. \end{aligned}$$

Finally, to show that for the partition  $\Pi_n$  the upper Lebesgue integral sum of the majorant  $g(t) = \|f(t)\|$  is convergent, note that the series  $\sum \sup_{\Delta_i} g \cdot \mu(\Delta_i)$  is dominated by the absolutely convergent series  $\sum \int_{\Delta_i} (g + \epsilon_n) d\mu$ . This finishes the proof.  $\square$

Now let us mention two properties of the RL integral proved in [2].

1. An RL-integrable function is Pettis-integrable and the values of both integrals coincide.
2. If the space  $X$  is separable, then RL-integrability is equivalent to Bochner integrability.

### 3 $RL_1(X, \mu)$ Space

Let us investigate the properties of the space  $RL_1(X, \mu)$ . As a closed subspace of the space  $\overline{L_1}(X, \mu)$ , it is a Banach space. Note that the space  $L_1(X, \mu)$  of Bochner-integrable  $X$ -valued functions is a closed subspace of  $RL_1(X, \mu)$ . It seems natural to ask: under what condition on the space  $X$ ,  $L_1(X, \mu)$  is complemented in  $RL_1(X, \mu)$ ? One sufficient condition can be formulated with the help of the following notion.

**Definition 3.1** *Let  $f : \Omega \rightarrow X$  be an RL-integrable function. We say that a function  $g : \Omega \rightarrow X$  is  $f$ 's Bochner-integrable equivalent if  $g$  is Bochner-integrable and for any measurable  $A \subset \Omega$ ,*

$$(RL) \int_A f d\mu = (Bochner) \int_A g d\mu.$$

It is easy to see that if every  $X$ -valued RL-integrable function has a Bochner-integrable equivalent, then  $L_1(X, \mu)$  is complemented in  $RL_1(X, \mu)$ . Indeed, the equivalent Bochner-integrable function for the given RL-integrable function  $f$  is  $f$ 's projection onto  $L_1(X, \mu)$ . In view of this, let us investigate under what conditions on the space  $X$  every RL-integrable function has a Bochner-integrable equivalent.

One easy sufficient condition for  $X$  is to have the Radon–Nikodým property (RNP).

**Theorem 3.1** *Let the Banach space  $X$  have the RNP. Then any RL-integrable function  $f : \Omega \rightarrow X$  has a Bochner-integrable equivalent.*

*Proof.* Consider the vector measure  $\nu : \Sigma \rightarrow X$  defined by  $\nu(A) = \int_A f d\mu$ . It is countably-additive and has bounded variation (this follows from the existence of integrable majorant of  $f$ ). Since  $X$  possesses the RNP there must exist a Bochner-integrable function  $g : \Omega \rightarrow X$  such that  $\nu(A) = \int_A g d\mu$  for any measurable  $A$ . Obviously,  $g$  is  $f$ 's Bochner-integrable equivalent.  $\square$

Another simple condition is the following:

**Theorem 3.2** *Let  $f : \Omega \rightarrow X$  be an RL-integrable function and the set  $\{\int_A f d\mu : A \in \Sigma\}$  be contained in a separable complemented subspace  $Y$  of  $X$ . Then  $f$  has a Bochner-integrable equivalent.*

*Proof.* Let  $P$  be the projection from  $X$  onto  $Y$ . Put  $g = Pf$ . Since  $g$  is an image of  $f$  under continuous linear map, it is RL-integrable. Moreover, the values of  $g$  lie in the separable space  $Y$ . Hence  $g$  is also Bochner integrable. Obviously,  $\int_A g d\mu = P \int_A f d\mu$  for any  $A \in \Sigma$ , and since all integrals  $\int_A f d\mu$  lie in  $Y$ ,  $P \int_A f d\mu = \int_A f d\mu$ . Thus  $g$  is the Bochner-integrable equivalent of  $f$ .  $\square$

Now let us note the following fact.

**Theorem 3.3** *Let  $f : \Omega \rightarrow X$  be an RL-integrable function. Then the set  $\{\int_A f d\mu : A \in \Sigma\}$  is separable.*

*Proof.* Fix a sequence of positive numbers  $\epsilon_n \rightarrow 0$ . Since  $f$  is RL-integrable, for each  $n$  there exists a partition  $\Pi_n$  such that for any two finer partitions  $\Gamma'$  and  $\Gamma''$  with any sets of sampling points  $T'$  and  $T''$  we have

$$\|S(f, \Gamma', T') - S(f, \Gamma'', T'')\| < \epsilon_n. \quad (3)$$

For each  $\Pi_n$  fix a set of sampling points  $T_n$ . Obviously,  $S(f, \Pi_n, T_n) \rightarrow \int f d\mu$ . Let  $U = \bigcup_{n=1}^{\infty} T_n$ . Since  $U$  is countable, the subspace  $Y = \overline{\text{Lin}(f(U))}$  is separable. Obviously,  $S(f, \Pi_n, T_n) \in \text{Lin}(f(U))$  and therefore  $\int f d\mu \in Y$ . Let us now show that in fact  $\int_A f d\mu \in Y$  for all  $A \in \Sigma$ .

Indeed, take an arbitrary  $A \in \Sigma$ . Since  $U$  is negligible,  $\int_A f d\mu = \int_{A \cup U} f d\mu$ . Therefore, without loss of generality we may and do assume that  $U \subset A$ . Denote by  $\Pi_n^A$  the partition of  $A$  formed by intersection of subsets of  $\Pi_n$  with  $A$ . Then for any two partitions of  $A$ ,  $\Gamma' \succ \Pi_n^A$  and  $\Gamma'' \succ \Pi_n^A$  with any sets of sampling points  $T'$  and  $T''$  respectively, condition (3) remains true. Therefore  $S(f, \Pi_n^A, T_n) \rightarrow \int_A f d\mu$  and hence,  $\int_A f d\mu \in Y$ .  $\square$

Theorems 3.2 and 3.3 together give us the following useful corollary:

**Theorem 3.4** *Let the Banach space  $X$  have the following property: every separable subspace of  $X$  is contained in a separable complemented subspace of  $X$  (the class of such spaces includes, for example, all the WCG spaces). Then any RL-integrable function  $f : \Omega \rightarrow X$  has a Bochner-integrable equivalent.*

Thus, we have shown that in quite a wide class of Banach spaces every RL-integrable function has a Bochner-integrable equivalent. However, there exist spaces where this is not true. Let us show an example.

*Example.* Consider the function  $f : [0; 1] \rightarrow L_\infty[0; 1]$  defined by  $f(t) = \chi_{[t; 1]}$ . We prove that it is RL-integrable, but does not have an equivalent Bochner-integrable function.

Indeed, let us show the following identity:

$$\left( \int_A f d\mu \right)(t) = \mu(A \cap [0; t]) \quad (4)$$

for any  $t \in [0; 1]$  and any Borel set  $A$ .

Since both sides of the equality contain  $L_\infty$ -valued countably-additive measures, it suffices to show the statement for  $A = [\frac{k}{2^n}; \frac{k+1}{2^n}]$ , where  $n \in \mathbb{N} \cup \{0\}$ ,  $0 \leq k \leq 2^n$ .

To this end, we take an arbitrary  $\epsilon > 0$  and find such a positive integer  $N$  that  $\frac{1}{2^{nN}} < \epsilon$ . Let us partition  $A$  into  $N$  equal intervals  $\{\Delta_i\}_{i=1}^N$  of length  $\frac{1}{2^{nN}}$  and let  $\{\Delta_{ij}\}_{i=1}^N \sum_{j=1}^{n_i}$  be an arbitrary finer partition into non-empty intervals, where  $\bigcup_{j=1}^{n_i} \Delta_{ij} = \Delta_i$ ,  $i = 1, \dots, N$ . Let also  $\{t_{ij}\}$  be a set of sampling points for this finer partition. We show that for the functions

$$S(t) = \left\langle \sum_{i=1}^N \sum_{j=1}^{n_i} f(t_{ij}) \mu(\Delta_{ij}), t \right\rangle$$

and

$$g(t) = \mu(A \cap [0; t])$$

the condition

$$|S(t) - g(t)| < \epsilon$$

holds for any  $t \in [0; 1]$ .

Indeed, for  $t \in [0; \frac{k}{2^n}]$  we have

$$S(t) = 0 = g(t). \quad (5)$$

If  $t \in [\frac{k+1}{2^n}; 1]$ , then

$$S(t) = \mu(A) = g(t) \quad (6)$$

For  $t \in \Delta_{i_0 j_0}$  we can estimate:

$$S(t) = \sum_{i=1}^{i_0} \sum_{j=1}^{j_0} \mu(\Delta_{ij}) \leq \sum_{i=1}^{i_0} \sum_{j=1}^{n_{i_0}} \mu(\Delta_{ij}) = \sum_{i=1}^{i_0} \mu(\Delta_i) = \frac{i_0}{2^{nN}},$$

$$S(t) \geq \sum_{i=1}^{i_0} \sum_{j=1}^{j_0-1} \mu(\Delta_{ij}) \geq \sum_{i=1}^{i_0-1} \sum_{j=1}^{n_{i_0-1}} \mu(\Delta_{ij}) = \sum_{i=1}^{i_0-1} \mu(\Delta_i) = \frac{i_0 - 1}{2^n N}, \text{ for } i_0 > 1,$$

$$S(t) \geq 0, \text{ for } i_0 = 1.$$

Analogously, the same estimates are checked for the function  $g$ . Thus, for  $t \in \Delta_{i_0 j_0}$  we have

$$|S(t) - g(t)| \leq \frac{1}{2^n N} < \epsilon. \quad (7)$$

Combining (5), (6) and (7), we obtain

$$\|S - g\|_\infty < \epsilon.$$

Since  $\epsilon$ ,  $\{\Delta_{ij}\}$  and  $\{t_{ij}\}$  have been chosen arbitrarily, equality (4) is proved, meaning by  $\int_A f d\mu$  the Riemann integral (for  $A$  of the form  $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ ). But since the Riemann integrability implies RL-integrability, we have shown a stronger result.

It remains to prove that  $f$  does not allow a Bochner-integrable equivalent. Since the Bochner integral is differentiable as the function of the upper limit, it suffices to show that the function

$$G(t) = \int_0^t f d\mu$$

is not differentiable at any point  $t \in (0; 1)$ .

Let  $t_0 \in (0; 1)$ ,  $\Delta t > 0$ . Then

$$\frac{G(t_0 + \Delta t) - G(t_0)}{\Delta t} = \frac{1}{\Delta t} \int_{t_0}^{t_0 + \Delta t} f d\mu.$$

But equality (4) shows that the functions  $\frac{1}{\Delta t} \int_{t_0}^{t_0 + \Delta t} f d\mu$  do not form a fundamental family, as  $\Delta t \rightarrow 0$ . So, this family has no limit and hence  $G(t)$  is not differentiable.

In the example above, note that the values of  $f$  are functions with at most one discontinuity of the first order. The space of all such functions is isomorphic to a subspace of the space  $C(K)$ , where  $K$  is the topological

space known as “two arrows of Alexandrov”. On the other hand, the values of  $\int_A f d\mu$  are all contained in  $C[0; 1]$ . Since  $f$  has been shown to have no a Bochner-integrable equivalent, theorem 3.2 implies the following interesting corollary:

**Corollary 3.5** *The space  $C[0; 1]$  is not complemented in  $C$  on “two arrows” and is not contained in any separable complemented subspace thereof.*

Another question that seems natural to ask is under what condition  $RL_1(X, \mu)$  coincides with  $L_1(X, \mu)$ , i.e. when every RL-integrable function is also Bochner-integrable. Let us present one sufficient condition.

**Theorem 3.6** *Let  $X$  be a Banach space such that every  $X$ -valued RL-integrable function has a Bochner-integrable equivalent. Let  $X$  have a countable set of functionals separating the points of  $X$ . Then every RL-integrable  $X$ -valued function is Bochner-integrable.*

*Proof.* Let  $f : \Omega \rightarrow X$  be an RL-integrable function and  $g \in L_1(\Omega, X)$  its Bochner-integrable equivalent. We show that  $f = g$  almost everywhere. Indeed, for any  $x^* \in X^*$  and any measurable  $A \subset \Omega$  consider:

$$\begin{aligned} \int_A x^*(f - g)d\mu &= \int_A x^* f d\mu - \int_A x^* g d\mu = \\ &= x^*\left(\int_A f d\mu\right) - x^*\left(\int_A g d\mu\right) = 0, \end{aligned}$$

where  $\int_A f d\mu$  denotes  $f$ 's RL-integral and  $\int_A g d\mu$  denotes  $g$ 's Bochner integral. These two integrals are equal, since  $g$  is  $f$ 's Bochner-integrable equivalent. Thus the Lebesgue integral of  $x^*(f - g)$  is zero over any measurable set  $A$ , which means that  $x^*(f - g) = 0$  a.e.

Now let  $\{x_n^*\}_{n=1}^\infty$  be the countable set of functionals separating the points of  $X$ . Put  $A_n = \{t \in \Omega : x_n^*(f - g) \neq 0\}$  and let  $A = \bigcup_{n=1}^\infty A_n$ . Note that for any  $t \in \Omega \setminus A$  and any functional  $x_n^*$ ,  $x_n^*(f - g)(t) = 0$ . Since  $A$  is negligible,  $x_n^*(f - g) = 0$  a.e., therefore  $f$  itself is Bochner-integrable.  $\square$

Let us further investigate the isomorphic structure of  $RL_1(X, \mu)$ . We show that this space can be “very large”, meaning it can contain an isomorphic copy of the space  $l_\infty(\Gamma)$ .

*Example.* Suppose  $X = l_2([0, 1])$  and  $\mu$  is the Lebesgue measure. Then  $RL_1(X, \mu)$  contains an isometrical copy of  $l_\infty([0, 1])$ .

First note that there is a continuum-cardinality family of non-measurable mutually disjoint sets  $\{A_t\}_{t \in [0,1]}$ , where  $A_t \subset [0; 1]$  and the outer measure of each  $A_t$  equals 1. Such a construction can be found, for example, in [2], in the proof of theorem 2.16.

Now consider the following function  $f : [0, 1] \rightarrow l_2([0, 1])$ :  $f(t) = e_t$ , where  $e_t$  is an ort of  $l_2([0, 1])$ :  $e_t = \chi_{\{t\}}$ . Define the linear map  $U : l_\infty([0, 1]) \rightarrow RL_1(X, \mu)$  as follows:

$$U(\alpha) = \sum_{t \in [0,1]} \alpha_t f \cdot \chi_{A_t},$$

whenever  $\alpha = (\alpha_t)_{t \in [0,1]} \in l_\infty([0, 1])$ .

To see that  $U(\alpha)$  is indeed in  $RL_1(X, \mu)$  for every  $\alpha \in l_\infty([0, 1])$ , take any partition  $\Pi = \{\Delta_i\}$  and any sample points  $U = \{t_i\}$  and estimate:

$$\begin{aligned} \|S(U(\alpha), \Pi, T)\| &\leq \|\alpha\|_\infty \left\| \sum_{i=1}^{\infty} e_{t_i} \mu(\Delta_i) \right\| = \\ &= \left( \sum_{i=1}^{\infty} \mu(\Delta_i)^2 \right)^{1/2} \leq \left( \sum_{i=1}^{\infty} d(\Pi) \cdot \mu(\Delta_i) \right)^{1/2} = \\ &= \sqrt{d(\Pi)} \left( \sum_{i=1}^{\infty} \mu(\Delta_i) \right)^{1/2} \leq \sqrt{d(\Pi)} \left( \sum_{i=1}^{\infty} \mu(\Delta_i) \right)^{1/2} = \sqrt{d(\Pi)}, \end{aligned}$$

where  $d(\Pi)$  denotes  $\sup\{\mu(\Delta_i)\}$ . Now take an arbitrary  $\epsilon > 0$ . Fix a partition  $\Pi$  with  $d(\Pi) < \epsilon$ . Then for any  $\Pi' \succ \Pi$ , any  $T'$ ,  $\|S(U(\alpha), \Pi', T')\| \leq \sqrt{d(\Pi')} \leq \sqrt{d(\Pi)} \leq \epsilon$ . Thus  $U(\alpha)$  is RL integrable and the integral equals 0.

Further, it is clear that  $\|U(\alpha)(t)\| \leq \|\alpha\|_\infty$  for every  $t \in [0, 1]$ . So  $\|U(\alpha)\|_{RL} \leq \|\alpha\|_\infty$ . On the other hand, for a fixed  $n \in \mathbb{N}$  any integrable majorant of  $\|U(\alpha)(t)\|$  must be not less than  $\|\alpha\|_\infty - \frac{1}{n}$  on a set of full outer measure, and therefore almost everywhere. Therefore  $\|U(\alpha)\|_{RL} \geq \|\alpha\|_\infty - \frac{1}{n}$ ,  $n \in \mathbb{N}$ . So  $\|U(\alpha)\|_{RL} = \|\alpha\|_\infty$  and we are done.

## 4 Limit Set $I(f)$

Even if a function is not RL-integrable, we still can consider the set of limit points of its RL-integral sums, which, in a sense, plays the role of RL-integral.

**Definition 4.1** Let  $f : \Omega \rightarrow X$  be an arbitrary function with values in a Banach space  $X$ . We say that a point  $x \in X$  belongs to the limit set  $I(f)$  if for every  $\epsilon > 0$  and any partition  $\Pi$  there exists a partition  $\Gamma \succ \Pi$  and a set of sampling points  $T$  such that  $\|S(f, \Gamma, T) - x\| < \epsilon$ ,  $S(f, \Gamma, T)$  being an absolute integral sum. In other words,  $I(f)$  is the set of all limit points of the net of  $f$ 's absolute integral sums.

Let us point out two important facts about the set  $I(f)$  (the proofs can be found in [2]):

1. The limit set  $I(f)$  is always convex.
2. For an arbitrary convex closed set  $S$  of no more than continuum cardinality in a Banach space  $X$ , there exists a function  $f : [0; 1] \rightarrow X$  such that  $I(f) = S$ .

We will show below that the problem of existence of such limit sets is not always solved in positive. However, as the following theorem shows, for a large class of Banach spaces it is.

**Theorem 4.1** Let  $(\Omega, \Sigma, \mu)$  be a measure space,  $X$  a WCG space and  $f \in \overline{L}_1(\Omega, \Sigma, \mu, X)$  a function. Then  $I(f)$  is not empty.

The proof is almost entirely contained in the following lemmas.

**Lma 4.2** Let  $(\Omega, \Sigma, \mu)$  be a probability measure space,  $X$  a WCG space generated by a convex balanced weakly-compact set  $K$ . Suppose  $f : \Omega \rightarrow X$  is an arbitrary function, not necessarily measurable. Then for any sequence of positive real numbers  $\epsilon_k$  there exists a sequence of functions  $g_k : \Omega \rightarrow X$ , sequence of subsets  $A_k \subset \Omega$  and sequence of integers  $n_k \in \mathbb{N}$  so that the following properties hold:

1.  $\mu^*(\bigcap_{i=k}^N A_i) > 1 - \sum_{i=k}^N \epsilon_i$  for any  $k \leq N$ ;
2.  $g_k(A_k) \subset n_k K$ ;
3.  $\|g_k(t) - f(t)\| \leq \epsilon_k$  for  $t \in A_k$  and  $g_k(t) = 0$  for  $t \in \Omega \setminus A_k$ .

*Proof.* Note that  $\bigcup_{n=1}^{\infty} (nK + \epsilon_k B(X)) = X$  for any  $k$ . Due to this, for any  $\epsilon_k$  we can choose an index  $n$  so large that  $\mu^*(f^{-1}(nK + \epsilon_k B(X))) > 1 - \epsilon_k$ . Moreover, for an arbitrary  $B \subset \Omega$  we can choose  $n$  so large that  $\mu^*((f^{-1}(nK + \epsilon_k B(X))) \cap B) \geq \mu^*(B) - \epsilon_k$ . These two observations allow us to construct the sequence  $\{n_k\} \subset \mathbb{N}$  by induction, so that if  $A_k = f^{-1}(n_k K + \epsilon_k B(X))$ , then

$$\mu^*(A_k) > 1 - \epsilon_k,$$

for any  $k$ , and

$$\mu^*(A_k \cap (\bigcap_{i=j}^{k-1} A_i)) \geq \mu^*(\bigcap_{i=j}^{k-1} A_i) - \epsilon_k,$$

for any  $k$  and any  $j < k$ .

It is easy to see now that the first of lemma's conditions is satisfied:

$$\begin{aligned} \mu^*\left(\bigcap_{i=k}^N A_i\right) &= \mu^*(A_N \cap (\bigcap_{i=k}^{N-1} A_i)) \geq \mu^*(\bigcap_{i=k}^{N-1} A_i) - \epsilon_N \\ &\geq \mu^*(\bigcap_{i=k}^{N-2} A_i) - \epsilon_N - \epsilon_{N-1} \geq \dots \geq \mu^*(A_k) - \sum_{i=k+1}^N \epsilon_i \\ &\geq 1 - \sum_{i=k}^N \epsilon_i. \end{aligned}$$

Now note that by the construction,

$$f(A_k) \subset n_k K + \epsilon_k B(X). \quad (8)$$

This allows us to construct functions  $g_k$  satisfying conditions 2 and 3 of the lemma. Indeed, we put  $g_k(t) = 0$  for  $t \in \Omega \setminus A_k$ . If  $t \in A_k$ , then by (8), there exists a point  $y \in n_k K$  such that  $\|f(t) - y\| \leq \epsilon_k$ . We put then  $g_k(t) = y$ . This finishes the proof.  $\square$

**Definition 4.2** Let  $g : \Omega \rightarrow X$  be a function,  $A \subset \Omega$ . We denote by  $\sigma(g|A)$  the set of all integral sums of  $g$ , controlled by  $A$ , that is, a sampling point is always chosen in  $A$ , whenever the partition subset intersects with  $A$ . We also denote by  $I(g|A)$  the set of limits of the integral sums from  $\sigma(g|A)$ .

**Lma 4.3** Suppose a function  $f : \Omega \rightarrow X$  has an integrable majorant  $h$ :  $\|f(t)\| \leq h(t)$ ,  $h \in L_1(\Omega, \Sigma, \mu)$ . Let the real numbers  $\epsilon_n > 0$  be so small that if  $A \in \Sigma$ ,  $\mu(A) < \epsilon_n$ , then

$$\int_A h d\mu < 2^{-n-1}. \quad (9)$$

Assume also that  $\epsilon_1 < \frac{1}{4}$  and  $\epsilon_{n+1} < \frac{1}{2^n} \epsilon_n$ . Then, under the conditions of lemma 4.2, there exists a convergent sequence  $\{x_n\} \subset X$  such that  $x_n \in I(g_n|A_n)$ .

*Proof.* On the set of all partitions define an ultrafilter  $\mathcal{U}$ , which dominates the filter of the refinement direction.

For every partition  $\Gamma = \{\Delta_i\}_{i=1}^\infty$  define a number  $N(\Gamma) \in \mathbb{N}$  so that  $N(\Gamma) \rightarrow \infty$ , as  $\Gamma$  gets finer (e.g.,  $N(\Gamma) = [1/\sup_i \mu(\Delta_i)]$ ). Construct the integral sums

$$S_k(\Gamma) = \sum_{j=1}^{\infty} g_k(t_j^{(k)}(\Gamma)) \mu(\Delta_j) \in \sigma(g_k|A_k),$$

where  $k = 1, 2, \dots, N(\Gamma)$ , so that the following condition is satisfied: if for some  $j \in \mathbb{N}$  there exists an index  $s \leq N(\Gamma)$  such that

$$\Delta_j \cap \bigcap_{i=s}^{N(\Gamma)} A_i \neq \emptyset$$

(denote the smallest such index by  $s(j)$ ), then

$$t_j^{(k)}(\Gamma) = t_j^{(s(j))}(\Gamma) \in \bigcap_{i=s(j)}^{N(\Gamma)} A_i$$

for  $k = s(j), s(j) + 1, \dots, N(\Gamma)$ .

Note that under such a choice of sampling points, in view of the first condition of lemma 4.2, for each  $k$  the total measure of all those  $\Delta_j$  where  $t_j^{(k)} = t_j^{(k+1)}$ , is bounded below by

$$1 - \sum_{i=k}^{N(\Gamma)} \epsilon_i > 1 - \epsilon_{k-1}.$$

Since  $h + \epsilon_k$  is a majorant for  $g_k$ , we have

$$\lim_{\mathcal{U}} \|S_k(\Gamma) - S_{k+1}(\Gamma)\| \leq 2\epsilon_k + 2^{-k}. \quad (10)$$

Since for any  $k$ , given a sufficiently fine partition  $\Gamma$ , the number  $N(\Gamma)$  is greater than  $k$ , the sum  $S_k(\Gamma)$  will be correctly defined and, due to condition 2 of lemma 4.2,  $S_k(\Gamma) \in n_k K$ . From the compactness argument we infer that there exists a weak limit  $w - \lim_{\mathcal{U}} S_k(\Gamma)$ . Denote this limit by  $x_k$ . It was proved in [2] that actually  $x_k \in I(g_k|A_k)$ . Due to (10),  $\{x_k\}$  forms a fundamental sequence. Thus  $\{x_k\}$  is required.  $\square$

*Proof of theorem 4.1.*

Let  $\{x_k\}$  be the sequence from lemma 4.3,  $x = \lim_{k \rightarrow \infty} x_k$ . Due to the condition  $\mu^*(A_k) > 1 - \epsilon_k$ , condition 3 of lemma 4.2 and (9), each  $x_k$  can be approximated with the precision of  $\epsilon_k + \frac{1}{2^k}$  by arbitrary fine integral sums of  $f$ . Therefore  $x \in I(f)$ , which is to be proved.  $\square$

So, we have shown that  $I(f)$  is non-empty for any  $f \in \overline{L_1}(X)$  for quite a large class of WCG spaces. This result supersedes the one of [2], where an analogous theorem is shown for separable and reflexive spaces  $X$ . However, there exist spaces where  $I(f)$  can be empty for certain functions  $f \in \overline{L_1}(X)$ . Let us exhibit an example.

*Example.* Consider the space  $X = l_1([0; 1])$  and function  $f : [0; 1] \rightarrow l_1([0; 1])$  given by  $f(t) = e_t$ , i.e.  $f(t) = \chi_{\{t\}}$ . Obviously,  $f$  is bounded and therefore  $f \in \overline{L_1}(X)$ . It is easy to see that the  $l_1$ -norm of any integral sum  $S(f, \Gamma, T)$  of  $f$  equals 1. Suppose that there exists an  $x \in I(f)$ . Then  $\|x\|_{l_1} = 1$ . Consider the coordinate functionals  $\delta_t \in (l_1[0; 1])^*$ :  $\delta_t(g) = g(t)$ . For every  $t \in [0; 1]$  we have  $\delta_t(f(\tau)) = 0$  for almost all  $\tau \in [0; 1]$  (in fact, for all  $\tau \neq t$ ). Since  $x \neq 0$ , there exists a  $t_0 \in [0; 1]$  such that  $\delta_{t_0}(x) \neq 0$ . On the other hand,

$$\delta_{t_0}(x) \in \delta_{t_0}(I(f)) \subset I(\delta_{t_0} \circ f) = \{0\}.$$

This contradiction shows that  $I(f) = \emptyset$ .

If a function  $f$  is RL-integrable, then  $I(f)$  obviously consists of a single point,  $f$ 's integral. The converse is not always true: a function may have a single-point  $I(f)$  and still not be RL-integrable. The following weaker statement is true, however.

**Theorem 4.4** *Let  $X$  be a Banach space such that every function  $f \in \overline{L_1}(\Omega, \Sigma, \mu, X)$  from any measure space  $(\Omega, \Sigma, \mu)$  has a non-empty limit set  $I(f)$ . Let a function  $f \in \overline{L_1}(\Omega, \Sigma, \mu, X)$  be such that  $I(f)$  consists of a single point  $x$ . Then  $f$  is Pettis integrable and  $x$  is  $f$ 's Pettis integral.*

Let us prove some lemmas first.

**Lma 4.5** *Let  $X$  and  $f$  be such as in theorem 4.4 and let  $A \in \Sigma$ . Then for the restriction  $f|_A$  of  $f$  on  $A$ ,  $I(f|_A)$  is a singleton.*

*Proof.* Due to the properties of  $X$ ,  $I(f|_A)$  cannot be empty. To prove that  $I(f|_A)$  is a singleton, assume there are two distinct points  $x_1, x_2 \in I(f|_A)$ . Fix an  $\epsilon > 0$ . Denote  $B = \Omega \setminus A$  and pick any  $y \in I(f|_B)$ .

Now take an arbitrary partition  $\Pi$  of  $\Omega$ . Let  $\Pi'$  be a partition that is finer than both  $\Pi$  and the partition of  $\Omega$  into two sets  $A$  and  $B$ . Then every member set of  $\Pi'$  is either a subset of  $A$  or a subset of  $B$ . Therefore we can consider the partition  $\Pi^A$  of  $A$  formed by those members of  $\Pi'$  that lie within  $A$ , and a partition  $\Pi^B$  of  $B$  formed by those members of  $\Pi'$  that lie within  $B$ .

Since  $x_1 \in I(f|_A)$ , there exists a partition  $\Pi_1^A \succ \Pi^A$  and a set of sampling points  $T_1^A$  for it such that  $\|S(f|_A, \Pi_1^A, T_1^A) - x_1\| < \epsilon/2$ . Since also  $x_2 \in I(f|_A)$ , there exists a partition  $\Pi_2^A \succ \Pi^A$  and a set of sampling points  $T_2^A$  for it such that  $\|S(f|_A, \Pi_2^A, T_2^A) - x_2\| < \epsilon/2$ . And since  $y \in I(f|_B)$ , there exists a partition  $\Pi_1^B \succ \Pi^B$  and a set of sampling points  $T_1^B$  for it such that  $\|S(f|_B, \Pi_1^B, T_1^B) - y\| < \epsilon/2$ . Combine the partitions  $\Pi_1^A$  and  $\Pi_1^B$  into partition  $\Pi_1$  of the entire  $\Omega$ , and put  $T_1 = T_1^A \cup T_1^B$ . Then  $\|S(f, \Pi_1, T_1) - (x_1 + y)\| < \epsilon$ . At the same time combine the partitions  $\Pi_2^A$  and  $\Pi_1^B$  into partition  $\Pi_2$  of the entire  $\Omega$ , and put  $T_2 = T_2^A \cup T_1^B$ . Then  $\|S(f, \Pi_2, T_2) - (x_2 + y)\| < \epsilon$ . Since  $\Pi_1 \succ \Pi$  and  $\Pi_2 \succ \Pi$ , both  $x_1 + y$  and  $x_2 + y$  belong to  $I(f)$ , which is impossible. Hence  $I(f|_A)$  consists of a single point.  $\square$

**Lma 4.6** *Let  $X$  and  $f$  be such as in theorem 4.4. Then  $f$  is weakly measurable.*

*Proof.* Assume the contrary. Then there exists a functional  $x^* \in X^*$  such that  $x^*f$  is a non-measurable function. Since  $f \in \overline{L_1}(\Omega, \Sigma, \mu, X)$ ,  $x^*f$  must have an integrable (and hence measurable) majorant. Therefore, there must exist the smallest measurable majorant  $f_2$  of  $x^*f$  and the largest measurable

minorant  $f_1$  of  $x^*f$ . In other words, if  $g : \Omega \rightarrow \mathbb{R}$  is measurable and  $x^*f \leq g$  a.e., then  $f_2 \leq g$  a.e., and similarly, if  $g : \Omega \rightarrow \mathbb{R}$  is measurable and  $x^*f \geq g$  a.e., then  $f_1 \geq g$  a.e. Note that  $f_1$  and  $f_2$  cannot coincide almost everywhere, since that would mean that  $f_1 = x^*f = f_2$  a.e. and  $x^*f$  would be measurable. Note that  $\{t : f_1(t) \neq f_2(t)\} = \bigcup_{n=1}^{\infty} \{t : f_2(t) - f_1(t) > 1/n\}$ . Since the set at the left-hand side is non-negligible, one of the sets on the right-hand side must be non-negligible too. Therefore, there exists a non-negligible measurable set  $A$  and an  $\epsilon > 0$  such that  $f_1(t) < f_2(t) - \epsilon$  for any  $t \in A$ .

Consider  $f|_A$ . Due to lemma 4.5,  $I(f|_A)$  consists of a single point. Consider the following two sets:

$$\begin{aligned} A_1 &= \{t \in A : x^*f(t) > \frac{2}{3}f_2(t) + \frac{1}{3}f_1(t)\} \\ A_2 &= \{t \in A : x^*f(t) < \frac{2}{3}f_1(t) + \frac{1}{3}f_2(t)\} \end{aligned}$$

and let  $B_1 = A \setminus A_1$ ,  $B_2 = A \setminus A_2$ . Note that  $B_1$  cannot contain any measurable non-negligible set. Indeed, assume that  $C$  is a measurable set,  $\mu(C) > 0$  and  $C \subset B_1$ . This means that for any  $t \in C$ ,  $x^*f(t) \leq \frac{2}{3}f_2(t) + \frac{1}{3}f_1(t) < f_2(t)$ . But now consider function  $g$ , which is equal to  $\frac{2}{3}f_2(t) + \frac{1}{3}f_1(t)$  for  $t \in C$  and coincides with  $f_2$  outside of  $C$ . This function is measurable, it is a majorant of  $x^*f$ , but  $g < f$  on a non-negligible set  $C$ . This contradicts the definition of  $f_2$  as the smallest measurable majorant of  $x^*f$ . Thus we have shown that  $B_1$  contains no measurable non-negligible subset, which means that  $\mu_*(B_1) = 0$  and  $\mu^*(A_1) = \mu(A)$ . It is easy to apply the same argument to  $B_2$  and  $A_2$  to show that  $\mu^*(A_2) = \mu(A)$ .

Now consider a  $\sigma$ -field  $\Sigma_{A_1}$  of subsets of  $A_1$  of the form  $C \cap A_1$ , where  $C \in \Sigma$ . Define  $\mu|_A(C \cap A_1) = \mu(C \cap A)$ . So, we obtain a measure space  $(A_1, \Sigma_{A_1}, \mu|_A)$ . It is easy to verify that this space is correctly defined, since  $\mu^*(A_1) = \mu(A)$ . The restriction  $f|_{A_1}$  is a function from this measure space to the Banach space  $X$ . Due to the properties of  $X$ , there exists an  $x_1 \in I(f|_{A_1})$ . By an analogous argument we can construct the measure space  $(A_2, \Sigma_{A_2}, \mu|_A)$  and find a point  $x_2 \in I(f|_{A_2})$ .

Let us show that  $x_1 \neq x_2$ . Indeed, note that  $x^*(x_1) \in I(x^*f|_{A_1})$  and  $x^*(x_2) \in I(x^*f|_{A_2})$ . Consider an integral sum of  $x^*f|_{A_1}$ . It has the form  $\sum x^*f(t_i)\mu(\Delta_i \cap A_1) = \sum x^*f(t_i)\mu(\Delta_i)$ , where  $\{\Delta_i\}$  is a partition of  $A$  and  $t_i \in \Delta_i \cap A_1$ . Thus all integral sums of  $x^*f|_{A_1}$  dominate the integral sums of the function  $\frac{2}{3}f_2(t) + \frac{1}{3}f_1(t)$  over  $A$ . On the other hand, the same argument shows that all integral sums of  $x^*f|_{A_2}$  are dominated by the integral sums of

the function  $\frac{2}{3}f_1(t) + \frac{1}{3}f_2(t)$  over  $A$ . Since the values of these two functions differ by at least  $\epsilon/3$  at all points of  $A$ , this implies that  $x^*(x_1) > x^*(x_2)$ , which means that  $x_1 \neq x_2$ .

Let us show that  $x_1, x_2 \in I(f|_A)$ . Indeed, since  $\mu^*(A_1) = \mu^*(A_2) = \mu(A)$ , any integral sum over  $A_1$  of the form  $\sum f(t_i)\mu(\Delta_i \cap A_1)$  is equal to the integral sum  $\sum f(t_i)\mu(\Delta_i)$  over  $A$ , and the same is true for  $A_2$ . So, we have found two different points  $x_1$  and  $x_2$  in  $I(f|_A)$ , which is impossible. This contradiction proves that  $f$  is weakly measurable.  $\square$

*Proof of theorem 4.4.*

The previous lemma shows that the function  $f$  is weakly measurable. Take any measurable  $A \subset \Omega$ . Take an  $x^* \in X^*$ . The real-valued function  $x^*f|_A$  is measurable. Since  $f$  has an integrable majorant, so does  $x^*f|_A$ . Therefore  $x^*f|_A$  is Lebesgue-integrable and thus, RL-integrable. Hence  $I(x^*f|_A)$  consists of a single point,  $\int_A x^*f d\mu$ . Let  $x_A$  be the only point of  $I(f|_A)$ . Since  $x^*(x_A) \in I(x^*f|_A)$ , we have  $x^*(x_A) = \int_A x^*f d\mu$ , and this holds for any functional  $x^* \in X^*$  and any subset  $A \in \Sigma$ . So  $f$  is Pettis integrable and  $\int f d\mu = x$ .  $\square$

## References

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