OPERATORS AND INTEGRALS IN BANACH SPACES

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ABSTRACT
This work consists of three parts.
In the first part we develop the general theory of Daugavet spaces, i.e those where the Daugavet equation \( \| \Id + T \| = 1 + \| T \| \) holds for all 1-dimensional operators. The following are results proved:

- The Daugavet equation holds for all operators not fixing a copy of \( \ell_1 \);
- If \( X \subset Y \) and \( X \) has the Daugavet property, then \( Y \) can be renormed so that the pair \( (X,Y) \) has the Daugavet property;
- No Daugavet space can be decomposed into an unconditional sum of spaces not containing a copy of \( \ell_1 \);
- There is an almost narrow quotient map \( Q: L_1[0,1] \rightarrow L_1[0,1]/Y \), which is not narrow.

In the second part we create a new technic of proving Krein-Šmulian-type theorems using the notion of a Riemann-Lebesgue integral sum. In particular, we show that if \( \tau \) is a topology on a WCG-space weaker than the norm topology, then the \( \tau \)-closed convex hull of every norm bounded \( \tau \)-compact in it is \( \tau \)-compact. Applications to the Boundary Problem are obtained.

The third part is devoted to strongly continuous operator semigroups. Our main result says that a semigroup \( \{ e^{tA} \}_{t \leq 0} \) on a Banach space \( X \) is hyperbolic if and only if \( s \rightarrow (is-A)^{-1} \) defines an operator-valued Fourier multiplier on \( L_p(\mathbb{R}, X) \). Analogues for the \( \alpha \)-smooth domains are found.
# TABLE OF CONTENTS

ACKNOWLEDGMENTS .................................................................................. ii
ABSTRACT ........................................................................................................ iii

1 The Daugavet property of Banach spaces ................................................. 1
  1.1 Introduction ............................................................................................. 1
  1.2 Basic theory ............................................................................................ 5
    1.2.1 Definition and equivalent reformulations ....................................... 5
    1.2.2 Examples .......................................................................................... 9
    1.2.3 Copies of $\ell_1$ and $L_1$ ................................................................. 12
  1.3 Narrow and almost narrow operators ....................................................... 14
    1.3.1 Extension of the Daugavet equation .............................................. 14
    1.3.2 Application to unconditional decompositions ............................... 18
    1.3.3 Narrow operators and rich subspaces ............................................ 20
    1.3.4 Example of almost narrow, but not narrow operator ................. 23
    1.3.5 $C$-narrow operators in $C(K)$ and $C(K,E)$ .................................. 27
    1.3.6 $L_1$-narrow and other operators on $L_1(\Omega)$ .......................... 35
  1.4 Daugavet renormings .............................................................................. 40
    1.4.1 Subspaces of $C(K)$ ......................................................................... 41
    1.4.2 Auxiliary space $m_0(K)$ ................................................................. 42
    1.4.3 The main renorming theorem .......................................................... 44
    1.4.4 More on pairs $(C(K), Y)$ ............................................................... 47
    1.4.5 More on pairs $(L_1(\Omega), Y)$ ........................................................ 48
  1.5 Hereditary Daugavet equation ................................................................. 50
    1.5.1 Hereditary Daugavet equation in $L_1(\Omega)$ ............................... 50
    1.5.2 Hereditary Daugavet equation in $C(K)$ ......................................... 55

2 On Riemann-Lebesgue integral sums and limit sets ........................... 58
  2.1 Introduction and basic definitions .......................................................... 58
  2.2 Limit sets in locally convex spaces ......................................................... 61
    2.2.1 Preliminaries .................................................................................. 61
    2.2.2 Krein-Šmulian-type theorems ....................................................... 63
    2.2.3 Another proof of B. Cascales, G. Manjabacas and G. Vera’s result .... 67
  2.3 Limit sets in Banach spaces ................................................................... 72
    2.3.1 $\aleph$-convex sets .......................................................................... 73
    2.3.2 The main theorem ............................................................................ 75
### 3 Hyperbolic semigroups and Fourier multipliers  

3.1 Introduction  
3.2 Characterization of hyperbolicity  
3.3 Extension to the case $\alpha > 0$  
3.4 An $\alpha$-analogue of hyperbolicity  
3.5 Strong $\alpha$-hyperbolicity  

REFERENCES
Chapter 1

The Daugavet property of Banach spaces

1.1 Introduction

It is a remarkable result proved by I. Daugavet [21] that the identity, called the Daugavet equation,

\[ \| \text{Id} + T \| = 1 + \| T \|, \]

(1.1.1)

holds for every compact operator \( T \) on \( C[0,1] \). Shortly afterwards, the same result was proved in \( L_1[0,1] \) by G. Lozanovskii, [57]. In a while the Daugavet equation was extended to all weakly compact operators on atomless \( L_1(\mu) \) and \( C(K) \) (see [5, 29, 38, 39]).

Are \( L_1(\mu) \) and \( C(K) \) the only Banach spaces where all compact operators satisfy (1.1.1)? At first glance naive, this question stood open for certain time until Y. Abramovich [2] discovered other examples such as \( L_1(\mu) \oplus_\infty L_1(\nu) \) and \( L_\infty(\mu) \oplus_1 L_\infty(\nu) \).

The appearance of new spaces spured to development of a general theory, and the following notion was introduced:

\( X \) is called a Daugavet space (or \( X \) has the Daugavet property) if the Daugavet equation (1.1.1) holds for all compact operators on \( X \).

Soon after Abramovich’s work, P. Wojtaszczyk showed that \( \ell_1 \) and \( \ell_\infty \)-sums of any number of Daugavet spaces was again a Daugavet space, [89].
In the same work he observed that every Daugavet space failed to contain a strongly exposed point on its unit ball, thus proving that a Daugavet space does not have the Radon-Nikodým property, and in particular, cannot be reflexive.

Partially generalizing the classical theorem of A. Pelczyński on impossibility to embed \( C[0, 1] \) or \( L_1[0, 1] \) into a space with an unconditional basis, [61], V. Kadets proved by means of a simple lemma (see Lemma 1.3.8) that no space with the Daugavet property may possess an unconditional Schauder decomposition. This result had been previously obtained by N. Kalton for \( L_1[0, 1] \), [50].

Meanwhile, in [44, 48, 63, 87, 89] and more recent works [8, 9, 10, 59] new Daugavet spaces were found among uniform and C*-algebras, certain "large" subspaces of \( L_1(\mu) \) and \( C(K) \), and their vector-valued counterparts (see Section 1.2.2). Besides, a large class of narrow operators satisfying (1.1.1) on \( L_1[0, 1] \) and including those not fixing a copy of \( L_1[0, 1] \) was introduced by A. Plichko and M. Popov in [63]. Later, V. Kadets and M. Popov proved (1.1.1) for a new class of narrow operators on \( C[0, 1] \), which also included those not fixing a copy of \( C[0, 1] \) (for the latter class the Daugavet equation was shown earlier by D. Werner and L. Weis in [84]).

From these results in much the same way we deduce that neither \( C[0, 1] \) nor \( L_1[0, 1] \) is representable as an unconditional sum of spaces without copies of \( C[0, 1] \) and \( L_1[0, 1] \), respectively.

One may still wonder whether \( C[0, 1] \) or \( L_1[0, 1] \) can be embedded into such a sum. Pelczyński’s theorem suggests they cannot, however to prove it via Kadets’s scheme we have to be able to renorm the given space \( Y \) containing \( C[0, 1] \) (\( L_1[0, 1] \)) in such a way that the new norm coincides with the old one on \( C[0, 1] \) (\( L_1[0, 1] \)) and every operator \( T: C[0, 1] \mapsto Y \) (resp. \( T: L_1[0, 1] \mapsto Y \)) not fixing a copy of the corresponding space satisfies the Daugavet equation with \( \text{Id} \) replaced by the inclusion map.

This problem, first of all, motivated the study of Daugavet property not only for a single space, but also for a pair of embedded Banach spaces (this approach is adapted in the present work):

A pair \((X, Y), X \subseteq Y\) has the Daugavet property if the equality

\[ \| J + T \| = 1 + \| T \| \]

holds for every compact operator \( T: X \mapsto Y \), where \( J: X \mapsto Y \) is the inclusion map.
Secondly, it demanded the appropriate renorming theorem for pairs.

This program was partially completed in [42, 45, 71], and now presented in full generality in Section 1.4. In particular, we prove (Theorem 1.4.9) that if $X$ is a Daugavet space and a $Y$ contains $X$, then $Y$ can be renormed in such a way that $(X, Y)$ has the Daugavet property and the new norm coincides with the old one on $X$ (see [73] for another version of this result for pairs).

The next layer of the theory was laid in works [49, 71, 73]. Its novelty consists in reformulation of equation (1.1.1) in terms of an easy to verify geometric condition (*): 

For every $x$ and $x^*$ on the unit spheres of $X$ and $X^*$, respectively, and for every $\varepsilon > 0$, there is a $y \in X$, $\|y\| = 1$, such that $\|x + y\| > 2 - \varepsilon$ and $x^*(y) > 1 - \varepsilon$.

Taking $x$ such that $x^*(x) > \alpha$ and applying (*)& to $-x$ one easily checks that every slice $\{x \in X : \|x\| \leq 1, x^*(x) > \alpha\}$ has diameter 2, which recovers Wojtaszczyk’s result on the absence of strongly exposed points. Also, using this condition one can easily prove the Daugavet property for almost all known examples in a much easier way (see Section 1.2.2).

Next, via repeated application of a slightly modified equivalent condition, one finds asymptotically isometric copies of $\ell_1$ in every Daugavet space, and due to [24, Theorem 2], an isometric copy of $L_1[0,1]$ in its dual (see Section 1.2.3).

The acquired tool enables us to extend the Daugavet equation to other classes of operators on Daugavet spaces. It will turn out that (1.1.1) holds for all strong Radon-Nikodým operators and operators not fixing a copy of $\ell_1$, if it merely holds for 1-dimensional ones. In fact, following [47] we define a class of almost narrow operators for which (1.1.1) holds trivially, and then prove that, it contains all the mentioned classes when considered on a Daugavet space, (Propositions 1.3.2 and 1.3.4)\(^1\).

Another important question is to find subspaces of a Daugavet space, which inherit the Daugavet property. We try to locate those examples from certain class of "large" subspaces, where the "largeness" is measured by "smallness" of the corresponding factors or their quotient maps. For this purpose we introduce the class of narrow operators, of course containing the

\(^1\)The proof of Proposition 1.3.4 is rather involved and was found without author’s contribution. Therefore, we do not include it (see [47, Section 4]). Instead, we show directly that any operator not fixing a copy of $\ell_1$ satisfies (1.1.1) (see Theorem 1.3.5)
class of almost narrow ones, for which the desired assertion holds (Proposition 1.3.13):

If $X$ has the Daugavet property and $T: X \mapsto X/Y$ is narrow, then $Y$ has the Daugavet property too. We call such a $Y$ rich.

Remark that there is an example of an almost narrow quotient map $Q: L_1[0,1] \mapsto L_1[0,1]/Y$ with the non-daugavetian kernel $Y$ (see Section 1.3.4).

Further we notice that the considered strong Radon-Nikodým operators and operators not fixing a copy of $\ell_1$ are, in fact, narrow, thus proving that every subspace of a Daugavet space, whose factor is either Radon-Nikodým or $\ell_1$-free is rich and as such inherits the Daugavet property (Section 1.3.3).

The relationship between the new and earlier introduced in [44, 63] old notions of a narrow operator on $C(K)$ and $L_1(\mu)$ (called $C$-narrow and $L_1$-narrow, resp.) is discussed in Sections 1.3.5 and 1.3.6.

In the $C(K)$ case the situation is simple (Proposition 1.3.24): the classes of almost narrow, narrow and $C$-narrow operators coincide. However, it is not that simple in $L_1(\mu)$. In general, every $L_1$-narrow operator is narrow and every narrow is almost narrow (Proposition 1.3.40); while there is an example of an almost narrow not narrow operator on $L_1[0,1]$ into another space $(L_1[0,1]/Y)$ (see Section 1.3.4). In all other possible cases no conclusive answer is known.

We further prove that the classes of $C$-narrow and $L_1$-narrow operators are stable under taking strongly unconditional sums. Same property is proved for operators not fixing a copy of $C[0,1]$ on $C[0,1]$ and not fixing a copy of $L_1[0,1]$ on $L_1[0,1]$. In particular, the latter generalizes the classical result of P. Enflo and T. Starbird, [26].

In Section 1.5.1 we address the question of finding the largest linear class of operator on $L_1(\mu)$ satisfying the Daugavet equation. This task appears to be unfeasible, as (1.1.1) itself is too weak to substantially influence structural properties of the operator. So, we introduce another type of a Daugavet equation in $L_1(\mu)$, called hereditary:

$$\|I\|_{L_1(A)} + \|T\|_{L_1(A)} = 1 + \|T\|_{L_1(A)},$$

for all measurable $A$, (1.1.2)

which, in particular, holds for all $L_1$-narrow operators. Then we prove that the desired largest linear class of operators satisfying (1.1.2) is defined by the following condition ([72]):
For every $\varepsilon > 0$ and measurable non-negligible $A$, there is a measurable $B \subset A$ with such that $\left\| \chi_B \cdot T \left( \frac{\chi_B}{\mu(B)} \right) \right\| < \varepsilon$.

At first glance, it reminds some kind of disjointness with the identity operator. And indeed, A. Schep has recently found that this condition is equivalent to $T$ to be disjoint with Id in the lattice sense (see [69]). He also obtained analogues of our operator class and the hereditary Daugavet equation in all $L_p(\mu)$.

Similar scheme is carried out for $C(K)$ in Section 1.5.2. An analogue of equation (1.1.2) is introduced and proved to imply a strong singular property of the operator, forcing it to be almost diffuse (see Definition 1.5.6) and, in particular, $C$-narrow.

Finally, we remark that the Daugavet equation is currently being studied under, in a sense, opposite conditions on the underlying Banach space. There is a theory of so-called anti-Daugavet property mainly dealing with spaces such as uniformly convex (see [1, 48, 49]). Other versions of (1.1.1) for the $L_p$-spaces appeared in works [59, 63, 64, 69, 76].

We hope that the Daugavet theory will gain more interest in the future and find applications in many other areas of mathematics.

1.2 Basic theory

In this section we introduce the Daugavet property for pairs of Banach spaces and establish some basic equivalent reformulations in terms of geometrical structure of their unit balls. As a consequence, we prove that every Daugavet space contains a copy of $\ell_1$.

We further present first classical examples of Daugavet spaces, such as $C(K)$, $L_1(\Omega)$, $L_\infty(\Omega)$ and their vector-valued counterparts.

1.2.1 Definition and equivalent reformulations

Throughout this section $X,Y$ denote Banach spaces. We assume that $X$ is a closed subspace of $Y$. The unit ball of $X$ is denoted by $B(X)$, and $S(X)$ stands for the unit sphere.

Definition 1.2.1. Let $J: X \hookrightarrow Y$ denote the inclusion map. We say that the pair $(X,Y)$ has the Daugavet property (or is a Daugavet pair) if for every
rank-1 bounded linear operator $T$ from $X$ to $Y$ the following identity

$$\|J + T\| = 1 + \|T\|,$$

(1.2.1)

which is called the Daugavet equation, holds. If (1.2.1) is satisfied by operators from some class $\mathcal{M}$ we say that $(X, Y)$ has the Daugavet property with respect to $\mathcal{M}$.

If $X = Y$ we simply call $X$ a Daugavet space.

Some short remarks are in order.

Firstly, it is not hard to show that (1.2.1) is enough to check only for operators of norm 1 (see, [1]).

Secondly, if $X^*$ has the Daugavet property, then $X$ does so too. The converse is not true (see Section 1.2.2).

Thirdly, it will turn out that the Daugavet equation extends automatically to much wider classes of operators such as weakly compact ones (Section 1.3).

**Definition 1.2.2.** A slice of the unit ball $B(X)$ is a set given by

$$S(x^*, \varepsilon) = \{x \in B(X) : x^*(x) > 1 - \varepsilon\},$$

where $x^* \in X^*$ and $\varepsilon > 0$. We always assume that $x^* \in S(X^*)$. If $X$ is a dual space and $x^*$ is taken from the predual, then $S(x^*, \varepsilon)$ is called a weak*-slice.

In the following lemma we restate the Daugavet property in terms of slices.

**Lemma 1.2.3.** The following conditions are equivalent:

(a) The pair $(X, Y)$ has the Daugavet property;

(b) For every $y_0 \in S(Y)$ and for every slice $S(x^*_0, \varepsilon_0)$ of $B(X)$ there is another slice $S(x^*_1, \varepsilon_1) \subset S(x^*_0, \varepsilon_0)$ of $B(X)$ such that for every $x \in S(x^*_1, \varepsilon_1)$ the inequality $\|x + y_0\| \geq 2 - \varepsilon_0$ holds;

(c) For every $x^*_0 \in S(X^*)$ and for every weak*-slice $S(y_0, \varepsilon_0)$ of $B(Y^*)$ there is another weak*-slice $S(y_1, \varepsilon_1) \subset S(y_0, \varepsilon_0)$ of $B(Y^*)$ such that for every $y^* \in S(y_1, \varepsilon_1)$ the inequality $\|x^*_0 + y^*_1\| \geq 2 - \varepsilon_0$ holds.
Proof. (a)⇒(b). Define \( T: X \to Y \) by \( Tx = x_0^*(x)y_0 \). Then \( \|J^* + T^*\| = \|J + T\| = 2 \), so there is a functional \( y^* \in S(Y^*) \) such that \( \|J^*y^* + T^*y^*\| \geq 2 - \varepsilon_0 \) and \( y^*(y_0) \geq 0 \). Put

\[
x_1^* = \frac{J^*y^* + T^*y^*}{\|J^*y^* + T^*y^*\|}, \quad \varepsilon_1 = 1 - \frac{2 - \varepsilon_0}{\|J^*y^* + T^*y^*\|}.
\]

Then for all \( x \in S(x_1^*, \varepsilon_1) \) we have

\[
\langle (J^* + T^*)y^*, x \rangle \geq (1 - \varepsilon_1)\|J^*y^* + T^*y^*\| = 2 - \varepsilon_0,
\]

therefore

\[
y^*(x) + y^*(y_0)x_0^*(x) \geq 2 - \varepsilon_0, \quad (1.2.2)
\]

which implies that \( x_0^*(x) \geq 1 - \varepsilon_0 \), i.e., \( x \in S(x_0^*, \varepsilon_0) \). Moreover, by (1.2.2) we have \( y^*(x) + y^*(y_0) \geq 2 - \varepsilon_0 \) and hence \( \|x + y_0\| \geq 2 - \varepsilon_0 \).

(b)⇒(a). Let \( T \in \mathcal{L}(X,Y) \), \( Tx = x_0^*(x)y_0 \) be a rank one operator. We can assume that \( \|T\| = 1 \) (see, for example, [1]) and \( \|x_0^*\| = \|y_0\| = 1 \). Fix any \( \varepsilon > 0 \). Then there is an \( x \in S(x_0^*, \frac{\varepsilon}{2}) \) such that \( \|x + y_0\| > 2 - \frac{\varepsilon}{2} \). So,

\[
\|J + T\| \geq \|x + x_0^*(x)y_0\| \geq \|x + y_0\| - |1 - x_0^*(x)| > 2 - \varepsilon.
\]

Let \( \varepsilon \) go to zero.

The proof of equivalence (a)⇔(c) is analogous.

In similar way it is not hard to see that the slices \( S(x_1^*, \varepsilon_1) \) and \( S(y_1, \varepsilon_1) \) in (b) and (c) can be replaced by single vectors \( x \) and \( y^* \). We will record this in the following lemma for future reference.

**Lemma 1.2.4.** The following assertions are equivalent:

(a) The pair \((X,Y)\) has the Daugavet property.

(b) For every \( y \in S(Y) \), \( x^* \in S(X^*) \) and \( \varepsilon > 0 \) there is some \( x \in S(X) \) such that \( x^*(x) \geq 1 - \varepsilon \) and \( \|x + y\| \geq 2 - \varepsilon \).

(c) For every \( y \in S(Y) \), \( x^* \in S(X^*) \) and \( \varepsilon > 0 \) there is some \( y^* \in S(Y^*) \) such that \( y^*(y) \geq 1 - \varepsilon \) and \( \|x^* + y^*\|_X \geq 2 - \varepsilon \).
As a consequence of these lemmas we get that every slice of $B_X$ and every weak*-slice of $B(X^*)$ has diameter 2 if $X$ has the Daugavet property. Thus, $X$ fails the Radon-Nikodým property, a fact originally due to P. Wojtaszczyk [89]. In particular, $X^*$ is non-reflexive. Likewise, $X^*$ fails the Radon-Nikodým property.

Our last result in this section is a version of Lemma 1.2.3 for traces of weak-open sets on the unit ball. It will prove useful later on in Section 1.3.

**Lemma 1.2.5.** The following assertions are equivalent:

(a) The pair $(X,Y)$ has the Daugavet property;

(b) For any given $\varepsilon > 0$, $y \in S(Y)$ and weak-open set $U$ in $X$ with $U \cap B(X) \neq \emptyset$ there is a weak-open set $V$ in $X$ with $V \cap B(X) \neq \emptyset$ and $V \cap B(X) \subseteq U \cap B(X)$ such that $\|v+y\| > 2-\varepsilon$, whenever $v \in V \cap B(X)$;

(c) For any given $\varepsilon > 0$, $x^* \in S(X^*)$ and weak*-open set $U$ in $Y^*$ with $U \cap B(Y^*) \neq \emptyset$ there is a weak*-open set $V$ in $Y^*$ with $V \cap B(Y^*) \neq \emptyset$ and $V \cap B(Y^*) \subseteq U \cap B(Y^*)$ such that $\|v_{|X} + x^*\| > 2 - \varepsilon$, whenever $v \in V \cap B(Y^*)$.

**Proof.** Let us prove that (a)$\Rightarrow$(b).

First we consider the weak*-open set $U_{**}$ in $X_{**}$ that induces $U$ on $X$, i.e. $U_{**} \cap X = U$. By the Krein-Milman Theorem, there is a convex combination of extreme points of $B(X_{**})$, $\sum_{i=1}^n \lambda_i x_{i}^{**}$, such that $\sum_{i=1}^n \lambda_i x_{i}^{**} \in U_{**}$. Clearly, we can find weak*-open neighborhoods $\{U_i\}_{i=1}^n$ of the points $\{x_{i}^{**}\}_{i=1}^n$ respectively, for which the following inclusion holds:

$$\sum_{i=1}^n \lambda_i (U_i \cap B(X_{**})) \subseteq U_{**}. \quad (1.2.3)$$

Now by the Choquet Lemma (weak*-slices containing an extreme point form a basis of its weak*-neighborhoods, [33, p.49]), we can assume that the sets $\{U_i \cap B(X_{**})\}_{i=1}^n$ are weak*-slices. Thus, inclusion (1.2.3) restricted on $X$ looks as follows: $\sum_{i=1}^n \lambda_i S_i \subseteq U$, where $S_i = U_i \cap B(X^*) \cap X$ are slices for all $i = 1, 2, \ldots, n$.

Employing Lemma 1.2.4(b) we find a vector $x_1 \in S_1$ with $\|\lambda_1 x_1 + y\| > (\lambda_1 + 1 - \varepsilon)$. Analogously, there is an $x_2 \in S_2$ with $\|\lambda_2 x_2 + \lambda_1 x_1 + y\| > (\lambda_2 + \lambda_1 + 1 - \varepsilon)$. Continuing in the same manner we finally find $x_n \in S_n$ with $\|\lambda_n x_n + \lambda_{n-1} x_{n-1} + \ldots + \lambda_1 x_1 + y\| > (\lambda_n + \lambda_{n-1} + \ldots + \lambda_1 + 1 - \varepsilon) = 2 - \varepsilon,$
\[ \sum_{i=1}^{n} \lambda_i x_i \in U. \] It remains to use the lower weak-semicontinuity of a norm to get the required weak open set \( V \).

This completes the proof of implication (a) \( \Rightarrow \) (b).

The implication (a) \( \Leftarrow \) (b) follows from Lemma 1.2.4, and the equivalence (a) \( \iff \) (c) is proved in the same way. \( \square \)

Remark 1.2.6. The above lemmas remain true also in the complex case. One only has to replace \( \ast \) \( x \) by \( \Re x \) in the definition of a slice.

1.2.2 Examples

Example 1.2.7. Let us apply Lemma 1.2.4 to show that if \( E \) is a Banach space and \( (\Omega, \Sigma, \nu) \) is an atomless positive measure space, then the space of \( E \)-valued Bochner \( \nu \)-integrable functions \( X := L_1(\Omega, E) \) has the Daugavet property. This is a special case of a result due to I. Nazarenko [58]. Our argument, however, is shorter even in the case of the scalar-valued function space \( L_1(\Omega) \), for which other proofs have appeared for instance in [5], [39] or [57].

In fact, let \( y \in S(X) \) and \( x^* \in S(X^*) \). The functional \( x^* \) can be represented by a weak*-measurable function \( \varphi \) taking values in \( E^* \). For \( \varepsilon > 0 \), find a measurable subset \( B \) of \( \Omega \) such that \( \|\chi_B y\|_{L_1} \leq \varepsilon/2 \) and \( \|\chi_B \varphi\|_{L_\infty} \geq 1 - \varepsilon/2 \), and pick \( x \in S(Y X) \) so that \( \chi_B x = x \) and \( \langle \varphi, x \rangle \geq 1 - \varepsilon \). Since clearly \( \|x + y\| \geq 2 - \varepsilon \), condition (b) of Lemma 1.2.4 is fulfilled.

It is worthwhile to indicate why the non-atomicity condition is necessary. Even in the case \( E = \mathbb{R} \), if \( \nu \) has an atom \( A \in \Sigma \), then it is easy to check that the rank-1 operator \( T f = -1 \cdot \frac{1}{\mu(A)} \int_A f d\mu \) does not satisfy (1.2.1).

Thus, \( \ell_1 \) and \( C[0,1] \) are examples of a non-Daugavet spaces. Nevertheless, as we see from our next example, \( C[0,1] \) is a Daugavet space. This was I. K. Daugavet’s original result [21].

Example 1.2.8. We reprove the result from [48] that the space of continuous \( E \)-valued functions \( X := C(K, E) \) has the Daugavet property if the compact space \( K \) has no isolated points.

Indeed, fix \( y \in S(X) \) and \( x^* \in S(X^*) \) as before. By the vector-valued version of the Riesz Representation Theorem, \( x^*(x) = \int_K x(k) dm(k), x \in X \), for some \( E^* \)-valued countably additive regular Borel measure \( m \) on \( K \) of bounded variation \( |m| \). For \( \varepsilon > 0 \), find an open set \( U \subset K \) such that \( |m|(U) < \varepsilon/2 \) and \( \|y(k) - x_0\| < \varepsilon \) for some \( x_0 \in S(E) \) and all \( k \in U \). Now
pick an \( x \in S(X) \) such that \( x(k) = x_0 \) for all \( k \in U \) and \( x^*(x) > 1 - \varepsilon \). In this case \( \|y + x\| > 2 - \varepsilon \) and condition (b) of Lemma 1.2.4 is fulfilled.

The absence of isolated points in \( K \) is necessary. Indeed, suppose that \( C(K) \) has the Daugavet property and \( k_0 \in K \) is an isolated point. Then, the rank-1 operator \( Tx = -\chi_{\{k_0\}} \cdot x(k_0) \), where \( \chi_{\{k_0\}} \) denotes the characteristic function of the singleton \( \{k_0\} \), does not satisfy the Daugavet equation.

For this reason, \( c_0 \) fails to be a Daugavet space.

**Example 1.2.9.** Even though it is harder to understand the dual to the space of \( E \)-valued strongly measurable bounded functions \( X := L_\infty(\Omega, E) \), we can show that it also has the Daugavet property provided the measure space \((\Omega, \Sigma, \nu)\) is atomless.

In fact, let us traditionally fix \( y \in S(X) \), \( x^* \in S_1(X^*) \) and \( \varepsilon > 0 \). Since \( y \) is a strongly measurable function, one can find a set \( A \in \Sigma \), \( \nu(A) \neq 0 \) such that \( \|y(\omega) - x_0\| < \varepsilon/2 \), for some \( x_0 \in S(E) \) and all \( \omega \in A \). Let us pick an auxiliary vector \( x_{aux} \in S(X) \) with \( x^*(x_{aux}) > 1 - \varepsilon/2 \). Then for some subset \( B \subset A \) of non-zero measure and \( x_1 \in B(E) \) we have \( \|x_{aux}(\omega) - x_1\| < \varepsilon/2 \), for all \( \omega \in B \). Now restrict \( x^* \) onto the subspace \((x_0 - x_1) \otimes L_\infty(\Omega)\). It can be treated as a functional on the scalar-valued \( L_\infty(\Omega) \). Thus, by the non-atomicity of \( \nu \) there is a further subset \( C \subset B \) and a non-negative function \( \varphi \in S(L_\infty(\Omega)) \) supported in \( C \) such that \( x^*((x_0 - x_1) \otimes \varphi) < \varepsilon/2 \).

One should take \( x \) to be the sum \( x_{aux} + (x_0 - x_1) \otimes \varphi \). Clearly, \( x^*(x) > 1 - \varepsilon \) and \( \|x\| \leq 1 + \varepsilon \). On the other hand, \( x \) is close to \( x_0 \) on a subset of \( C \) where \( f \) is close to 1. By our construction, on this set \( y \) is close to \( x_0 \) too. So, their sum is close to \( 2x_0 \). Consequently, we get \( \|y + x\| > 2 - \varepsilon \). Normalizing \( x \) and decreasing \( \varepsilon \) is necessary, we meet the requirements of condition (b) from Lemma 1.2.4.

The exact same example as in \( L_1 \) shows that non-atomicity of the underlying space is essential. In particular, \( \ell_1 \) does not has the Daugavet property.

**Example 1.2.10.** A similar idea of restricting a given functional on a simpler subspace is used to prove that \( C_w(K, E) \) and \( C_{w^*}(K, E^*) \), the spaces of weakly continuous and respectively weak*-continuous functions on a perfect compact set \( K \), has the Daugavet property. More generally, it has been recently proved by D. Bilik in his Master Thesis [8] that any sup-normed space of \( E \)-valued functions \( Y \) containing \( C(K, E) \) and stable under multiplication by a scalar function has the Daugavet property provided \( K \) has no isolated points and for every \( f \in Y \) the function \( \|f(k)\| \) is lower semicontinuous.
The reader may also find interesting to know that virtually all admissible pairs constructed from the previous examples are daugavetian. For instance, \( \{C(K, X); C_w(K, X)\} \) or \( \{C([0, 1], X); L_{\infty}([0, 1], X)\} \).

Other examples of Daugavet spaces including complex ones are known too. We list them with references below.

(i) A closed subalgebra \( A \subset C(K) \) containing the constant functions and separating the points of \( K \) is called a uniform algebra, [55, Ch. 4]. The Choquet boundary of \( A \) is the set of all \( k \in K \) such that the functional \( \delta_k : f \mapsto f(k) \) is an extreme point of \( B(A^*) \). If the Choquet boundary of \( A \) does not have isolated points, then \( A \) has the Daugavet property. Hence, the Daugavet property is possessed by the disk algebra and the algebra of bounded analytic functions \( H^\infty \). The proofs are found in [87, 89] (see also Example 1.3.26).

(ii) A real \( L_1 \)-predual space \( X \) has the Daugavet property if the set of extreme points \( \text{ext} B(X^*) \) has no isolated points. In the complex case one has to consider the quotient \( \text{ext} B(X^*)/\sim \) instead, where \( p^* \sim q^* \) if they are multiples of each other, [87].

(iii) It was proved in [59] that a \( C^* \)-algebra has the Daugavet property if and only if it is non-atomic. Hence, the predual of a non-atomic von Neumann algebra has the Daugavet property.

Now we show how to produce new Daugavet pairs and spaces from old ones.

**Lemma 1.2.11.** If \( (X_1, Y_1) \) and \( (X_2, Y_2) \) have the Daugavet property, then so do \( (X_1 \oplus_1 X_2, Y_1 \oplus_1 Y_2) \) and \( (X_1 \oplus_\infty X_2, Y_1 \oplus_\infty Y_2) \).

**Proof.** We first deal with \( (X_1 \oplus_\infty X_2, Y_1 \oplus_\infty Y_2) \). Let us consider \( x_j^* \in X_j^* \), \( y_j \in Y_j \) (\( j = 1, 2 \)) with \( \| (y_1, y_2) \| = \max\{\| y_1 \|, \| y_2 \| \} = 1 \), \( \| (x_1^*, x_2^*) \| = \| x_1^* \| + \| x_2^* \| = 1 \). Assume without loss of generality that \( \| y_1 \| = 1 \). By Lemma 1.2.4 there is, given \( \varepsilon > 0 \), some \( x_1 \in X_1 \) satisfying

\[
\| x_1 \| = 1, \quad x_1^*(x_1) \geq \| x_1^* \|(1 - \varepsilon), \quad \| x_1 + y_1 \| \geq 2 - \varepsilon.
\]

Also, pick \( x_2 \in X_2 \) such that

\[
\| x_2 \| = 1, \quad x_2^*(x_2) \geq \| x_2^* \|(1 - \varepsilon).
\]
Then \( \|(x_1, x_2)\| = 1, \langle (x^*_1, x^*_2), (x_1, x_2) \rangle \geq 1 - \varepsilon \) and
\[
\|(x_1, x_2) + (y_1, y_2)\| \geq \|x_1 + y_1\| \geq 2 - \varepsilon.
\]

Thus, \((X_1 \oplus \infty X_2, Y_1 \oplus \infty Y_2)\) has the Daugavet property.

A similar calculation, based on Lemma 1.2.4(c), shows that \((X_1 \oplus_1 X_2, Y_1 \oplus_1 Y_2)\) has the Daugavet property.

We remark that the converse of the above lemma is valid, too.

**Proposition 1.2.12.** Suppose that \((X_1, Y_1), (X_2, Y_2), \ldots\) are pairs of Banach spaces with the Daugavet property. Then \((c_0(X_j), c_0(Y_j))\) and \((\ell_1(X_j), \ell_1(Y_j))\) have the Daugavet property.

**Proof.** It follows from Lemma 1.2.11 that \((X_1 \oplus_\infty \cdots \oplus_\infty X_n, Y_1 \oplus_\infty \cdots \oplus_\infty Y_n)\), resp. \((X_1 \oplus_1 \cdots \oplus_1 X_n, Y_1 \oplus_1 \cdots \oplus_1 Y_n)\), have the Daugavet property for each \(n \in \mathbb{N}\). Since the union of these spaces is dense in \(c_0(X_j), c_0(Y_j), \ell_1(X_j)\) or \(\ell_1(Y_j)\), respectively, the result follows.

For \(X_j = Y_j\) these results were first proved by P. Wojtaszczyk [89] and, in a special case, by Y. Abramovich [2] using different approaches. Remarkably, the \(\ell_1\)- and \(\ell_\infty\)-sums of couples of Daugavet spaces are the only possible sums producing a Daugavet space. More precisely, suppose \(X = (X_1 \oplus X_2)_F\) is the \(F\)-sum of Daugavet spaces \(X_1\) and \(X_2\), where \(F\) is a 2-dimensional Banach space with 1-unconditional basis. If \(X\) has the Daugavet property, then \(F\) is either \(\ell^2_1\) or \(\ell^2_\infty\). This was proved by D. Bilik (see [9, Corollary 5.4] or [8]).

### 1.2.3 Copies of \(\ell_1\) and \(L_1\)

We will see below that any Daugavet space has an abundance of almost isometric copies of \(\ell_1\) and may not have any of \(L_1[0,1]\). Although the dual to a Daugavet space contains \(L_1[0,1]\) isometrically.

First we prove an extension of Lemma 1.2.3.

**Lemma 1.2.13.** If \((X,Y)\) has the Daugavet property, then for every finite-dimensional subspace \(Y_0\) of \(Y\), every \(\varepsilon_0 > 0\) and every slice \(S(x^*_0, \varepsilon_0)\) of \(B(X)\) there is a slice \(S(x^*_1, \varepsilon_1)\) of \(B(X)\) such that
\[
\|y + tx\| \geq (1 - \varepsilon_0)(\|y\| + |t|) \quad \forall y \in Y_0, \ x \in S(x^*_1, \varepsilon_1).
\]
Proof. Let $\delta = \varepsilon_0/2$ and pick a finite $\delta$-net $\{y_1, \ldots, y_n\}$ in $S(Y_0)$. By a repeated application of Lemma 1.2.3(a) we obtain a sequence of slices $S(x_0^*, \varepsilon_0) \supset S(x^{*({1})}, \varepsilon^{(1)}) \supset \ldots \supset S(x^{*({n})}, \varepsilon^{(n)})$ such that one has

$$\|y_k + x\| \geq 2 - \delta$$

(1.2.5)

for all $x \in S(x^{*({k})}, \varepsilon^{(k)})$. Put $x_1^* = x^{*({n})}$ and $\varepsilon_1 = \varepsilon^{(n)}$; then (1.2.5) is valid for every $x \in S(x_1^*, \varepsilon_1)$ and $k = 1, \ldots, n$. This implies that for every $x \in S(x_1^*, \varepsilon_1)$ and every $y \in S(Y_0)$ the condition

$$\|y + x\| \geq 2 - 2\delta = 2 - \varepsilon_0$$

holds.

Let $0 \leq t_1, t_2 \leq 1$ with $t_1 + t_2 = 1$. If $t_1 \geq t_2$, we have for $x$ and $y$ as above

$$\|t_1 x + t_2 y\| = \|t_1(x + y) + (t_2 - t_1)y\|$$

$$\geq t_1\|x + y\| - |t_2 - t_1|\|y\|$$

$$\geq t_1(2 - \varepsilon_0) + t_2 - t_1$$

$$= t_1 + t_2 - t_1\varepsilon_0 \geq 1 - \varepsilon_0,$$

and an analogous argument shows this estimate in case $t_1 < t_2$.

This implies (1.2.4), by the homogeneity of the norm and the symmetry of $S(Y_0)$.

Proposition 1.2.14. If $X$ has the Daugavet property, then $X$ contains a copy of $\ell_1$.

Proof. Using Lemma 1.2.13 inductively, it is easy to construct a sequence of vectors $e_1, e_2, \ldots$ and a sequence of slices $S(x_n^*, \varepsilon_n)$, $\varepsilon_n = 4^{-n}$, $n \in \mathbb{N}$, such that $e_{n+1} \in S(x_{n+1}^*, \varepsilon_{n+1})$ and every element of $S(x_{n+1}^*, \varepsilon_{n+1})$ is “up to $\varepsilon_n$” $\ell_1$-orthogonal to lin{$e_1, \ldots, e_n$}, which means

$$\|y + x\| \geq (1 - \varepsilon_n)(\|y\| + \|x\|) \quad \forall y \in \text{lin}\{e_1, \ldots, e_n\}, \ x \in S(x_{n+1}^*, \varepsilon_{n+1}).$$

The sequence $(\varepsilon_n)$ is then equivalent to the unit vector basis in $\ell_1$.

The proof shows even a stronger statement that $X$ contains a so-called asymptotically isometric copy of $\ell_1$. This notion was introduced by J. Hagler in [34]. One of the equivalent definitions is the following: there is a null
sequence \((\varepsilon_n)\) of positive numbers and there is a sequence \((x_n) \subset S(X)\) such that
\[
\sum_{k=1}^{n} (1 - \varepsilon_n)|a_n| \leq \left\Vert \sum_{k=1}^{n} a_n x_n \right\Vert \leq \sum_{k=1}^{n} |a_n|,
\]
for all reals \((a_n)\). By Theorem 2 from [24], the dual of a Banach space with an asymptotically isometric copy of \(\ell_1\) contains an isometric copy of \(L_1[0,1]\). Combining this with our Theorem 1.2.14, we see that the dual to every Daugavet space contains a copy of \(L_1[0,1]\). Unfortunately, not every Daugavet space itself contains a copy of \(L_1[0,1]\). An example is given by the quotient space \(L_1/Y\), with \(Y\) the space constructed by M. Talagrand [78] as a counterexample to the three-space problem for \(L_1\) (see [49] for details).

1.3 Narrow and almost narrow operators

This section introduces the notions of a narrow and almost narrow operator. We show how they naturally appear from attempt to extend the Daugavet equation to other classes of operators. As a result, we prove that all weakly compact operators satisfy the Daugavet equation and all subspaces with reflexive quotient inherit the Daugavet property.

1.3.1 Extension of the Daugavet equation

So far we have been working only with rank-1 operators. However, it is tempting to find other classes of operators satisfying (1.2.1) in a Daugavet pair. Before we look into this problem let us return to an arbitrary Banach pair \((X,Y)\), \(X \subset Y\) and find a simple condition on \(T: X \mapsto Y\) which guarantees (1.2.1).

To begin with, pick an \(x \in S(X)\) such that \(\|Tx\| \sim \|T\|\). Then try to replace the \(x\) by a 'better' vector \(z \in B(X)\) leaving \(Tx\) to be approximately \(Tz\) and so that \(\|z + Tx\| \sim 1 + \|Tx\|\). After this one should get
\[
\|z + Tz\| \sim \|z + Tx\| \sim 1 + \|Tx\| \sim 1 + \|T\|.
\]
So, the desired condition may be the following: for every \(\varepsilon > 0\) and \(x \in S(X)\) there is a \(z \in B(X)\) such that \(\|z + Tx\| > 1 + \|T\| - \varepsilon\) and \(\|Tx - Tz\| < \varepsilon\). However, sometimes we will be interested in operators acting from \(X\) into a third Banach space \(W\). Therefore we replace \(Tx\) by an arbitrary vector \(y \in Y\) in the first inequality and obtain the following definition.
Definition 1.3.1. Let $X, Y, W$ be Banach spaces and $X \subset Y$. An operator $T : X \mapsto W$ is called almost narrow (or strongly Daugavet) with respect to $Y$ if for every $\varepsilon > 0$, $x \in S(X)$ and $y \in S(Y)$ there is a $z \in B(X)$ such that $\|y + z\| > 2 - \varepsilon$ and $\|T x - T z\| < \varepsilon$. In case $X = Y$ we just say $T$ is almost narrow.

Our next step is to isolate subclasses of almost narrow operators in a Daugavet pair. We show that all weakly compact and even strong Radon-Nikodým operators are almost narrow.

Recall that a point $a_0 \in A \subset W$ is said to be strongly exposed on $A$ if there is a functional $w^* \in W^*$ such that diameter of the set $\{a \in A : w^*(a) > w^*(a_0) - \varepsilon\}$ decreases to zero as $\varepsilon$ goes to zero. An operator $T : X \mapsto W$ is called strong Radon-Nikodým if $T(B(X))$ is a Radon-Nikodým subset of $W$. In this case $\overline{T(B(X))}$ is the closed convex hull of its strongly exposed points ([7, Theorem 5.17]). If $W$ has the Radon-Nikodým property, then certainly every bounded linear operator with values in $W$ is strong Radon-Nikodým. A weakly compact operator is also an example of a strong Radon-Nikodým operator due to the fact that all weakly compact sets are Radon-Nikodým.

**Proposition 1.3.2.** Suppose $(X, Y)$ is a Daugavet pair. Let $T : X \mapsto W$ be such that the set $\overline{T(B(X))}$ coincides with the closed convex hull of its strongly exposed points. Then $T$ is almost narrow with respect to $Y$.

**Proof.** (This is a remake of the proof from [73, Theorem 6]). Let $\varepsilon, x, y$ be as in Definition 1.3.1. By our assumption on $T$, there is a convex combination $\sum_{i=1}^n \lambda_i a_i$ of strongly exposed points $\{a_i\}_{i=1}^n$ of $\overline{T(B(X))}$ such that

$$\left\| \sum_{i=1}^n \lambda_i a_i - Tx \right\| < \frac{\varepsilon}{2}. \quad (1.3.1)$$

Let $\{w_i^*\}_{i=1}^n \subset W^*$ be functionals exposing $\{a_i\}_{i=1}^n$, respectively, and let positive numbers $\{\varepsilon_i\}_{i=1}^n$ be chosen so that

$$\text{diam}\{a \in \overline{T(B(X))} : w_i^*(a) > w_i^*(a_i) - \varepsilon_i\} < \frac{\varepsilon}{2}. \quad (1.3.2)$$

Applying Lemma 1.2.13 we find $x_i \in \{b \in S(X) : T^*w_i^*(b) > w_i^*(a_i) - \varepsilon_i\}$ such that

$$\left\| \sum_{i=1}^n \lambda_i x_i + x \right\| > 2 - \varepsilon.$$
Put $z = \sum_{i=1}^{n} \lambda_i x_i$. Then $\|z + x\| > 2 - \varepsilon$. On the other hand, since $w_i^*(Tx_i) > w_i^*(a_i) - \varepsilon_i$, by (1.3.2) $\|a_i - Tx_i\| < \varepsilon/2$, for all $i = 1, ..., n$. Thus, in view of (3.2.11), we get

$$\|Tz - Tx\| \leq \left\| \sum_{i=1}^{n} \lambda_i Tx_i - \sum_{i=1}^{n} \lambda_i a_i \right\| + \frac{\varepsilon}{2} < \varepsilon,$$

which is the second needed inequality.

Our next object is an operator not fixing copies of $\ell_1$.

**Definition 1.3.3.** An operator $T: X \mapsto W$ is said to be *not fixing copies* of a Banach space $E$ or simply $E$-singular, if $T$ is not bounded from below on every subspace of $X$ isomorphic to $E$.

We know from Section 1.2.3 that a Banach space with the Daugavet property contains plenty of $\ell_1$-copies. So, $\ell_1$-singular operators defined on a Daugavet space are, in a sense, ”small”. This principle is reflected in the following proposition proved in [47].

**Proposition 1.3.4.** Let $(X, Y)$ be a Daugavet pair and $T: X \mapsto W$ an $\ell_1$-singular operator. Then $T$ is almost narrow with respect to $Y$. In particular, if $W = Y$, then $T$ satisfies the Daugavet equation.

We omit the proof in the present text for the reasons explained on page 3. Instead, bypassing the concept of almost narrowness, we show directly that the Daugavet equation holds for $\ell_1$-singular operators, [73].

**Proposition 1.3.5.** If the pair $(X, Y)$ has the Daugavet property, then every $\ell_1$-singular operator $T: X \mapsto Y$ satisfies the Daugavet equation.

**Proof.** Assume, for simplicity, that $\|T\| = 1$ and let $\varepsilon > 0$ be arbitrary.

Our considerations rely on the following “releasing principle”: suppose for some finite set of vectors $\{x_i\}_{i=1}^{n} \subset B(X)$ and some $\varepsilon > 0$ the inequalities

$$\left\| \sum_{i=1}^{n} \theta_i x_i \right\| > n - \varepsilon,$$

and

$$\left\| \sum_{i \in I_1} a_i x_i + \sum_{i \in I_2} a_i T x_i \right\| > \left( \sum_{i \in I_1 \cup I_2} a_i \right) (1 - \varepsilon)$$

16
hold for all non-negative reals $a_i$, signs $\theta_i$, and some disjoint sets $I_1, I_2 \subset \{1, 2, \ldots, n\}$. Then there is a weak-open set $U \subset X$ such that (1.3.3) and (1.3.4) remain true for all $x_n \in U \cap B(X)$.

Let us prove it. By the compactness argument, there is a $\delta > 0$ such that

$$\left\| \sum_{i \in I_1} a_i x_i + \sum_{i \in I_2} a_i T x_i \right\| > 1 - \varepsilon + \delta,$$  

(1.3.5)

whenever $\sum_{i \in I_1 \cup I_2} a_i = 1$ and $I_1, I_2$ as above. Fix a finite $\frac{\delta}{2}$-net

$$\{(a_{k,1}, a_{k,2}, \ldots, a_{k,n})\}_{k=1}^K$$

in the set

$$\left\{(a_1, a_2, \ldots, a_n) : \sum_{i=1}^n a_i = 1, \ a_i \geq 0\right\}$$

equipped with the $\ell_1$-metric. Using the lower weak-semicontinuity of a norm and weak-continuity of a bounded linear operator we conclude that there is a weak-open set $U$ such that both (1.3.3) and (1.3.5) hold for $a_i = a_{k,i}$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, K$ and all $x_n \in U \cap B(X)$. It is not hard to see that $U$ is desired.

Now we construct a sequence $\{x_i\}_{i=1}^\infty \subset B(X)$ which satisfies (1.3.3) and (1.3.4) for all non-negative reals $a_i$, signs $\theta_i$ and all disjoint finite sets $I_1, I_2 \subset \mathbb{N}$.

Assume that we have constructed such a sequence $\{x_i\}_{i=1}^n$ of length $n$. We want to prove now that altering only the last term $x_n$ one can find another vector $x_{n+1}$ such that the resulting sequence of length $n+1$ satisfies (1.3.3) and (1.3.4). Arguing in this way, we produce the desired infinite sequence if only take $x_1 \in S(X)$ with $\|Tx_1\| > 1 - \varepsilon$ on the first step.

Let us put $x'_{n+1} = x_n$ for a moment. Clearly, (1.3.4) remains true for the sequence $x_1, x_2, \ldots, x_n, x'_{n+1}$ and all $I_1, I_2$ with additional restriction: if one of them contains $n$, then the other does not contain $n + 1$. We get rid of this restriction by alteration of $x_n$ and $x'_{n+1}$. To this end, we use the “releasing principle” for $x'_{n+1}$ and find the corresponding weak open set $U \subset X$. Application of Lemma 1.2.5(b) several times yields a vector $x_{n+1} \in U \cap B(X)$ such that (1.3.3) is valid for the sequence $x_1, x_2, \ldots, x_n, x_{n+1}$ and (1.3.4) holds without the restriction: if $I_1$ contains $n + 1$, then $I_2$ does not contain $n$. Then we use the “releasing principle” to release $x_n$ so that both
(1.3.3) and (1.3.4) remain true. Appealing to Lemma 1.2.5(b) we finally get an $x'_n$ such that (1.3.4) holds for the sequence $x_1, x_2, \ldots, x'_n, x_{n+1}$ without any restrictions on $I_1$ and $I_2$. Inequality (1.3.3) is satisfied automatically.

The constructed sequence is $(1 - \varepsilon)$-equivalent to the canonical basis of $\ell_1$, for if $\sum_{i=1}^{n} |\lambda_i| = 1$, then by (1.3.3) we have

$$
\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| = \left\| \sum_{i=1}^{n} \text{sign} \lambda_i \cdot x_i + \sum_{i=1}^{n} (\lambda_i - \text{sign} \lambda_i) \cdot x_i \right\|
\geq n - \varepsilon - \sum_{i=1}^{n} |\lambda_i| + \sum_{i=1}^{n} |1 - |\lambda_i||
= n - \varepsilon - n + 1 = 1 - \varepsilon.
$$

Since $T$ fixes no copies of $\ell_1$, by Rosenthal’s Lemma [67] we may assume that the sequence $(Tx_n)_{n=1}^\infty$ is weakly Cauchy. Thus, $(Tx_{2n+1} - Tx_{2n})_{n=1}^\infty$ is weakly null. By the Mazur Theorem there are two finite disjoint sets $I_1, I_2 \subset N$ such that for some $p \in \text{conv}\{x_i : i \in I_1\}$ and $q \in \text{conv}\{x_i : i \in I_2\}$ we have $\|Tp - Tq\| < \varepsilon$. From this and (1.3.4) we finally obtain

$$
\|p + Tp\| > \|p + Tq\| - \varepsilon > 2(1 - \varepsilon) - \varepsilon = 2 - 3\varepsilon,
$$

which implies $\|J + T\| = 2$ in view of arbitrariness of $\varepsilon$.

This finishes the proof. 

**Corollary 1.3.6.** Let $X$ be a space with the Daugavet property and $X' \subset X$ is a complemented subspace. If $X/X'$ either possesses the Radon-Nikodým property or contains no copies of $\ell_1$, then any projection onto $X'$ has norm at least $2$.

**Proof.** Indeed, if $P : X \mapsto X$ is such a projection, then $Q = -\text{Id} + P$ is either strong Radon-Nikodým or $\ell_1$-singular. So, by Propositions 1.3.2, 1.3.5, respectively, we get $\|P\| = \|\text{Id} + Q\| = 1 + \|Q\| \geq 2$. 

**1.3.2 Application to unconditional decompositions**

As a vivid application of Propositions 1.3.2 and 1.3.5 we obtain the following result.

**Theorem 1.3.7.** No Banach space with the Daugavet property can be embedded into a direct unconditional sum of Banach spaces either possessing the Radon-Nikodým property or containing no copies of $\ell_1$. In particular, $X$ does not embed into a Banach space with an unconditional basis.
This is a considerable generalization of Pełczyński’s [61] result saying that neither $C[0, 1]$ nor $L_1[0, 1]$ embeds into a space with unconditional basis.

To prove Theorem 1.3.7 we need a lemma observed by V. Kadets in [48] (see also [44, 49])

**Lemma 1.3.8.** Let $X \subset Y$ be Banach spaces with $J: X \to Y$ the natural embedding. Suppose that the pair $(X, Y)$ has the Daugavet property with respect to a subspace $M \subset L(X, Y)$ of operators. Let $T = \sum_{n=1}^{\infty} T_n$ be a strongly unconditionally convergent series of operators $T_n \in M$. Then $\|J + T\| \geq 1$. In particular, $J$ cannot be represented as a strongly unconditionally convergent sum $J = \sum_{n=1}^{\infty} T_n$ with $T_n \in M$ for all $n \in \mathbb{N}$.

**Proof.** Denote by $\text{FIN}(\mathbb{N})$ the set of finite subsets of $\mathbb{N}$. By the Banach-Steinhaus Theorem, the quantity

$$\alpha = \sup \left\{ \left\| \sum_{n \in A} T_n \right\| : A \in \text{FIN}(\mathbb{N}) \right\}$$

is finite, and whenever $B \subset \mathbb{N}$, then

$$\left\| \sum_{n \in B} T_n \right\| \leq \sup \left\{ \left\| \sum_{n \in A} T_n \right\| : A \in \text{FIN}(\mathbb{N}), A \subset B \right\} \leq \alpha.$$

Let $\varepsilon > 0$ and pick $A_0 \in \text{FIN}(\mathbb{N})$ such that $\left\| \sum_{n \in A_0} T_n \right\| \geq \alpha - \varepsilon$. Then we obtain from the Daugavet property

$$\|J + T\| \geq \left\| J + \sum_{n \in A_0} T_n \right\| - \left\| \sum_{n \notin A_0} T_n \right\| \geq 1 + \left\| \sum_{n \notin A_0} T_n \right\| - \alpha \geq 1 - \varepsilon,$$

which proves the lemma. \qed

**Proof of Theorem 1.3.7.** Assume that $X$ embeds into an unconditional sum of Banach spaces $Y = \oplus_{n=1}^{\infty} X_n$ with associated projections $P_n$ from $Y$ onto $X_n$. In Section 1.4 we develop a certain technic which allows to find an equivalent norm on $Y$ turning $(X, Y)$ into a Daugavet pair (Theorem 1.4.9). So we may and do assume that the pair $(X, Y)$ has the Daugavet property.

Another observation to make is that if each $X_n$ has the Radon-Nikodým property or each $X_n$ contains no copies of $\ell_1$, then the finite sums $\sum_{n \in A} P_n |_{X_n}$, $A \in \text{FIN}(\mathbb{N})$ are all strong Radon-Nikodým or $\ell_1$-singular, respectively, and
hence satisfy the Daugavet equation. Since the embedding operator $J: X \hookrightarrow Y$ has an expansion into a strongly unconditionally convergent series $J = \sum_{n=1}^{\infty} P_n|_X$, we deduce from Lemma 1.3.8 that $P_{n_0}|_X$ is not strong Radon-Nikodým or fixes a copy of $\ell_1$ for some $n_0$. This immediately leads to a contradiction.

**Corollary 1.3.9.** $C(K)$ does not embed into an unconditional sum of Banach spaces without copies of $C[0,1]$.

In fact, if $P_n|_{C(K)}$ is a $C[0,1]$-singular, then it is $\ell_1$-singular due to the following H. Rosenthal’s result, [66] (see the end of Section 1.3.5 for an alternative proof).

**Theorem 1.3.10.** Let $T: C(K) \hookrightarrow W$. If $T^*(W^*)$ is not separable, then $T$ fixes a copy of $C[0,1]$.

**Corollary 1.3.11.** If $C[0,1] = \bigoplus_{n=1}^{\infty} X_n$ unconditionally, then there is $X_{n_0}$ isomorphic to $C[0,1]$.

Indeed, by the previous corollary, there is $X_{n_0}$ containing a copy of $C[0,1]$. If so, due to A. Pelczyński’s result [62], there can be found a complemented copy of $C[0,1]$ in $X_{n_0}$. By another classical A. Pelczyński’s result, $X_{n_0}$ itself is isomorphic to $C[0,1]$.

Corollary 1.3.9 is not valid in $L_1$, for there is a subspace $Y \subset L_1[0,1]$ constructed by M. Talagrand in [78] such that neither $Y$ nor $L_1[0,1]/Y$ contains a copy of $L_1[0,1]$, but $L_1[0,1]$ embeds into $Y \oplus L_1[0,1]/Y$. However, the analogue of Corollary 1.3.11, has been known earlier in the case of finite number of summands, [26], and now is proved in full in Section 1.3.6.

### 1.3.3 Narrow operators and rich subspaces

Having extended the Daugavet equation to larger classes of operators, we are now looking for subspaces which inherit the Daugavet property. Our strategy is to isolate a class of "small" operators whose kernels are big enough to differ little from the ambient Daugavet space. Of course, the first candidate would be the class of almost narrow operators studied in the previous section. However, it appears to be too loose. In fact, in Section 1.3.4 we construct an almost narrow quotient map $Q: L_1[0,1] \mapsto L_1[0,1]/Y$ with the non-Daugavet kernel $Y$.

So, to suit our purposes, we introduce the notion of a narrow operator.
**Definition 1.3.12.** Let $X, Y, W$ be Banach spaces and $X \subset Y$. An operator $T: X \hookrightarrow W$ is called narrow with respect to $Y$ if for every $x^* \in X^*$ the operator $T \oplus x^*: X \hookrightarrow W \oplus \mathbb{R}$ defined by

$$T \oplus x^*(x) = (Tx, x^*(x))$$

is almost narrow with respect to $Y$. If $X = Y$ we simply say that $T$ is narrow.

Examples of narrow operators are provided by Propositions 1.3.2, 1.3.4. Indeed, if $T$ is strong Radon-Nikodým or $\ell_1$-singular, then so is $T \oplus x^*$ for every $x^* \in X^*$. More generally, it is proved in [47, Theorem 3.13] that $T \oplus \bar{T}$ is narrow (with respect to $Y$) provided $T$ is narrow (with respect to $Y$) and $\bar{T}$ is strong Radon-Nikodým.

The desired result is contained in our next proposition.

**Proposition 1.3.13.** Let $(X, Y)$ be a Daugavet pair of Banach spaces and $X'$ a closed subspace of $X$. If the quotient map $Q: X \hookrightarrow X/X'$ is narrow with respect to $Y$, then the pair $(X', Y)$ has the Daugavet property.

**Proof.** Take an arbitrary $y \in S(Y)$ and a slice $S(x^*, \varepsilon)$ of the unit ball of $X'$. We pick an $x \in S(x^*, \varepsilon/2)$ and extend $x^*$ to all of $X$ preserving its norm. According to Definition 1.3.12, we can find a $z \in S(X)$ such that $\|y + z\| > 2 - \varepsilon$ and $\|Qz - Qx\| + |x^*(z) - x^*(x)| < \varepsilon/2$. Then $z \in S(x^*, \varepsilon)$ and $\text{dist}\{z, X'\} < \varepsilon$. Now one approximates $z$ by an element from $S(X')$ with similar properties and applies Lemma 1.2.4 to get the result. \(\square\)

Proposition 1.3.13 suggests the following definition.

**Definition 1.3.14.** Let $X$ be a Daugavet space. A closed subspace $X' \subset X$ is called rich if the quotient map $Q: X \hookrightarrow X/X'$ is narrow.

Thus, all rich subspaces inherit the Daugavet property and due to our Propositions 1.3.2, 1.3.4 $X'$ is rich if either of the following conditions holds ((2) implies (3)):

1. $X/X'$ has the Radon-Nikodým property;
2. $(X/X')^*$ has the Radon-Nikodým property;
3. $X/X'$ contains no copies of $\ell_1$. 

21
Let us consider an example of a rich subspace frequently appearing in harmonic analysis.

**Example 1.3.15.** On the unit circle \( \mathbb{T} \) consider the space \( L^1(\mathbb{T}) \). Denote by \( \hat{f} \) the Fourier transform of \( f \). A subset \( \Lambda \subset \mathbb{Z} \) is called Sidon if the operator \( q: f \mapsto \hat{f}_\Lambda \) maps \( L^1(\mathbb{T}) \) onto \( c_0(\Lambda) \). For example, \( \{2^n\}_{n \in \mathbb{N}} \) is Sidon set. Denote \( \Lambda' = \mathbb{Z}/\Lambda \) and let \( L^1(\mathbb{T})/L^1_{\Lambda'} \) be the kernel of \( q \). Then \( L^1(\mathbb{T})/L^1_{\Lambda'} \) is isomorphic to \( c_0(\Lambda) \) and, in particular, does not contain a copy of \( \ell_1 \). By case (3), \( L^1_{\Lambda'} \) is a rich subspace.

Furthermore, not only rich subspaces may inherit the Daugavet property. For instance, if a Daugavet space \( X \) is the \( \ell_1 \)- or \( \ell_\infty \)-sum of subspaces \( X_1 \) and \( X_2 \), then \( X_1 \) and \( X_2 \) have the Daugavet property too. Indeed, extend a rank-1 operator \( T: X_1 \mapsto X_1 \) to \( X \) by \( \tilde{T}(x_1 + x_2) = Tx_1, \quad x_1 \in X_1, \ x_2 \in X_2. \) Then, \( \|\tilde{T}\| = \|T\| \). Since \( \|\text{Id} + \tilde{T}\| = 1 + \|\tilde{T}\| \), we can find an \( x \in S(X) \), \( x = x_1 + x_2 \) such that \( \|x_1 + x_2 + Tx_1\| > 1 + \|T\| - \varepsilon \). Then \( \|x_1 + Tx_1\| > 1 + \|T\| - 2\varepsilon \), and we are done.

A more general result is also true.

**Proposition 1.3.16 ([49]).** Let \( X \) be a Daugavet space and \( X_0 \subset X \) is such that \( X_0^\perp \) has an \( \ell_1 \)-complement in \( X^* \) (such a subspace is called M-ideal). Then \( X_0 \) has the Daugavet property.

It turns out that M-ideals provide the right setting for the tree-space problem.

**Proposition 1.3.17 ([49]).** Suppose \( X_0 \) is an M-ideal in \( X \). If \( X_0 \) and \( X/X_0 \) share the Daugavet property, then so does \( X \).

**Problem 1.3.18.** Not much is known about factors of a Daugavet space. What "smallness" property of a subspace \( X' \subset X \) assures that \( X/X' \) has the Daugavet property provided \( X \) does? The only result we could find so far is the following.

**Proposition 1.3.19.** Suppose \( X \) is a Daugavet space and \( X' \) is a reflexive subspace of \( X \). Then \( X/X' \) has the Daugavet property.

It is proved by dualizing the proof of Proposition 1.3.2 using Lemma 1.2.4(c) (see [73, Theorem 6]).
1.3.4 Example of almost narrow, but not narrow operator

The aim of this section is to construct an example of an almost narrow operator on $L_1 := L_1[0,1]$ which is not narrow. In fact, we shall define a subspace $Y \subset L_1$ so that the quotient map $Q: L_1 \to L_1/Y$ is a strong Daugavet operator, but $Y$ fails the Daugavet property. By Proposition 1.3.13, $Q$ cannot be narrow.

For convenience, we denote by $|\cdot|$ the Lebesgue measure on $[0,1]$.

Let $I_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ for $n \in \mathbb{N}_0$ and $k = 1, 2, \ldots, 2^n$. Fix $N \in \mathbb{N}$. We define

$$
g_{0,1} = (2^N - 1)\chi_{I_{N,1}} - \chi_{I_{0,1}\setminus I_{N,1}}$$
$$
g_{1,1} = (2^{N^2-N} - 1)\chi_{I_{N^2,1}} - \chi_{I_{N,1}\setminus I_{N^2,1}}$$
$$
g_{1,k} = g_{1,1}(t - \frac{k-1}{2^N}), \quad k = 2, \ldots, 2^N,$$
$$
\vdots$$
$$
g_{n,1} = (2^{N^{n+1}-N^n} - 1)\chi_{I_{N^n+1,1}} - \chi_{I_{N^n,1}\setminus I_{N^n+1,1}}$$
$$
g_{n,k} = g_{n,1}(t - \frac{k-1}{2^{N^n}}), \quad k = 2, \ldots, 2^{N^n}.
$$

Denote by $P_n$ the “peak set” of the $n$’th generation, i.e.,

$$
P_n = \left\{ t \in [0,1] : \sum_{k=1}^{2^{N^n}} g_{n,k}(t) > 0 \right\},
$$

and $P = \bigcup_n P_n$. Clearly $|P_n| = 2^{N^n}/2^{N^{n+1}} = (1/2^{N-1})^N$ and $|P| \leq 1/(2^N - 1)$. Notice also that $\int_0^1 g_{n,k}(t) dt = 0$ for all $n$ and $k$.

First we formulate a lemma.

**Lemma 1.3.20.** Let

$$
g = \sum_{n=0}^{M} \sum_{k=1}^{2^{N^n}} a_{n,k} g_{n,k}.
$$

Then

$$
\|g\chi_{[0,1]\setminus P}\| \leq 3\|g\chi_P\|.
$$
Proof. Denote
\[ g'' = \sum_{\text{supp}(g_{n,k}) \subset P} a_{n,k}g_{n,k}, \quad g' = g - g''. \]
Since \( g' \) and \( g \) coincide off \( P \), we clearly have
\[ \|g'\chi_{[0,1] \setminus P}\| = \|g\chi_{[0,1] \setminus P}\|. \] (1.3.6)
We also have that
\[ \|g'\chi_P\| \leq \|g\chi_P\|. \] (1.3.7)
Indeed, we can write \( P \) as a countable union of disjoint (half-open) intervals; denote by \( I \) any one of these. Then \( g' \) is constant on \( I \), and
\[ \int_{0}^{1} g''(t) dt = 0. \] Hence
\[ \|g'\chi_I\| = \left| \int_{0}^{1} g'(t) \chi_{I}(t) dt \right| = \left| \int_{0}^{1} (g'(t) \chi_{I}(t) + g''(t) \chi_{I}(t)) dt \right| \leq \|g\chi_I\|. \]
Summing up over all \( I \) gives the result.
Next, we claim that
\[ \|g'\chi_{[0,1] \setminus P}\| \leq 3\|g'\chi_P\|. \] (1.3.8)
To see this, we label the intervals \( I \) from the previous paragraph as follows. For every \( l \in \mathbb{N} \) write \( B_0 = P_0 \) and \( B_l = P_l \setminus \bigcup_{i=1}^{l-1} P_i \). Each \( B_l \) can be written as \( \bigcup_{d \in D_l} I_{N^{l+1},d} \) where \( D_l \) is some subset of \( \{1, \ldots, 2^{N^{l+1}}\} \) with cardinality \( < 2^{N^l} \). Let us write \( g' = \sum_{n=0}^{M} \sum_{k=1}^{2^{N^l}} b_{n,k}g_{n,k} \). We then have the estimates
\[ \int_{0}^{1} |g'(t)\chi_{B_0}(t)| dt = |b_{0,1}| \frac{2^{N} - 1}{2^{N}} \]
and
\[ \int_{0}^{1} |g'(t)\chi_{B_l}(t)| dt \geq \sum_{d \in D_l} \int_{I_{N^{l+1},d}} - b_{0,1} - \sum_{n=1}^{l-1} \sum_{k=1}^{2^{N^l}} b_{n,k} \chi_{\text{supp}(g_{n,k})} + b_{l,(d-1)/(2^{N-1})N^{l+1}+1} \left( 2^{N^{l+1}} - N^l \right) \]
\[ \geq \sum_{k=1}^{2^{N^l}} \left( \frac{1}{2^{N^l}} - \frac{1}{2^{N^{l+1}}} \right) |b_{l,k}| \]
\[ - \frac{1}{(2^{N-1})N^l} |b_{0,1}| - \frac{1}{(2^{N-1})N^l} \sum_{n=1}^{l-1} \sum_{k=1}^{2^{N^l}} |b_{n,k}|. \]
Summing up over all $l$ gives us
\[
\int_P |g'(t)| \, dt \geq |b_{0,1}| \left( \frac{2N - 1}{2N} - \sum_{m=1}^{\infty} \frac{1}{(2N-1)^m} \right) \\
+ \sum_{l=1}^{\infty} \left( \frac{1}{2N^l} - \frac{1}{2N^{l+1}} \right) - \sum_{m=l+1}^{\infty} \frac{1}{(2N-1)^m} \right) \sum_{k=1}^{2N^l} |b_{l,k}| \\
\geq \frac{1}{2} |b_{0,1}| + \frac{1}{2} \sum_{l=1}^{\infty} \frac{1}{2N^l} \sum_{k=1}^{2N^l} |b_{l,k}|.
\]
On the other hand, by the triangle inequality
\[
\int_0^1 |g'(t)| \, dt \leq 2 \left( |b_{0,1}| + \sum_{l=1}^{\infty} \frac{1}{2N^l} \sum_{k=1}^{2N^l} |b_{l,k}| \right),
\]
hence the claim follows.

The lemma now results from (1.3.6)–(1.3.8) \[ \square \]

**Theorem 1.3.21.** Let $Y_N \subset L^1$ be the closed subspace generated by the system \{\(g_{n,k}\)\} and the constants. Then the quotient map $Q_N: L^1 \to L^1/Y_N$ is an almost narrow operator for all $N$, but $Y_N$ fails the Daugavet property if $N \geq 4$.

**Proof.** Let us fix $x, y \in S(L^1)$ and $\varepsilon > 0$. Without loss of generality we may assume that $x = \sum_{k=1}^{2N^n} a_{n,k} \chi_{n,k}$ for a big enough $n$ to be chosen later.

Put $h = \sum_{k=1}^{2N^n} a_{n,k} g_{n,k}$. Then
\[
x + h = \sum_{k=1}^{2N^n} 2^{N^n+1-N^n} \chi_{N^n+1,d_{n,k}} a_{n,k}
\]
with $d_{n,k} = 1 + (k - 1)(2^{N-1})^n$. So
\[
\|x + h\| = \sum_{k=1}^{2N^n} \frac{|a_{n,k}|}{2^{N^n}} = \|x\| = 1,
\]
and supp$(x + h) \subset P_n$. Since $|P_n| \to 0$ we can pick $n$ big enough to satisfy $\|x + h + y\| > 2 - \varepsilon$. This shows that $Q_N$ is an almost narrow operator.
To show that $Y_N$ fails the Daugavet property if $N \geq 4$, take $g^* = \chi_{[0,1]\setminus P} \in Y_N^*$ and $\varepsilon = 2|P|$. Since $1 \in S(Y_N)$, we get

$$\|g^*\| \geq g^*(1) = 1 - \varepsilon/2 > 1 - \varepsilon.$$  

Thus, $S(g^*, \varepsilon) \cap B(Y_N) \neq \emptyset$. We show that there is no $f$ in this slice such that $\|f - 1\| > 2 - \varepsilon$.

Suppose, on the contrary, that there is such an $f$. Without loss of generality we can assume that

$$f = a_0 1 + g$$

where $g$ is as in Lemma 1.3.20.

It follows from our conditions that

$$\|f \chi_P\| = \int_P |f(t)| \, dt = \|f\| - g^*(|f|) \leq 1 - g^*(f) < \varepsilon. \quad (1.3.9)$$

Hence,

$$1 \geq \int_0^1 f(t) \, dt = \int_P f(t) \, dt + g^*(f) > 1 - 2\varepsilon,$$

and since $\int_0^1 f(t) \, dt = a_0$, we get

$$1 - 2\varepsilon < a_0 \leq 1. \quad (1.3.10)$$

By (3.2.1) and (1.3.10),

$$\|g \chi_P\| \leq \varepsilon + |P| < 2\varepsilon, \quad (1.3.11)$$

thus (1.3.10) and (1.3.11) yield

$$\|g \chi_{[0,1]\setminus P}\| \geq \|g\| - 2\varepsilon = \|f - a_0 1\| - 2\varepsilon \geq \|f - 1\| - 4\varepsilon > 2 - 5\varepsilon.$$  

But now Lemma 1.3.20 and (1.3.11) imply

$$2 - 5\varepsilon < \|g \chi_{[0,1]\setminus P}\| \leq 3\|g \chi_P\| < 6\varepsilon,$$

which yields $\varepsilon > 2/11$, i.e., $|P| > 1/11$, which is false for $N \geq 4$. \qed

**Theorem 1.3.22.** On $L_1[0,1]$ the class of almost narrow operators does not coincide with the class of narrow operators.
1.3.5 \textit{C}-narrow operators in $C(K)$ and $C(K, E)$

Let us fix a Hausdorff compact set $K$ with no isolated points. In this case $C(K)$ is a Daugavet space, i.e. (1.2.1) holds for rank-1 operators. Like in the beginning of Section 1.3.1, we would like to expand the class of operators satisfying (1.2.1). For this purpose, we carry out the same heuristic argument, now using peculiar structure of the space $C(K)$. As a result, we will obtain an equivalent definition of narrowness (Proposition 1.3.24).

Suppose $T: C(K) \mapsto C(K)$. We want to find a condition on $T$ which guarantees (1.2.1). Let $f \in C(K)$ be such that $\|Tf\| \sim \|T\|$. Then $|Tf|(k) \sim \|T\|$ for all $k$ in some open set $U \subset K$. Let us choose $U$ so small that $f$ is approximately equal to a constant $\alpha$ on $U$, and $Tf$ has constant sign $\theta$. Switching $f$ to $-f$ if necessary, we may assume that $\theta = 1$.

Now if $T$ allows us to pick a positive function $\varphi \in S(C(K))$ supported on $U$ such that $\|T\varphi\| \sim 0$, then we put $g = f + (1 - \alpha)\varphi$. It is not hard to check that $\|g\| \sim 1$. Moreover, at a point where $\varphi = 1$ we have

$$Tg + g = f + (1 - \alpha)\varphi + Tf + (1 - \alpha)T\varphi \sim \alpha + (1 - \alpha) + 1 + 0 = 1 + \|T\|.$$ 

This proves equality (1.2.1).

So, a possible condition may be the following: for every open set $U \subset K$ and every $\varepsilon > 0$ there is a function $\varphi \in S(C(K))$ supported on $U$ such that $\|T\varphi\| < \varepsilon$. However it has some drawbacks.

Firstly, it will turn out that the positivity requirement is redundant (see Proposition 1.3.28). So, we may exclude it making our condition more applicable. Secondly, unlike in the preceding argument, the fact that $T$ takes values in $C(K)$ is not used in the condition itself. So, it makes sense to extend it on operators acting from $C(K)$ into an arbitrary Banach space. In fact, this will be particularly important in Section 1.4.

Taking into account these remarks we arrive at the following definition.

\textbf{Definition 1.3.23.} For an open set $U \subset K$ denote by $C_0(U)$ the space of continuous functions vanishing on $K \setminus U$. Let $W$ be a Banach space. An operator $T: C(K) \mapsto W$ is called \textit{C}-narrow, if for every open set $U \subset K$ and every $\varepsilon > 0$ there is a function $\varphi \in S(C_0(U))$ such that $\|T\varphi\| < \varepsilon$.

This notion was introduced by V. Kadets and M. Popov in [44].
An operator
Suppose
let
Sup
\( y \)
support condition such that

\[ \sup_{n \in \mathbb{N}} V_n \]
where
\( y \)
of
\( g \)
In particular
\( f \)
supported on
Proposition 1.3.28 we prove below, there is a positive
\( T \)
that
\( \in \)
Proof. Suppose \( T \) is \( C \)-narrow, and let \( x^* \in C(K) \). Then, \( x^* \) can be viewed as a regular Borel measure. Since \( K \) has no isolated points, in every open set \( U \subset K \) one can find an open subset \( V \subset U \) such that \( |x^*(f)| < \varepsilon \) for every \( f \in C_0(V) \). This implies that \( T \oplus x^* \) is \( C \)-narrow. So, it suffices to prove that \( T \) is almost narrow.

To this end, let us fix \( x \in S(C(K)), y \in S(C(K)) \) and \( \varepsilon > 0 \). Find an open set \( U \subset K \) so that \( |x - \alpha| < \varepsilon \) and \( \theta y > 1 - \varepsilon \) on \( U \) for some sign \( \theta \). By Proposition 1.3.28 we prove below, there is a positive \( \varphi \in C_0(U) \) such that

\[ \| T \varphi \| < \varepsilon. \]

Put \( z = x + (\theta - \alpha) \varphi \). Then

\[ \| z \| \leq 1 + \varepsilon \quad \text{and} \quad \| Tx - Tz \| \leq 2\varepsilon. \]
On the other hand, if \( \varphi(u_0) = 1 \), then

\[ |z + y(u_0)| = |\alpha + \theta - \alpha + y(u_0)| - \varepsilon > 2 - 2\varepsilon. \]

Making \( \varepsilon \) infinitely small and normalizing \( z \) if necessary, we fulfill the requested conditions.

Assume now that \( T \) is an almost narrow operator. Fix a closed subset \( F \subset K \) and \( 0 < \varepsilon < 1/4 \). According to the definition it is sufficient to prove that there is a function \( f \in S(C(K)) \) for which the restriction to \( F \) is less than \( 2\varepsilon \) and \( \| Tf \| < 2\varepsilon \). Let us fix a neighborhood \( U \) of \( F \) and an open set \( V \subset K, V \cap U = \emptyset \). Select inductively functions \( x_n, y_n \in S(C(K)) \) and \( f_n, g_n \in C(K) \) as follows.

All the \( x_n \) are supported on \( U \), and the \( y_n \) are non-negative functions supported on \( V \). Given \( x_n \) and \( y_n \) pick a \( z_n \in S(C(K)) \) such that

\[ \| z_n + y_n \| > 2 - \varepsilon \quad \text{and} \quad \| Tx_n - Tz_n \| < \varepsilon. \]

Put \( f_n = z_n - x_n \) and let \( g_n = f_1 + \ldots + f_n \). In particular \( f_n \) automatically satisfies

\[ \| f_n + x_n \| = 1, \quad \| f_n + x_n + y_n \| > 2 - \varepsilon \quad \text{and} \quad \| T f_n \| < \varepsilon. \]
Then choose \( x_{n+1} \in S(C(K)) \) subject to the above support condition such that

\[ \sup_{t \in F} |g_n(t)| x_{n+1} \]
coincides on \( F \) with \( g_n \), and let \( y_{n+1} \) be a non-negative continuous function supported on the subset of \( V \) where \( g_n \) attains its supremum on \( V \) up to \( \varepsilon \), i.e., on the set \( \{ t \in V : g_n(t) > \sup_{s \in V} g_n(s) - \varepsilon \} \), etc. (There is no initial restriction on the choice of \( y_1 \) and \( x_1 \) apart from the support and positivity conditions.)

We first claim that

\[ \| g_n \|_F := \sup_{t \in F} |g_n(t)| \leq 3. \]

28
This is certainly true for \( n = 1 \) since \( \|f_1 + x_1\| = 1 \). Now induction yields for \( t \in F \)

\[
|g_{n+1}(t)| = |g_n(t) + f_{n+1}(t)| = \|g_n\|_F x_{n+1}(t) + f_{n+1}(t) \\
= |x_{n+1}(t) + f_{n+1}(t) + (\|g_n\|_F - 1)x_{n+1}(t)| \\
\leq |x_{n+1}(t) + f_{n+1}(t)| + \|g_n\|_F - 1 \leq 1 + 2 = 3.
\]

Next, we have that 

\[
\sup_{t \in V} g_n(t) > n(1 - 2\varepsilon).
\]

Indeed, the functions \( x_1 \) and \( y_1 \) are disjointly supported; hence there is a point in the support of \( y_1 \) at which \( f_1 = g_1 \) is bigger than \( 1 - \varepsilon \). Thus, the above inequality holds for \( n = 1 \). To perform the induction step we use the same argument to find a point \( t_0 \) in the support of \( y_{n+1} \) at which \( f_{n+1} \) exceeds \( 1 - \varepsilon \). At this point \( t_0 \) the function \( g_n \) attains its supremum on \( V \) up to \( \varepsilon \). So

\[
\sup_{t \in V} g_{n+1}(t) \geq g_{n+1}(t_0) = g_n(t_0) + f_{n+1}(t_0) > \sup_{t \in V} g_n(t) + 1 - 2\varepsilon \\
> n(1 - 2\varepsilon) + 1 - 2\varepsilon = (n + 1)(1 - 2\varepsilon).
\]

Therefore \( \|g_{n+1}\| > (n + 1)(1 - 2\varepsilon) \), and on the other hand we have \( \|Tg_n\| \leq \sum_{k=1}^{n} \|Tf_n\| < n\varepsilon \). So for \( n \) big enough the function \( f = g_n/\|g_n\| \) will satisfy the desired conditions. \( \square \)

If \( C(K) \) is a closed subspace of \( Y \), then the ’with respect to \( Y \)’ version of just proved result may not be true. Nevertheless, the following proposition holds.

**Proposition 1.3.25.** Let \( C(K) \) be closed subspace of a Banach space \( Y \). Then there is an equivalent norm on \( Y \), which remains the same on \( C(K) \) and such that every operator \( T: C(K) \to W \) is \( C \)-narrow if and only if it is narrow with respect to \( Y \). In particular, the pair \( (C(K), Y) \) has the Daugavet property with respect to \( C \)-narrow operators.

This last conclusion of the proposition was proved in [45].

We postpone the proof till Section 1.4.4, after a suitable renorming technic is developed.
Example 1.3.26 ([88]). Consider a uniform algebra $A \subset C(K)$. The weak$^*$-closure of its Choquet boundary is called Šilov boundary. Suppose that the Šilov boundary of $A$ is $K$. We claim that the quotient map $Q: C(K) \mapsto C(K)/A$ is narrow, and hence $A$ is a reach subspace. Indeed, by Exercise 5 from [55, p. 82], for every open set $U \subset K$ one can find a function $f \in S(A)$ with $|f| < \varepsilon$ on $K/U$. An obvious modification of $f$ yields a function $g \in S(C_0(U))$ whose distance to $A$ is less than $\varepsilon$. The claim follows from our Proposition 1.3.24.

Suppose now that the Choquet boundary of $A$ does not contain isolated points, as in example (i) on page 11. Then its Šilov boundary $S$ does not contain isolated points either. Moreover, the restriction of $A$ onto $S$ is an isometric embedding. So, one can replace $K$ by a smaller compact set $S$ to fulfill the above conditions. Thus, $A$ has the Daugavet property.

Let us now adapt Definition 1.3.23 to the vector-valued case.

**Definition 1.3.27.** An operator $T: C(K, E) \mapsto W$ is called $C$-narrow if there is a constant $\lambda$ such that given any $\varepsilon > 0$, $x \in S(E)$ and open set $U \subset K$ there is a function $f \in C(K, E)$, $\|f\| \leq \lambda$, satisfying the following conditions:

(a) $\text{supp}(f) \subset U$,

(b) $f^{-1}(B(x, \varepsilon)) \neq \emptyset$, where $B(x, \varepsilon) = \{z \in E: \|z - x\| < \varepsilon\}$,

(c) $\|Tf\| < \varepsilon$.

As in the scalar case, one can prove that every $C$-narrow operator is narrow. However, for the converse statement to be true $E$ necessarily has to have special geometric properties, for example, absence of non-zero almost narrow operators (see [9, Proposition 4.3]). Thus, if $E$ has the Daugavet property, then there exists a narrow operator on $C(K, E)$, which is not $C$-narrow.

We do not pursue this topic any further. Instead, let us focus on structural properties of $C$-narrow operators.

As the following proposition shows, condition (b) of the previous definition can be substantially strengthened. In particular, the size of the constant $\lambda$ is immaterial; but introducing this constant in the definition allows for more flexibility in applications.
Proposition 1.3.28. If $T$ is a $C$-narrow operator, then for every $\varepsilon > 0$, $x \in S(E)$ and open set $U \subset K$ there is a function $f$ of the form $g \otimes x$, where $g \in C_0(U)$, $\|g\| = 1$ and $g$ is nonnegative, such that $\|Tf\| < \varepsilon$.

Proof. Let us fix $\varepsilon > 0$, an open set $U$ in $K$ and $x \in S(E)$. According to Definition 1.3.27 we find a function $f_1 \in C(K, E)$ corresponding to $\varepsilon$, $U$ and $x$. Put $U_1 = U$ and $U_2 = f_1^{-1}(B(x, \frac{1}{2}))$. As above, there is a function $f_2$ corresponding to $\varepsilon$, $U_2$ and $x$. We denote $U_3 = f_2^{-1}(B(x, \frac{1}{4}))$ and continue the process. In the $r$th step we get the set $U_r = f_{r-1}^{-1}(B(x, \frac{1}{2r}))$ and apply Definition 1.3.27 to obtain a function $f_r$ corresponding to $U_r$.

Choose $n \in \mathbb{N}$ so that $(\lambda + 2)/n < \varepsilon$ and put $f = \frac{1}{n}(f_1 + f_2 + \cdots + f_n)$. Now using the Urysohn Lemma we find a continuous function $g$ satisfying $\frac{k-1}{n} \leq g(t) \leq \frac{k}{n}$ for all $t \in U_k$, $k = 1, \ldots, n$, $\|g\| = 1$ and vanishing outside $U_1$. We claim that $\|f - g \otimes x\| < \varepsilon$. Indeed, by our construction, if $t \in K \setminus U_1$, then $\|(f - g \otimes x)(t)\| = 0$, and if $t \in U_k \setminus U_{k+1}$ (with the understanding that $U_{n+1}$ stands for $\emptyset$), then

$$\|(f - g \otimes x)(t)\| = \left\|\frac{1}{n}(f_1 + \cdots + f_k)(t) - g(t) \cdot x\right\|$$

$$\leq \left\|\frac{1}{n}(1(f_1(t) - x) + \cdots + (f_{k-1}(t) - x) + f_k(t))\right\| + \frac{1}{n}$$

$$\leq \frac{1}{n}\left(\frac{1}{2} + \cdots + \frac{1}{2^{k-1}} + \lambda\right) + \frac{1}{n} < \frac{\lambda + 2}{n} < \varepsilon.$$ 

Moreover,

$$\|Tf\| \leq \frac{1}{n} (\|Tf_1\| + \|Tf_2\| + \cdots + \|Tf_n\|) < \varepsilon.$$ 

Thus $\|T(g \otimes x)\| < \varepsilon + \varepsilon\|T\|$, and since $\varepsilon$ was chosen arbitrarily, we are done.

Another way to express this proposition is to say that $T$: $C(K, E) \hookrightarrow W$ is $C$-narrow if and only if, for each $x \in E$, the restriction $T_x$: $C(K) \hookrightarrow W$, $T_x(g) = T(g \otimes x)$, is $C$-narrow.

One of the fundamental properties of $C$-narrow operators is stated in our next theorem.

Theorem 1.3.29. Suppose that operators $T, T_n$: $C(K, E) \hookrightarrow W$ are such that the series $\sum_{n=1}^{\infty} w^*(T_n f)$ converges absolutely to $w^*(T f)$, for every $w^* \in W^*$ and $f \in C(K, E)$. If all the $T_n$ are $C$-narrow, then so is $T$. In particular, the sum of two $C$-narrow operators is a $C$-narrow operator.
**Corollary 1.3.30.** A strongly unconditionally convergent sum of narrow operators on \( C(K) \) is a narrow operator itself.

Indeed, this follows from Theorem 1.3.29 and Proposition 1.3.24. We remark that the case of a sum of two narrow operators on \( C(K) \) was treated earlier in [44] and [47], but the assertion about infinite sums is new even there. It was proved in [49] for a pointwise unconditionally convergent sum \( T = \sum_{n=1}^{\infty} T_n \) on a space with the Daugavet property that

\[
\| \text{Id} + T \| \geq 1
\]

whenever \( \| \text{Id} + S \| = 1 + \|S\| \) for every \( S \) in the linear span of the \( T_n \)'s. In the context of Theorem 1.3.29 we even obtain

\[
\| \text{Id} + T \| = 1 + \|T\| \quad (1.3.12)
\]

when all the \( T_n \) are narrow on \( C(K) \). Another consequence of Corollary 1.3.30 is that the identity on \( C(K) \) cannot be represented as an unconditional sum of narrow operators, since obviously (1.3.12) fails for \( T = -\text{Id} \); see [44] and [49] for more results along these lines.

We now turn to the proof of Theorem 1.3.29 for which we need an auxiliary concept. A similar idea has appeared in [44].

**Definition 1.3.31.** Let \( G \) be a closed \( G_\delta \)-set in \( K \) and \( T: C(K) \to W \). We say that \( G \) is a vanishing set of \( T \) if there is a sequence of open sets \( (U_i)_{i \in \mathbb{N}} \) in \( K \) and a sequence of functions \( (f_i)_{i \in \mathbb{N}} \) in \( S(C(K)) \) such that

\begin{align*}
(a) \quad G &= \bigcap_{i=1}^{\infty} U_i; \\
(b) \quad \text{supp}(f_i) &\subseteq U_i; \\
(c) \quad \lim_{i \to \infty} f_i = \chi_G \text{ pointwise}; \\
(d) \quad \lim_{i \to \infty} \|T f_i\| = 0.
\end{align*}

The collection of all vanishing sets of \( T \) is denoted by \( \text{van}_T \).

Let \( T: C(K) \to W \). By the Riesz Representation Theorem, \( T^*w^* \) can be identified a regular measure on the Borel subsets of \( K \), whenever \( w^* \in W^* \).

**Lemma 1.3.32.** Suppose \( G \) is a closed \( G_\delta \)-set in \( K \) and \( T: C(K) \to W \). Then \( G \in \text{van}_T \) if and only if \( T^*w^*(G) = 0 \) for all \( w^* \in W^* \).
Proof. Let $G \in \text{van}T$, and pick functions $(f_i)_{i \in \mathbb{N}}$ as in Definition 1.3.31. Then by the Lebesgue Dominated Convergence Theorem, for any given $w^* \in W^*$ we have

$$T^*w^*(G) = \int_K \chi_G dT^*w^* = \lim_{i \to \infty} \int_K f_i dT^*w^* = \lim_{i \to \infty} w^*(Tf_i) = 0.$$ 

Conversely, let $(U_i)_{i \in \mathbb{N}}$ be a sequence of open sets in $K$ such that $\overline{U}_{i+1} \subset U_i$ and $G = \bigcap_{i=1}^\infty U_i$. By the Urysohn Lemma there exist functions $(f_i)_{i \in \mathbb{N}}$ having the following properties: $0 \leq f_i(t) \leq 1$ for all $t \in K$, supp$(f_i) \subset U_i$, and $f_i(t) = 1$ if $t \in \overline{U}_{i+1}$. Clearly, $\lim_{i \to \infty} f_i = \chi_G$ pointwise and

$$\lim_{i \to \infty} w^*(Tf_i) = \lim_{i \to \infty} T^*w^*(f_i) = T^*w^*(G) = 0$$

whenever $w^* \in W^*$. This means that the sequence $(Tf_i)_{i \in \mathbb{N}}$ is weakly null. Applying the Mazur Theorem we finally obtain a sequence of convex combinations of the functions $(f_i)_{i \in \mathbb{N}}$ which satisfies all the conditions of Definition 1.3.31.

This completes the proof. \hfill $\Box$

Lemma 1.3.33. An operator $T: C(K) \to W$ is $C$-narrow if and only if every nonvoid open set $U \subset K$ contains a nonvoid vanishing set of $T$. Moreover, if $(T_n)_{n \in \mathbb{N}}$ is a sequence of $C$-narrow operators acting from $C(K)$ into $W$, then every open set $U \neq \emptyset$ contains a set $G \neq \emptyset$ that is a vanishing set for all $T_n$ simultaneously.

Proof. We first prove the more general “moreover” part. Put $U_{1,1} = U$. By the definition of a $C$-narrow operator and Proposition 1.3.28 there is a function $f_{1,1} \in S(C(K))$ with supp$(f_{1,1}) \subset U_{1,1}$, $U_{1,2} := f_{1,1}^{-1}(\frac{3}{2}, 1] \neq \emptyset$ and $\|T_1f_{1,1}\| < \frac{1}{2}$. Obviously, $\overline{U}_{1,2} \subset f_{1,1}^{-1}[\frac{2}{3}, 1] \subset U_{1,1}$. Again applying the definition we find $f_{1,2} \in S(C(K))$ with supp$(f_{1,2}) \subset U_{1,2}$, $U_{2,1} = f_{1,2}^{-1}(\frac{2}{3}, 1] \neq \emptyset$ and $\|T_1f_{1,2}\| < \frac{1}{3}$. As above $\overline{U}_{2,1} \subset U_{1,2}$.

In view of the $C$-narrowness of $T_2$ there exists a function $f_{2,1} \in S(C(K))$ with supp$(f_{2,1}) \subset U_{2,1}$, $U_{1,3} = f_{2,1}^{-1}(\frac{3}{4}, 1] \neq \emptyset$ and $\|T_2f_{2,1}\| < \frac{1}{4}$. In the next step we construct $f_{1,3} \in S(C(K))$ such that $U_{2,2} = f_{1,3}^{-1}(\frac{3}{5}, 1] \neq \emptyset$ and $\|T_1f_{1,3}\| < \frac{1}{5}$.

Proceeding in the same way, in the $n^{th}$ step we find a set of functions $(f_{k,l})_{k+l=n} \subset S(C(K))$ and nonempty open sets $(U_{k,l})_{k+l=n}$ in $K$ such that supp$(f_{k,l}) \subset U_{k,l}$, $\|T_kf_{k,n-k}\| < \frac{1}{n}$ and $U_{k,l} = f_{k-1,n-k}^{-1}(\frac{n-k}{n}, 1]$, if $k \neq 1$. Then we put $U_{1,n} = f_{n-1,1}^{-1}(\frac{n-1}{n}, 1]$ to start the next step.
It remains to show that the set \( G = \bigcap_{k,l \in \mathbb{N}} U_{k,l} = \bigcap_{k,l \in \mathbb{N}} U_{k,l} \) is as desired. Indeed, \( G \) is clearly a nonempty closed \( G_\delta \)-set and \( G = \bigcap_{i=1}^\infty U_{n,i} \) for every \( n \in \mathbb{N} \). It is easily seen that the sequences \((f_{n,i})_{i \in \mathbb{N}}\) and \((U_{n,i})_{i \in \mathbb{N}}\) meet the conditions of Definition 1.3.31 for the operator \( T_n \). So, \( G \in \text{van } T_n \) for every \( n \in \mathbb{N} \).

To prove the converse, let \( U \neq \emptyset \) be any open set in \( K \) and let \( \varepsilon > 0 \). By assumption on \( \text{van } T \) we can find a closed \( G_\delta \)-set \( \emptyset \neq G \subset U \), \( G \in \text{van } T \). Consider the open sets \((U_i)_{i \in \mathbb{N}}\) and functions \((f_i)_{i \in \mathbb{N}}\) provided by Definition 1.3.31. For sufficiently large \( i \in \mathbb{N} \) we have \( U_i \subset U \) and \( \|T f_i\| < \varepsilon \) so that \( f_i \) may serve as a function as required in Definition 1.3.27.

This finishes the proof.

**Proof of Theorem 1.3.29.** By virtue of Proposition 1.3.28, we may assume that \( E = \mathbb{R} \). By Lemma 1.3.33 it suffices to show that \( \bigcap_{n=1}^\infty \text{van } T_n \subset \text{van } T \).

Suppose \( G \in \bigcap_{n=1}^\infty \text{van } T_n \). According to Lemma 1.3.32 we need to prove that \( T^* w^*(G) = 0 \) for all \( w^* \in W^* \). By the condition of the theorem, the series \( \sum_{n=1}^\infty T_n^* w^* \) is weak*-unconditionally Cauchy and hence weakly unconditionally Cauchy. Since \( C(K)^* \) does not contain a copy of \( c_0 \), it is actually unconditionally norm convergent by the Bessaga-Pelczynski Theorem [23]. This implies that for the bounded sequence of functions \((f_i)_{i \in \mathbb{N}}\) converging to \( \chi_G \) pointwise constructed in the proof of Lemma 1.3.32, we have

\[
T^* w^*(G) = \lim_{i \to \infty} T^* w^*(f_i) = \lim_{i \to \infty} \sum_{n=1}^\infty T_n^* w^*(f_i) = \sum_{n=1}^\infty T_n^* w^*(\chi_G) = \sum_{n=1}^\infty T_n^* w^*(G) = 0.
\]

The proof is complete.

Now using Theorem 1.3.29 we get an alternative proof of Corollary 1.3.9.

Indeed, if, on the contrary, each \( P_n|_{C(K)} \) is \( C[0,1] \)-singular, then it is \( C \)-narrow. By Theorem 1.3.29 the inclusion map cannot be represented as an unconditional sum of \( C \)-narrow operators. Thus, we get a contradiction.

A similar result for \( C[0,1] \)-singular operators is proved below.

**Proposition 1.3.34.** Suppose \( T, T_n : C[0,1] \to W \) are such that \( T = \sum_{n=1}^\infty T_n \) strongly unconditionally. If all \( T_n \)'s are \( C[0,1] \)-singular, then so is \( T \).
Proof. First, let us prove the proposition for the case of two operators (and hence, any finite number of operators).

Assume the contrary. Then by Pełczyński’s classical result from [61], there is a subspace $X \subset C[0, 1]$ such that $T|_X$ is an isomorphism into and $T(X)$ is complemented. Fix a projection $P: C[0, 1] \rightarrow X$ and isomorphisms $V: C[0, 1] \rightarrow X$, $\tilde{V}: T(X) \rightarrow C[0, 1]$. Then

$$Id = V^{-1}T^{-1}PT_1V + V^{-1}T^{-1}PT_2V.$$ 

Since both summands are $C[0, 1]$-singular, this contradicts the statement of Lemma 1.3.8.

To prove the general case, we also start with the contrary. Using the same operators as introduced above, we get

$$Id = \sum_{n=1}^{\infty} V^{-1}T^{-1}PT_nV.$$ 

Since we have proved that the sum of any finite number of $C[0, 1]$-singular operators is again $C[0, 1]$-singular, a contradiction with Lemma 1.3.8 is obvious.

1.3.6 $L_1$-narrow and other operators on $L_1(\Omega)$

Here $\Omega$ denotes a complete separable metric space with an atomless probability measure $\nu$ defined on the family of Borel subsets $\Sigma$. Recall that $L_1(\Omega)$ is then isometrically isomorphic to $L_1[0, 1]$. So, we sometimes write $L_1$ for $L_1(\Omega)$.

Let also $\Sigma^+ = \{A \in \Sigma : \nu(A) > 0\}$. If $A \in \Sigma^+$, $L_1(A)$ stands for the space of (classes of) real-valued $\nu$-integrable functions supported on $A$. If $T$ is a bounded linear operator on $L_1(\Omega)$ and $A \in \Sigma^+$, we denote by $T_A$ the restriction of $T$ onto $L_1(A)$. Thus, $\text{Id}_A$ is the inclusion map from $L_1(A)$ into $L_1(\Omega)$.

We have already seen in Section 1.3.4 that the classes of narrow and almost narrow operators on $L_1$ are different. There is another important type of narrow operators, which we will introduce in this section. Following [49] we call it $L_1$-narrow, although in the literature it appeared under the name of narrow.
Let us begin with mentioning an important result obtained by combining Theorem 5.5 from N. Kalton’s paper [50] and Theorem 1.5 from H. Rosenthal’s work [68].

**Theorem 1.3.35.** For a bounded operator $T : L_1(\Omega) \to L_1(\Omega)$ the following assertions are equivalent:

(i) There is a set $A \in \Sigma^+$ and an $\varepsilon > 0$ such that $\|Tf\| \geq \varepsilon$ for all $f \in L_1(\Omega)$ with $\int \! f \, d\nu = 0$, $|f| = \chi_A$.

(ii) There is a set $A \in \Sigma^+$ such that $T_A$ is an isomorphism into.

(iii) There is a set $A \in \Sigma^+$ such that $T_A$ is an isomorphism into and $T(L_1(A))$ is complemented in $L_1(\Omega)$.

According to [68], operators satisfying condition (i) are called sign-preserving.

Let us remark that for this theorem to hold it is essential that the range space of $T$ is $L_1(\Omega)$. Otherwise, same Talagrand’s construction [78] works as a counterexample. Still a weaker statement proved by J. Bourgain and H. Rosenthal in [11] holds.

**Theorem 1.3.36.** Let $W$ be an arbitrary Banach space. If $T : L_1(\Omega) \to W$ is sign-preserving, then $T$ fixes a copy of $\ell_1$.

The opposite to the notion of a sign-preserving operator was considered by A. Plichko, M. Popov and V. Kadets in their papers [43, 44, 63].

**Definition 1.3.37.** An operator $T : L_1(\Omega) \to W$ is called $L_1$-narrow if it is not sign-preserving. In other words, $T$ is $L_1$-narrow if for every set $A \in \Sigma^+$ and $\varepsilon > 0$ there is a function $f \in L_1(\Omega)$ with $\int \! f \, d\nu = 0$, $|f| = \chi_A$ such that $\|Tf\| \geq \varepsilon$.

Due to Theorem 1.3.36, every $\ell_1$-singular operator is $L_1$-narrow.

In the following we establish relationship between $L_1$-narrow and narrow operators on $L_1(\Omega)$.

A function $f \in L_1(\Omega)$ is said to be a balanced $\varepsilon$-peak on $A \subseteq \Sigma$ if $f \geq -1$, $\text{supp} \, f \subseteq A$, $\int_{\Omega} f \, d\nu = 0$ and $\nu \{ t : f(t) = -1 \} > \nu(A) - \varepsilon$. The collection of all balanced $\varepsilon$-peaks on $A$ will be denoted by $P(A, \varepsilon)$. 


Lemma 1.3.38. If $T: L_1(\Omega) \rightarrow W$ is $L_1$-narrow, then it is not bounded from below on every $P(A, \varepsilon)$.

The proof consists of repeated application of Definition 1.3.37 (see [63, p.55, Lemma 1]).

Lemma 1.3.39. An operator $T: L_1(\Omega) \rightarrow W$ is narrow if and only if it is not bounded from below on every $P(A, \varepsilon)$.

Proof. Suppose $T$ is narrow. Let $\delta, \varepsilon > 0$, and $A \in \Sigma$ be arbitrary. Consider a slice in $L_1(\Omega)$ of the form

$$S = \{ f \in B(L_1(\Omega)) : \int_A f \, d\nu > 1 - \delta \}.$$ 

By Definitions 1.3.1, 1.3.12, applied to the elements $x = \chi_A/\nu(A)$, $y = -\chi_A/\nu(A)$ and $\delta$ we get a function $v \in S$ such that

$$\|v - \chi_A/\nu(A)\| > 2 - \delta, \quad \|T(v - \chi_A/\nu(A))\| < \delta. \quad (1.3.13)$$

Denote by $B$ the set $\{t \in A : v(t) > 0\}$. The condition $v \in S$ implies that $\|v - \chi_B v\| < \delta$, so

$$\|v\chi_B - \chi_A/\nu(A)\| > 2 - 2\delta.$$ 

Next, introduce $C = \{t \in B : v(t) > 1/\nu(A)\}$. By the last inequality

$$\|v\chi_C - \chi_A/\nu(A)\| > 2 - 2\delta, \quad \|v - \chi_C v\| < 3\delta$$

and

$$\nu(C) < 2\delta \nu(A); \quad (1.3.14)$$

to see this observe that

$$2 - 2\delta < \|\chi_B v - \chi_A/\nu(A)\| \leq \|\chi_C v - \chi_C/\mu(A)\| + \|\chi_{B \setminus C} v - \chi_{A \setminus C}/\nu(A)\|$$

$$= \int_C (\chi_B v - \frac{1}{\nu(A)}) \, d\nu + \int_{A \setminus C} \frac{1}{\nu(A)} - \chi_B v \, d\nu$$

$$\leq \|\chi_C v\| + \frac{1}{\nu(A)}(\nu(A) - \nu(C)).$$

So, $\|\chi_C v\| \geq 1 - 2\delta$ and $1 - \frac{\nu(C)}{\nu(A)} \geq 1 - 2\delta$. 

37
Put $f = (\nu(A)/\beta) \chi_C v - \chi_A$ with $\beta = \int_C v \, d\nu$ so that $\int_A f \, d\nu = 0$. Since $\int_A v \, d\mu > 1 - \delta$ we have from $\|v - \chi_C v\| < 3\delta$ that $\beta \geq 1 - 4\delta$. By (1.3.13) we conclude that
\[
\|Tf\| = \nu(A) \left\| T \left( \frac{\chi_C v}{\beta} - \frac{\chi_A}{\nu(A)} \right) \right\| \leq \nu(A) \left( \|T\| \left\| \frac{\chi_C v}{\beta} - v \right\| + \delta \right)
\]
and
\[
\left\| \frac{\chi_C v}{\beta} - v \right\| \leq \left\| \frac{\chi_C v - v}{\beta} \right\| + \left\| \frac{v}{\beta} - v \right\| \leq \frac{3\delta}{\beta} + \left( \frac{1}{\beta} - 1 \right) \leq \frac{7\delta}{1 - 4\delta},
\]
and if $\delta$ is small enough, by (1.3.14) $f \in P(A, \varepsilon)$. This proves the unboundedness from below.

To prove the converse, let us assume that $T$ unbounded from below on every $P(A, \varepsilon)$. Let $x, y \in S(L_1(\Omega))$, $x^* \in S(L_\infty(\Omega))$ and $\varepsilon > 0$ be such that $\langle x^*, x \rangle = \alpha$. Without loss of generality we may assume that there is a partition $A_1, \ldots, A_n$ of $\Omega$ such that the restrictions of $x, y$ and $x^*$ on $A_k$ are constants, say $a_k, b_k$ and $c_k$ respectively. By our assumption $T$ is unbounded from below on each of the $P(A_k, \delta)$ for every $\delta > 0$, $k = 1, \ldots, n$. Let us fix functions $f_k \in P(A_k, \delta)$ such that $\|Tf_k\| < \delta$, $k = 1, \ldots, n$, and put
\[
z = x + \sum_{k=1}^n a_k f_k.
\]
By definition of balanced $\delta$-peaks, $\langle x^*, z \rangle = \alpha$, $\|z\| = 1$, and $\|T(x - z)\|$ and $\mu(\text{supp} v)$ become arbitrarily small when $\delta$ is small enough. Thus $\delta$ can be chosen so that $z$ fulfills the conditions $\|T(x - z)\| < \varepsilon$ and $\|y + z\| > 2 - \varepsilon$. □

Lemmas 1.3.38 and 1.3.39 along with Theorem 1.3.35 yield the following result.

**Proposition 1.3.40.** Let $T : L_1(\Omega) \to W$ be a bounded operator. If $T$ is $L_1$-narrow, then it is narrow. If $W = L_1(\Omega)$, then the inverse implication is also true. In particular, all $L_1$-narrow operators satisfy the Daugavet equation.

**Problem 1.3.41.** Is a narrow operator from $L_1(\Omega)$ into a general Banach space $W$ $L_1$-narrow? It is also interesting to know whether every almost narrow operator is narrow in case $W = L_1(\Omega)$ (for $W \neq L_1(\Omega)$ a counterexample is constructed in Section 1.3.4).

The following version of Proposition 1.3.40 will be proved in Section 1.4.5.

38
Proposition 1.3.42. Suppose $L_1(\Omega)$ is a closed subspace of a Banach space $Y$. Then there is an equivalent norm on $Y$ remaining the same on $L_1(\Omega)$ and such that every narrow (hence, every $L_1$-narrow) operator $T: L_1(\Omega) \mapsto W$ is narrow with respect to $Y$. In particular, the pair $(L_1(\Omega), Y)$ has the Daugavet property with respect to $L_1$-narrow operators.

The last conclusion of this proposition was proved in [45].

It is a fundamental question to know whether the sum of two "small" operators is also "small". It was proved by P. Enflo and T. Starbird [26] that this is true for $L_1$-singular operators from $L_1$ into $L_1$. Their result implies that $L_1$ is prime, i.e. whenever it splits into a direct sum of two subspaces, one of them is isomorphic to $L_1$. Now using the Daugavet properties of "small" operators on $L_1$, we are able to give an alternative proof of P. Enflo and T. Starbird's result and the analogous result for $L_1$-narrow operators, actually generalizing them to infinite unconditional sums.

Proposition 1.3.43. Let $T, T_n: L_1(\Omega) \mapsto L_1(\Omega)$ be bounded operators, $n \in \mathbb{N}$. Suppose $T = \sum_{n=1}^{\infty} T_n$ strongly unconditionally.

(i) If all $T_n$'s are $L_1$-singular, then so is $T$;

(ii) If all $T_n$'s are $L_1$-narrow, then so is $T$.

Proof. (i). Let us prove this in the case of two operators, as we did for Proposition 1.3.34.

So suppose $T = T_1 + T_2$ and $T$ is fixes a copy of $L_1$. Then by [26, Theorem 4.2] there is a subspace $X \subset L_1$ isomorphic to $L_1$ such that $T|_X$ is an isomorphism and $T(X)$ is complemented in $L_1$. Denote by $P: L_1(\Omega) \mapsto L_1(\Omega)$ a projection onto $T(X)$. Also let $U: L_1 \mapsto X$ and $V: T(X) \mapsto L_1$ be isomorphisms. Then, clearly,

$$VPTU = VPT_1U + VPT_2U$$

and $VPTU$ is an isomorphism of $L_1$ onto itself. Denote $T'_i = (VPTU)^{-1}VPT_iU$, $i = 1, 2$. Then we have $\text{Id} = T'_1 + T'_2$, where each $T'_i$ is $L_1$-singular. By Proposition 1.3.40 $T'_i$ satisfy the Daugavet equation. Hence,

$$1 + \|T'_2\| = 1 + \|\text{Id} - T'_1\| = 2 + \|T'_1\| = 2 + \|\text{Id} - T'_2\| = 3 + \|T'_2\|,$$

which is a contradiction.
In the general case, we analogously define $U, V$ and $P$, and put $T'_n = (VPTU)^{-1}VPTnU$, $n \in \mathbb{N}$. So, $\text{Id} = \sum_{n=1}^{\infty} T'_n$ strongly unconditionally. By the previous, every finite sum of $T'_n$’s is an $L_1$-singular operator and hence satisfies the Daugavet equation. This contradicts Lemma 1.3.8.

The proof of (ii) goes similarly. First, we show that a sum of two $L_1$-narrow operators is $L_1$-narrow. Assume $T = T_1 + T_2$ is such a sum and $T$ is not $L_1$-narrow. Then using Theorem 1.3.35 find an $A \in \Sigma^+$ so that $T_A$ is an isomorphism into and $T(L_1(A))$ is complemented in $L_1$. Denote by $P$ a projection from $L_1$ onto $T(L_1(A))$ and put $U = (PT_A)^{-1}$. Then $\text{Id} = U(T_1)_A + U(T_2)_A$. Since the summands are still $L_1$-narrow, we have a contradiction.

The general case is now proved as in (i).

We remark that part (ii) was proved by A. Plichko and M. Popov in [63], but their argument contained a mistake.

**Corollary 1.3.44.** If $L_1(\Omega) = \oplus_{n=1}^{\infty} X_n$ unconditionally, then there is some $X_{n_0}$ isomorphic to $L_1$.

*Proof.* Denote by $P_n$ the projection onto $X_n$. Then $\text{Id} = \sum_{n=1}^{\infty} P_n$ strongly unconditionally. By Proposition 1.3.43 (I) some $P_{n_0}$ fixes a copy of $L_1$. In view of [26, Theorem 4.2] $X_{n_0}$ contains a complemented copy of $L_1$. Since $X_{n_0}$ itself is complemented in $L_1$, it is actually isomorphic to $L_1$. □

Using part (ii) of Proposition 1.3.43 we may even claim that some $L_1(A)$ projects isomorphically into $X_{n_0}$.

**Remark 1.3.45.** Recently V. Kadets has found an alternative more ”daugavetian” proof of Proposition 1.3.43 and Corollary 1.3.30, which avoids Theorem 5.5 from N. Kalton’s paper [50] and employs Theorems 7 and 12 from [45].

### 1.4 Daugavet renormings

In this section we show that the Daugavet property extends to all larger spaces after appropriate renorming. Our main result says that if $X$ is a Daugavet space and $X \subset Y$, then there is an equivalent norm on $Y$ such that $(X, Y)$ becomes a Daugavet pair (Theorem 1.4.9). In some cases this is shown to be true even without any renorming, for instance if $Y = C(B(X^*))$ (Proposition 1.4.2).
Roughly, our plan is to introduce an auxiliary space \( m_0 \) built upon \( X \) so that \((X, m_0)\) preserves the Daugavet property. \( m_0 \) turns out to be big enough to have a copy of any other space \( Y \) containing \( X \) with separable quotient \( Y/X \). Then we construct an embedding of \( Y \) into \( m_0 \) which leaves \( X \) untouched. So, the norm inherited by \( Y \) from its copy in \( m_0 \) fulfills the requested conditions.

If \( Y/X \) is not separable, we build the analogous \( m_0 \) upon the \( \ell_\infty \)-sum of a sufficient number of \( C(B(X^*)) \). Now \( m_0 \) becomes large enough to engulf \( Y \) and still manages not to loose the Daugavet property of \((X, m_0)\). Then the norm is constructed analogously.

In further subsections we consider \( X = C(K) \) and \( X = L_1 \), and show that for those spaces our renorming yields the Daugavet property even with respect to \( C \)-narrow and \( L_1 \)-narrow operators, respectively. Finally, we prove Propositions 1.3.25 and 1.3.42 as promised.

### 1.4.1 Subspaces of \( C(K) \)

We start with characterization of subspaces in \( C(K) \) that form a Daugavet pair with \( C(K) \). Although we assume here \( K \) to be a general compact Hausdorff space without isolated points, our main target is the sets \( K = B(X^*) \) and \( K = \text{ext}B(X^*) \), where ”ext” stands for extreme points and the bar means closure in the weak*-topology.

As in example (i) on page 11, we denote by \( \delta_k, k \in K \), the functional on \( C(K) \) defined by \( \delta_k(f) = f(k) \), \( f \in C(K) \).

**Lemma 1.4.1.** Let \( X \) be a subspace of \( C(K) \). The following conditions are equivalent:

(a) The pair \((X, C(K))\) has the Daugavet property;

(b) For every \( \varepsilon > 0 \), \( x^* \in S(X^*) \) and open set \( U \) in \( K \) there exists a point \( u \in U \) such that \( \|x^* + \delta_u\| > 2 - \varepsilon \);

(c) For every \( x^* \in S(X^*) \) and open set \( U \) in \( K \) there exists a (closed) \( G_\delta \)-set \( G \) in \( U \) such that \( \|x^* + \delta_u\| = 2 \), whenever \( u \in G \).

**Proof.** (a)⇒(b). Let \( f \in S(C(K)) \) be a function vanishing outside \( U \). By Lemma 1.2.3(c), there is a slice \( S \subset S(f, \frac{1}{2}) \) such that \( \|x^* + \mu\| > 2 - \varepsilon \), for all \( \mu \in S \). Pick any \( \delta_u \in S \). Clearly, \( \delta_u(f) = f(u) > \frac{1}{2} \) and hence, \( u \in U \). So, \( u \) is the required point.
\[(b) \Rightarrow (c).\] Apply part \((b)\) countably many times and use the lower weak*-semicontinuity of a dual norm and the regularity of a Hausdorff compact set.

\[(c) \Rightarrow (a).\] We apply Lemma 1.2.3 again. Pick an arbitrary \(x^* \in S(X^*)\) and weak*-slice \(S(f, \varepsilon)\) in \(B(C^*(K))\). Let \(U = \{k \in K : f(k) > 1 - \varepsilon\}\). By condition \((c)\), we can find a point \(u \in U\) such that \(|x^* + \delta u|_X| = 2\). Moreover, we have \(\delta u(f) = f(u) > 1 - \varepsilon\) and hence, \(\delta u \in S(f, \varepsilon)\). This completes the proof. \(\square\)

Of course, not every pair \((X, C(K))\) has the Daugavet property provided \(X\) does, e.g. \(\{C[0, 1]; C([0, 1] \cup [2, 3])\}\). However, there always exists a \(C(K)\)-space containing \(X\) isometrically and such that the pair \((X, C(K))\) preserves the Daugavet property.

**Proposition 1.4.2.** If the pair \((X, Y)\) has the Daugavet property and \(K\) is either \(B(Y^*)\) or \(\text{ext} B(Y^*)\), then the pair \((X, C(K))\) also has the Daugavet property.

**Proof.** In both cases we verify condition \((b)\) of Lemma 1.4.1.

First, consider \(K = B(Y^*)\). Fix arbitrary \(\varepsilon > 0\), open set \(U \subset K\) and \(x^* \in S(X^*)\). By Lemma 1.2.5\((c)\) there is \(y^* \in U\) such that \(|x^* + y^*|_X| > 2 - \varepsilon\). We denote by \(u\) the functional \(y^*\) regarding it as a point of topological space \(K\). It remains to notice that \(\delta u|_X = y^*|_X\).

Let now \(K = \text{ext} B(Y^*)\). Fix \(\varepsilon, U\) and \(x^*\) as above. By the Choquet Lemma [33, p.49] we may assume that \(U\) is induced by a slice \(S\). By Lemma 1.2.3\((c)\) there is another slice \(S_1 \subset S\), and hence, there is a \(y^* \in S_1 \cap K\) such that \(|x^* + y^*|_X| > 2 - \varepsilon\) holds for all \(y^* \in S_1\). Since every slice contains an extreme point, we may pick a \(y^* \in S_1 \cap K\) and put \(u = y^*\). \(\square\)

### 1.4.2 Auxiliary space \(m_0(K)\)

Let again \(K\) be a general compact Hausdorff space without isolated points. Recall that a first category set in \(K\) is a countable union of nowhere dense sets. All other sets belong to the second category. By the Baire Category Theorem, every open set in \(K\) is of the second category.

**Lemma 1.4.3 ([51]).** For every second category set \(B\) there is an open set \(O\) such that whenever \(O'\) is another open set intersecting \(O\), \(O' \cap B\) is of the second category.
Let us introduce the following two spaces
\[ \ell_\infty(K) = \{ f : K \to R, \| f \|_\infty = \sup(\| f(s) \|, s \in K) < \infty \}, \]
\[ m(K) = \{ f \in \ell_\infty(K) : \text{supp} f \text{ is a first category set} \} \]

$m(K)$ is a closed linear subspace of $\ell_\infty(K)$. To see this observe the following
\[ \text{supp}(\lambda f) = \text{supp}(f), \lambda \neq 0; \]
\[ \text{supp}(f_1 + f_2) \subset \text{supp}(f_1) \cup \text{supp}(f_2); \]
\[ \text{if } \| f_n - f \| \to 0, \text{ then supp}(f) \subset \bigcup_{n=1}^{\infty} \text{supp}(f_n). \]

Now we define $m_0(K)$ to be the quotient $\ell_\infty(K)/m(K)$ endowed with the factor-norm
\[ ||[f]|| = \inf \{ \sup(\| f(s) \|, s \in K \setminus F) : F \text{ is a first category set} \}. \]

**Lemma 1.4.4.** The quotient map $J : \ell_\infty(K) \to m_0(K)$ restricted to $C(K)$ is an isometric embedding.

**Proof.** It is enough to prove that $\| f \|_\infty = ||[f]||$, $f \in C(K)$. In fact, $||[f]|| \leq \| f \|_\infty$ is obvious. To show the converse, we fix an $\varepsilon > 0$ and choose an open set $U \subset K$ such that $|f(k)| > \| f \|_\infty - \varepsilon$, for all $k \in U$. Since $U$ is of second category, we obtain
\[ ||[f]|| \geq \inf \{ \sup(\| f(s) \|, s \in U \setminus F) : F \text{ is a first category set} \} > \| f \|_\infty - \varepsilon. \]

Since $m(K)$ is an ideal in $\ell_\infty(K)$, $m_0(K)$ is a real $C^*$-algebra, and hence, is a $C(Q)$-space. The appropriate compact set $Q = Q_K$ can be defined as the set of all real homomorphisms on $m_0(K)$ endowed with the induced weak* topology. This is precisely the set of limits by ultrafilters on $K$, which do not contain the first category sets. Let $\mathcal{U}$ be such an ultrafilter. We denote by $\lim \mathcal{U}$ the point in $K$ to which it converges and by $\mathcal{U}_\mathcal{V}$ the real homomorphism on $m_0(K)$ it generates ($\mathcal{U}_\mathcal{V} \in Q_K$).

**Lemma 1.4.5.** Suppose $U$ is an open set in $Q_K$, then there is an open set subset $V$ in $K$ such that for every $v \in V$ one can find an ultrafilter $\mathcal{U}_v$ on $K$ with $\lim \mathcal{U}_v = v$ and $\mathcal{U}_v \lim \in U$.

**Proof.** By the construction of $Q_K$ we may assume there are a finite sequence $(f_i)_{i=1}^n \subset m_0(K)$, $\varepsilon > 0$ and ultrafilter $\mathcal{U}_0$ on $K$ such that $U = \{ \varphi \in Q_K :
|φ(f_i) − Ψ_0 -lim(f_i)| < ε}. Denote b_i = Ψ_0 -lim(f_i). We fix a second category set \( B ∈ Ψ_0 \) with the following property:

\[
f_i(B) ⊂ (b_i - ε, b_i + ε), \quad i = 1, 2, \ldots, n.
\]

(1.4.1)

Applying Lemma 1.4.3 we find an open set \( V \) in \( K \) such that for any open \( V' ⊂ V \), \( V' ∩ B \) is a second category set. It remains to show that \( V \) is required.

Indeed, let \( v ∈ V \). Consider an ultrafilter \( Ψ_v \) containing

\[
\{V' ∩ B: V' \text{ is an open neighborhood of } v\}.
\]

Plainly, \( \lim Ψ_v = v \). On the other hand, in view of (1.4.1) we have \( Ψ_v -\lim(f_i) ∈ (b_i - ε, b_i + ε), i = 1, 2, \ldots, n \). This means \( Ψ_v -\lim ∈ Ψ \). \( \square \)

**Lemma 1.4.6.** If the pair \((X, C(K)\)) has the Daugavet property, then so does the pair \((X, m_0(K))\).

**Proof.** We apply Lemma 1.4.1 again using the interpretation of \( m_0(K) \) as a \( C(Q) \)-space. To this end, we fix \( ε > 0 \), open set \( U ⊂ Q_K \) and \( x^* ∈ S(X^*) \). Applying Lemma 1.4.5 to \( U \) we find the corresponding open set \( V ⊂ K \). Lemma 1.4.1 applied to the pair \((X, C(K))\) yields \( v ∈ V \) such that \( \|x^* + δ_v|_X\| > 2 - ε \). Consider the ultrafilter \( Ψ_v \) with \( \lim Ψ_v = v \) and \( Ψ_v -\lim ∈ U \), and denote \( u = Ψ_v -\lim \). So, \( δ_v|_X = δ_u|_X \) and \( u ∈ U \). Hence, the point \( u \) is desired. \( \square \)

**Corollary 1.4.7.** The pair \((C(K), m_0(K))\) has the Daugavet property.

**Proposition 1.4.8.** Let \((X, Y)\) have the Daugavet property and \( K \) be either \( B(Y^*) \) or \( \overline{B(Y^*)} \). Then the pair \((X, m_0(K))\) has the Daugavet property too.

Combine Proposition 1.4.2 and the previous lemma.

### 1.4.3 The main renorming theorem

Now we are in a position to prove our main result.

**Theorem 1.4.9.** Let \( X \) and \( Y \) be Banach spaces such that \( X ⊂ Y \). If \( X \) has the Daugavet property, then there is an equivalent norm on \( Y \) remaining the same on \( X \) in which the pair \((X, Y)\) possesses the Daugavet property.
According to our plan, we prove a lemma which establishes, in some sense, a property of universality for \( m_0(K) \), where \( K \) is the unit ball of a dual space. Since in the sequel we often deal with density character of a Banach space \( X \) (the minimal cardinality of a dense set in \( X \)), we denote it by \( \text{dens}(X) \).

**Lemma 1.4.10.** Let \( X \) be a closed subspace of Banach spaces \( Y \) and \( X_{\text{big}} \). Let also \( \text{dens}(Y/X) = \beta \), where \( \beta \) is an ordinal. Suppose \( B(X_{\text{big}}^*) \) contains a family \( \{B_\alpha\}_{\alpha<\beta} \) of disjoint second category sets such that if \( B' = \bigcup_{\alpha<\beta} B_\alpha \), then \( B' \cap -B' = \emptyset \). Then there is an isomorphic embedding \( \mathcal{E}: Y \mapsto m_0(B(X_{\text{big}}^*)) \), which coincides with the natural one on \( X \).

**Proof.** Let us fix a dense set \( \{[y_\alpha]\}_{\alpha<\beta} \subset B(Y/X) \) with \( \|y_\alpha\| \leq 1 \), and for every \( \alpha < \beta \) find a functional \( \varphi_\alpha \in S(Y^*) \) so that \( \varphi_\alpha(y_\alpha) = \|y_\alpha\| \). Also to every \( w^* \in X_{\text{big}}^* \) we assign a functional \( \overline{\varphi} \) obtained by restricting \( w^* \) on \( Y \) and then extending it to all of \( Y \) by the Hahn-Banach Theorem.

First, we embed \( Y \) into \( \ell_\infty(B(X_{\text{big}}^*)) \) so that every element from the image of \( S(Y) \) takes values greater than 1/8 on a second category set. To this end, for each \( y \in Y \) we define a function \( f_y \in \ell_\infty(B(X_{\text{big}}^*)) \) as follows:

\[
f_y(w^*) = \begin{cases} \overline{\varphi}^*(y), & w^* \in B(X_{\text{big}}^*) \setminus B_0 \\ \overline{\varphi}^*(y) + 8\varphi_\alpha(y), & w^* \in B_\alpha \end{cases}
\]

Clearly the mapping \( \mathcal{E}: y \mapsto f_y \) is linear and bounded. Moreover, \( f_y(w^*) = w^*(y) \), if \( y \in Y \). So, \( \mathcal{E}|_X \) is the natural inclusion of \( X \) into \( \ell_\infty(B(X_{\text{big}}^*)) \) (even into \( C(B(X_{\text{big}}^*)) \)).

Suppose now \( y \in Y \) with \( \|y\| = 1 \). Then either \( \|\varphi_\alpha\| \leq \frac{1}{4} \) or \( \|\varphi_\alpha\| > \frac{1}{4} \). In the former case there is an \( x \in X \) such that \( \|y - x\| < \frac{1}{8} \). Because of the condition imposed on \( B' \), the set \( \{w^* \in B(X_{\text{big}}^*) \setminus B': w^*(x) > \|x\| - \frac{1}{8} \} \) is of the second category, and for every its element \( w^* \) we have

\[
|f_y(w^*)| = |\overline{\varphi}^*(y)| > |\overline{\varphi}^*(x)| - \frac{3}{8} = |w^*(x)| - \frac{3}{8} = \|x\| - \frac{1}{2} > \frac{1}{8}.
\]

So, \( |f_y(w^*)| > \frac{1}{8} \) hold for all \( w^* \) from a second category set.

In the case \( \|\varphi_\alpha\| > \frac{1}{4} \), there is an ordinal \( \alpha, \alpha < \beta \), and \( x \in X \) such that \( \|y_\alpha\| > \frac{1}{4} \) and \( \|y - y_\alpha - x\| < \frac{1}{16} \). From this we get for all \( w^* \in B_\alpha \)

\[
|f_y(w^*)| = |\overline{\varphi}^*(y) + 8\varphi_\alpha^*(y)| \\
> |8\varphi_\alpha^*(y_\alpha - x)| - \frac{1}{2} - |\overline{\varphi}^*(y)| = 8\|y_\alpha\| - \frac{3}{2} \\
> \frac{8}{4} - \frac{3}{2} = \frac{1}{2}.
\]

45
We define the desired isomorphic embedding $\mathcal{E}: Y \mapsto m_0(B(X^*_{\text{big}}))$ by $\mathcal{E}(y) = [\mathfrak{f}(y)]$, $y \in Y$. 

It is not hard to construct a countable number of such sets $B_\alpha$ in the dual ball of every infinite-dimensional Banach space. So, in the special case when $Y/X$ is separable and $X = X_{\text{big}}$, we obtain the following corollary.

**Corollary 1.4.11.** Let $X$ be a closed subspace of $Y$ such that $Y/X$ is separable. Then there exists an isomorphic embedding of $Y$ into $m_0(B(X^*))$, which coincides with the natural one on $X$.

**Proof of Theorem 1.4.9.** Suppose $X$ is a Daugavet space and $Y$ is some Banach space containing $X$. If $B(X^*)$ had enough disjoint second category sets to meet the condition of Lemma 1.4.10 with $X = X_{\text{big}}$, there would exist an isomorphic embedding $\mathcal{E}$ of $Y$ into $m_0(B(X^*))$. Then by Corollary 1.4.8, the equivalent norm $|||y||| = \|\mathcal{E}(y)\|$ would be desired.

However, that is not always the case, for example when $\text{dens}(Y)$ exceeds $\text{dens}(m_0(B(X^*)))$. So, we should replace $X$ by a bigger space, say $X_{\text{big}}$, which fulfills the conditions of Lemma 1.4.10 and at the same time preserves the Daugavet property for $(X, X_{\text{big}})$. Then there is an embedding $\mathcal{E}: Y \mapsto m_0(B(X^*_{\text{big}}))$, with $\mathcal{E}|_X = \text{Id}$. Since the pair $\{X, m_0(B(X^*_{\text{big}}))\}$ still has the Daugavet property by Proposition 1.4.8, the norm $|||y||| = \|\mathcal{E}(y)\|$ is desired.

Let $\beta$ be as in Lemma 1.4.10. We define $X_{\text{big}}$ to be the $\ell_\infty$-sum of $\beta$ copies of $C(B(X^*))$, i.e.

$$X_{\text{big}} = \left\{ (f_\alpha)_{\alpha < \beta} : f_\alpha \in C(B(X^*)) \text{ and } \|(f_\alpha)\| = \sup_{\alpha < \beta} \|f_\alpha\| < \infty \right\}.$$

$X$ embeds into $X_{\text{big}}$ by

$$x \to (r_\alpha)_{\alpha < \beta}, \quad x \in X$$

$$r_\alpha(s) = s(x), \quad s \in B(X^*).$$

So, $X$ can be regarded as a subspace of $X_{\text{big}}$. It is not difficult to prove that the pair $(X, X_{\text{big}})$ has the Daugavet property. Indeed, fix $y \in S(X_{\text{big}})$, slice $S \subset B(X)$ and $\varepsilon > 0$. Find a $\gamma < \beta$ for which $\|f_\gamma\| > 1 - \varepsilon/2$. According to Proposition 1.4.2, the pair $\{X, C(B(X^*))\}$ has the Daugavet property and hence, there is an $x \in S$ such that $\|x + f_\gamma\| > 1 - \varepsilon$. Then clearly $\|x + y\| > 1 - \varepsilon$ and condition (b) of Lemma 1.2.4 is verified.
Now fix $f \in C(B(X^*))$, $\|f\| = 1$, and for every $\alpha$, $\alpha < \beta$, define a vector $\omega_\alpha \in X_{\text{big}}$ by $w_\alpha = (f_{\alpha'})_{\alpha' < \beta}$ so that $f_{\alpha'} = f$, if $\alpha' = \alpha$, and $f_{\alpha'} = 0$ otherwise. Put $B_\alpha$ to be $S(w_\alpha, \frac{1}{\beta})$, the weak*-slice of the dual unit ball $B(X_{\text{big}}^*)$. Since every $B_\alpha$ is weak*-open, it is a second category set. Next, $B_{\alpha'} \cap B_{\alpha''} = \emptyset$, $\alpha' \neq \alpha''$, for otherwise every $w^* \in B_{\alpha'} \cap B_{\alpha''}$ would have norm bigger than 1. For the same reason, $B' = \cup_{\alpha < \beta} B_\alpha$ is disjoint with $-B'$.

So, we have constructed the space satisfying all our requirements and this finishes the proof. \hfill $\square$

**Problem 1.4.12.** It would be interesting answer the following question. If $X$ is a Daugavet space contained in $Y$, can $Y$ be renormed to have the Daugavet property? We can assert that such a renorming cannot be accomplished leaving the norm on $X$ unchanged. In fact, let $X = L_{\infty}[0,1]$. It is 1-complemented in every containing Banach space. Since every 1-codimensional subspace of a Daugavet space is at least 2-complemented, $L_{\infty}[0,1] \oplus \mathbb{R}$ cannot be renormed to have the Daugavet property so that the equivalent norm remains the same on $L_{\infty}[0,1]$.

### 1.4.4 More on pairs $(C(K), Y)$

This section we present the proof of Proposition 1.3.25. Our aim is to show that in the norm provided by the proof of Theorem 1.4.9 every $C$-narrow operator $T$: $C(K) \mapsto W$ is narrow with respect to $Y$.

So, suppose $T$: $C(K) \mapsto W$ is a $C$-narrow operator and denote $X := C(K)$ for convenience. Recall that the norm on $Y$ is inherited from $m_0(B(X_{\text{big}}^*))$, where $X_{\text{big}}$ constructed in the proof of Theorem 1.4.9. Thus, we prove the proposition if we show that $T$ is with respect to $m_0(B(X_{\text{big}}^*))$. Moreover, since $T \oplus x^*$ is also $C$-narrow for all $x^* \in X^*$, it is enough to prove only almost narrowness.

To this end, let us fix an $f \in S(X)$, $y \in S(m_0(B(X_{\text{big}}^*)))$ and $\varepsilon > 0$. It suffices to find $g \in X$ with $\|g\| < 1 + \varepsilon$ such that $\|Tf - Tg\| < \varepsilon$ and $\|y + g\| > 1 - 3\varepsilon$ (here we identify $g$ with its image in $m_0(B(X_{\text{big}}^*))$).

Our proof starts with isolating a special weak*-dense subset of $B(X_{\text{big}}^*)$.

Define the evaluation functional $\delta_{s,\gamma}$, $s \in B(X^*)$, $\gamma < \beta$ on $X_{\text{big}}$ by $\delta_{s,\gamma}((f_{\alpha})_{\alpha < \beta}) = f_{\gamma}(s)$. Clearly, $\delta_{s,\gamma} \in B(X_{\text{big}}^*)$.

**Lemma 1.4.13.** In every weak*-open set of $B(X_{\text{big}}^*)$ there is an element of the form $\sum_{i=1}^{n} \mu_i \delta_{s_i,\gamma_i}$, where $s_i = \sum_{j=1}^{m_i} \lambda_{ij} \delta_{k_{ij}}$, $\sum_{i=1}^{n} |\mu_i| = \sum_{j=1}^{m_i} |\lambda_{ij}| = 1$, \hfill 47
\(k_{ij} \in K\). Moreover, each \(k_{ij}\) can be chosen in some \(G_{ij} \in \text{van}\) \(T\), where \(G_{ij}\)'s lie in open sets \(U_{ij}\) with disjoint closures, respectively.

Proof. For the first part of the lemma we simply prove that the set of functionals \(\delta_{s,\gamma}^{\text{big}}\) with \(s = \sum \lambda_i \delta_{i,j}\) is 1-norming. Indeed, for \((f_\alpha)_{\alpha < \beta} \in S(X_{\text{big}})\) and \(\delta > 0\), pick a \(\gamma < \beta\) with \(\|f_\gamma\| > 1 - \delta\). Then find an open \(U \subset B(X^*)\) with \(\|f_\gamma(s)\| > 1 - \delta\), for all \(s \in U\). Since sums \(\sum \lambda_i \delta_{i,j}\) are weak*-dense in \(B(X^*)\) we can find one in \(U\). Denote it by \(s\). Then \(\delta_{s,\gamma}^{\text{big}}((f_\alpha)_{\alpha < \beta}) > 1 - \delta\).

To prove the second part, in any given open set \(O \subset B(X^*_{\text{big}})\) find a sum as in the first part. Then chose open neighborhoods \(U_{ij}\) of \(k_{ij}\), respectively, such that for all other \(k'_{ij} \in U_{ij}\) we also have \(\sum_{i,j} \mu_i \delta_{i,j}^{\text{big}} \delta_{s',\gamma} \in O\), where \(s'_i = \sum_{j=1}^m \lambda_i \delta_{i,j}^\gamma\). We can pick \(k'_{ij}\)'s all different and enclose them into further open subsets \(U'_{ij} \subset U_{ij}\) now with disjoint closures. By Lemma 1.3.33, each \(U'_{ij}\) contains a vanishing set \(G_{ij}\). So, take \(k'_{ij} \in G_{ij}\). \(\square\)

Going back to our proof, assume that the set

\[B = \{t \in B(X^*_{\text{big}}) : y(t) > 1 - \varepsilon\},\]

is of second category. Otherwise, we consider \(\{t \in B(X^*_{\text{big}}) : y(t) < -1 + \varepsilon\}\) and the the rest of the proof goes similarly.

Let \(O\) be as in Lemma 1.4.3 applied to \(B\). Then, there exists a \(t_0 \in O\) of the form described in Lemma 1.4.13. Moreover we can assume that \(\|f - f(k_{ij})\| < \varepsilon\) on \(U_{ij}\), respectively. According to Definition 1.3.31, there are positive \(\varphi_{ij} \in S(C_0(U_{ij}))\) such that \(\varphi_{ij}(k_{ij}) > 1 - \varepsilon\) and \(\|T\varphi_{ij}\| < \frac{\varepsilon}{2^{(m_1 + m_2 + \ldots + m_n)}}\).

Put \(g = f + \sum_{i,j} [\text{sign}(\mu_i \lambda_{i,j}) - f(k_{ij})] \varphi_{ij}\). Then it easily follows from above that \(\|g\| < 1 + \varepsilon\), and \(\|Tf - Tg\| < \varepsilon\). On the other hand, we get

\[g(t_0) = \sum_{i,j} f(k_{ij}) \mu_i \lambda_{ij} + \sum_{i,j} |\mu_i||\lambda_{ij}| \varphi_{ij}(k_{ij})\]

\[-\sum_{i,j} f(k_{ij}) \varphi_{ij}(k_{ij}) \mu_i \lambda_{ij} > 1 - 2\varepsilon.\]

Hence, there is an open neighborhood \(O'\) of \(t_0\) intersecting \(B\) by a second category set, on which \(g > 1 - 2\varepsilon\). Then \(y + g > 1 - 3\varepsilon\) on \(O' \cap B\) and the proof is finished.

48
1.4.5 More on pairs \((L_1(\Omega), Y)\)

In this section we prove Proposition 1.3.42 by showing that in the norm on \(Y\) given by Theorem 1.4.9 every narrow (hence, every \(L_1\)-narrow) operator \(T: L_1(\Omega) \mapsto W\) is narrow with respect to \(Y\). In fact, as in the previous section, it is enough to show only almost narrowness, because \(T \oplus x^*\) remains \(L_1\)-narrow for every \(x^* \in L_\infty(\Omega)\). For convenience we denote \(X := L_1(\Omega)\) in the sequel.

So, let us fix \(f \in S(X), \ y \in S(m_0(B(X^*_{big})))\) and \(\varepsilon > 0\). Our aim is to find a \(g \in S(X)\) such that \(\|y + g\| > 2 - 2\varepsilon\) and \(\|Tf - Tg\| < \varepsilon\). Without loss of generality we assume that \(f = \sum_{k=1}^p a_k \chi_{A_k}\). And as before we carry out the argument when \(B = \{ t \in B(X^*_{big}) : y(t) > 1 - \varepsilon \}\) is a second category set, the opposite case being treated similarly. Let us denote by \(O\) the open set provided by Lemma 1.4.3. Just by the same argument as in Lemma 1.4.13, there is a point in \(O\) of the form \(\sum_{i=1}^n \mu_i \delta_{s_i, \gamma_i}^{big}\), where \(s_i \in B(X^*)\). According to Lemma 1.3.39 there are functions \(f_{m,k} \in P(A_k, \frac{1}{m})\) such that \(\sum_{k=1}^p |a_k||Tf_{m,k}| < \varepsilon\), for all \(m \in \mathbb{N}\). Put \(g_m = f + \sum_{k=1}^p a_k f_{m,k}\). Then automatically \(\|Tf - Tg_m\| < \varepsilon\) and we only need to find an \(m\) for which \(\|g_m + y\| > 2 - \varepsilon\). Let us consider \(A_{m,k} = A_k \cap \{ f_{m,k} \neq -1 \}\). Then \(\nu(A_{m,k}) < \frac{1}{m}\) and hence \(\nu(\bigcup_{k=1}^p A_{m,k}) < \frac{p}{m}\). Modify \(s_i\) as follows:

\[
s_{i,m} = \sum_{k=1}^p \text{sign}(a_k \mu_i) \chi_{A_{m,k}} + s_i \cdot \chi_{\Omega \setminus \bigcup_{k=1}^p A_{m,k}}.
\]

Clearly, \(s_{i,m}\) tends to \(s_i\) weakly* and hence there exists an \(m\) with \(t = \sum_{i=1}^n \mu_i \delta_{s_{i,m}, \gamma_i}^{big} \in \mathcal{O}\).

We claim that \(s_{i,m}(g_m) = \text{sign}(\mu_i)\). Indeed, by the definition of \(P(A_k, \frac{1}{m})\),

\[
s_{i,m}(g_m) = \sum_{k=1}^p \int_{A_{m,k}} |a_k|(1 + f_{m,k}) d\nu \cdot \text{sign}(\mu_i) = \sum_{k=1}^p \int_{A_k} |a_k|(1 + f_{m,k}) d\nu \cdot \text{sign}(\mu_i) = \text{sign}(\mu_i).
\]

Using this we get \(g_m(t) = \sum_{i=1}^n |\mu_i| = 1\). Take a neighborhood \(\mathcal{O}'\) of \(t\) such that \(g_m > 1 - \varepsilon\) on \(\mathcal{O}'\). Then \(g_m + y > 2 - 2\varepsilon\) on the second category set \(\mathcal{O}' \cap B\).
This completes the proof.

Problem 1.4.14. We do not know whether every narrow operator $T: X \mapsto W$ is narrow with respect to $m_0(B(X^*_\text{big}))$ or even $m_0(B(X^*))$. The positive answer to this question would naturally generalize Propositions 1.3.25 and 1.3.42.

1.5 Hereditary Daugavet equation

In this section a hereditary Daugavet equation on $L_1(\Omega)$ and $C(K)$ is introduced.

1.5.1 Hereditary Daugavet equation in $L_1(\Omega)$

Historically, the study of the Daugavet equation on $L_1$ started with G. Lozanovskii’s work [57], where he proved it for all compact operators. This was also reproved later by V. Babenko and S. Pičugov in [5]. Then J. Holub generalized their result to all weakly compact operators on atomless $L_1$, [39]. Finally, the class of $L_1$-narrow operators introduced by A. Plichko and M. Popov revealed the Daugavet equation for a wide range of other operators including $L_1[0, 1]$-singular ones, [63].

So, it is natural to conclude this line of results by finding the largest linear class of operators for which (1.2.1) holds. However, the Daugavet equation itself appears to be not strong enough to effect structural properties of the operator. Therefore, we consider a more restrictive so-called hereditary Daugavet equation, which nevertheless is satisfied by all $L_1$-narrow operators.

Let $(\Omega, \Sigma, \nu)$ be an arbitrary atomless positive measure space. We adopt the notation from Section 1.3.6. Besides, we write $B \in \Sigma_A^+$ if $B \in \Sigma^+$ and $B \subset A$.

Definition 1.5.1. We say that an operator $T: L_1(\Omega) \mapsto L_1(\Omega)$ satisfies the hereditary Daugavet equation if

$$\|\text{Id}_A \pm T_A\| = 1 + \|T_A\|$$

holds for all $A \in \Sigma^+$. 

50
Using the proof of Lemma 1.3.39 it is not difficult to see that all $L_1$-narrow operators satisfy the hereditary Daugavet equation.

We define $\mathcal{M}$ as the set of all operators $T: L_1(\Omega) \to L_1(\Omega)$ that meet the following condition:

For every $\varepsilon > 0$ and $A \in \Sigma^+$ there is a $B \in \Sigma_A^+$ with $\mu(B) < \infty$ such that

$$\left\| \chi_B \cdot T \left( \frac{\chi_B}{\mu(B)} \right) \right\| < \varepsilon.$$ \hfill (1.5.2)

This condition simply means that the operator $T$ can shift sufficiently many functions off their supports. A prototypical representative of the class $\mathcal{M}$ is the translation mapping $f \to f(\cdot + t)$ on the real line. However, as we will see in a moment all $L_1$-narrow operators also belong to $\mathcal{M}$.

**Theorem 1.5.2.** A bounded linear operator on $L_1(\Omega)$ satisfies the hereditary Daugavet equation if and only if it belongs to $\mathcal{M}$ and $\mathcal{M}$ is closed under addition. Thus, $\mathcal{M}$ is the largest linear class of operators for which (1.5.1) holds.

The main ingredient in the proof is the following lemma.

**Lemma 1.5.3.** For an operator $T \in L(L_1(\Omega))$ the following conditions are equivalent:

(i) $T$ satisfies the hereditary Daugavet equation;

(ii) For every $\varepsilon > 0$ and $A \in \Sigma^+$ there is an $A' \in \Sigma_A^+$ such that if $B \in \Sigma_{A'}^+$ then we can find a $B' \in \Sigma_B^+$ with the following properties:

a) $\left\| \frac{\chi_{B'}}{\nu(B')} - \frac{\chi_B}{\nu(B)} \right\| < \varepsilon$;

b) $\left\| \chi_{B'} \cdot T \left( \frac{\chi_B}{\nu(B')} \right) \right\| < \varepsilon$;

(iii) $T \in \mathcal{M}$.

**Proof.** (i) implies (ii). We begin with the following observation.

Suppose $S : L_1(A) \to L_1(\Omega)$ is a bounded linear operator, then for any given $\varepsilon > 0$ there is a set $A_1 \in \Sigma^+_A$ with $\mu(A_1) < \infty$ such that for every non-negative function $f \in S(L_1(A_1))$ we have $\|Sf\| \geq \|S\| - \varepsilon$.  

51
Indeed, we can assume that \( \nu(A) < \infty \) and choose \( g^* \in S(L_1(\Omega)) \) so that \( \|S^*g^*\| > \|S\| - \varepsilon \). Then, regarding \( S^*g^* \) as an element of \( L_\infty(A) \) we find a set \( A_1 \in \Sigma_A^+ \) with \( \theta S^*g^*(A_1) \subset (\|S\| - \varepsilon, \|S\|] \), where \( \theta \) is a sign. Now, if \( f \in S(L_1(A)) \), \( f \geq 0 \) and \( \text{supp}(f) \subset A_1 \), then \( \|Sf\| > \theta g^*(Sf) = \theta S^*g^*(f) > \|S\| - \varepsilon \), from where the observation follows.

We know that \( \|\text{Id}_A + T_A\| = 1 + \|T_A\| \). By scaling, without loss of generality we can and do assume that \( \|T_A\| = 1 \). So there is an \( A_1 \in \Sigma_A^+ \) with \( \nu(A_1) < \infty \) such that

\[
\left\| \frac{\chi_B}{\nu(B)} + T \left( \frac{\chi_B}{\nu(B)} \right) \right\| > 2 - \varepsilon, \tag{1.5.3}
\]

whenever \( B \in \Sigma_A^+ \). We also know that \( \|\text{Id}_{A_1} - T_{A_1}\| = 1 + \|T_{A_1}\| > 2 - \varepsilon \). Thus there exists an \( A' \in \Sigma_{A_1}^+ \) such that

\[
\left\| \frac{\chi_B}{\nu(B)} - T \left( \frac{\chi_B}{\nu(B)} \right) \right\| > 2 - \varepsilon, \tag{1.5.4}
\]

whenever \( B \in \Sigma_{A'}^+ \).

We prove that \( A' \) is the desired set.

To this end, let us fix \( B \in \Sigma_{A'}^+ \). It follows from (1.5.3), (1.5.4) and a theorem of Dor [25] that there are two disjoint measurable sets \( \Omega_1 \) and \( \Omega_2 \) in \( \Omega \) such that

\[
\int_{\Omega_1} \left| T \left( \frac{\chi_B}{\nu(B)} \right) \right| d\nu > (1 - \varepsilon)^2 \tag{1.5.5}
\]

and

\[
\int_{\Omega_2} \frac{\chi_B}{\nu(B)} d\nu > (1 - \varepsilon)^2.
\]

The last inequality implies

\[
\nu(B \cap \Omega_1) = \nu(B) \int_{B \cap \Omega_1} \frac{\chi_B}{\nu(B)} d\nu < \nu(B) \int_{\Omega_1 \cap \Omega_2} \frac{\chi_B}{\nu(B)} d\nu < (1 - (1 - \varepsilon)^2)\nu(B) = (2\varepsilon - \varepsilon^2)\nu(B). \tag{1.5.6}
\]

Let us put \( B' = B \setminus \Omega_1 \) and show that \( B' \) meets conditions a) and b).

First,

\[
\left\| \frac{\chi_{B'}}{\nu(B')} - \frac{\chi_B}{\nu(B)} \right\| = \int_{\Omega} \left| \frac{\chi_{B'}}{\nu(B')} - \frac{\chi_B}{\nu(B)} \right| d\nu \leq 1 - \frac{\nu(B')}{\nu(B)} + \frac{\nu(B \cap \Omega_1)}{\nu(B)} = 2 \frac{\nu(B \cap \Omega_1)}{\nu(B)},
\]

52
and taking into account (1.5.6), we obtain
\[ \left\| \frac{\chi_{B'}}{\nu(B')} \right\| - \chi_{B} \left( \frac{B'}{\nu(B)} \right) < 2(2\varepsilon - \varepsilon^2). \] (1.5.7)

Second, from (1.5.5), (1.5.7) and $\|T_A\| = 1$ it follows that
\[ \left\| \chi_{B'} \cdot T \left( \frac{\chi_{B'}}{\nu(B')} \right) \right\| = \int_{B'} \left| T \left( \frac{\chi_{B'}}{\nu(B')} \right) \right| d\nu < \int_{B'} \left| T \left( \frac{\chi_{B}}{\nu(B)} \right) \right| d\nu + 2(2\varepsilon - \varepsilon^2) \leq \int_{\Omega \setminus B} \left| T \left( \frac{\chi_{B}}{\nu(B)} \right) \right| d\nu + 3(2\varepsilon - \varepsilon^2).
\]

In view of arbitrariness of $\varepsilon$, this gives the desired result.

It is obvious that (iii) follows from (ii).

Let us finally prove that (iii) implies (i). Since $\mathcal{M}$ is stable under scalar multiplication, it is sufficient to prove (1.2.1) only for $T$.

To this end, we fix an arbitrary $A \in \Sigma^+$ and as above for any given $\varepsilon > 0$ we find an $A' \in \Sigma_A^+$ with $\nu(A') < \infty$ such that for every $B \in \Sigma_A^+$,
\[ \left\| \frac{\chi_{B}}{\nu(B)} \right\| > \left\| T_A \right\| - \varepsilon. \] By condition (1.5.2), there is a $B_0 \in \Sigma_A^+$ such that $\left\| \chi_{B_0} \cdot T \left( \frac{\chi_{B_0}}{\nu(B_0)} \right) \right\| < \varepsilon$. This means that $\chi_{B_0}$ and $T \left( \frac{\chi_{B_0}}{\nu(B_0)} \right)$ are almost disjoint functions, and as a consequence we have the following estimate:
\[ \left\| \text{Id}_A + T_A \right\| \geq \left\| \frac{\chi_{B_0}}{\nu(B_0)} \right\| + T \left( \frac{\chi_{B_0}}{\nu(B_0)} \right) > 1 - \varepsilon + \left\| T_A \right\| - \varepsilon - \varepsilon = 1 + \left\| T_A \right\| - 3\varepsilon. \]

This finishes the proof. \[ \square \]
Now we are in a position to prove our main result.

**Proof of Theorem 1.5.2.** Lemma 3.3.9 implies that $\mathcal{M}$ consists of operators satisfying (1.2.1) for all $A \in \Sigma^+$, and that every linear space of such operators is contained in $\mathcal{M}$. $\mathcal{M}$ is obviously closed and stable under scaling. So, the only thing we have to prove is that if operators $T_1$ and $T_2$ belong to $\mathcal{M}$, then their sum belongs to $\mathcal{M}$ too. To show this, we check condition (ii) of Lemma 3.3.9 for $T_1 + T_2$. Further on, we assume that $\|T_2\| \leq 1$.

Indeed, let $A \in \Sigma^+$ and $\varepsilon > 0$ be arbitrary. Applying Lemma 3.3.9 to the operator $T_1$ we find a set $A' \in \Sigma_A^+$ as in condition (ii). Then, by the same proposition applied to $V$ we find a set $A'' \in \Sigma_{A'}^+$ with the corresponding properties. To show that $A''$ is the required set, suppose $B \in \Sigma_{A''}^+$. By the choice of $A''$ there is a $B' \in \Sigma_B^+$ such that

$$\left\| \frac{\chi_{B'}}{\nu(B')} - \frac{\chi_B}{\nu(B)} \right\| < \frac{\varepsilon}{4}, \quad (1.5.8)$$

and

$$\left\| \chi_{B'} \cdot T_2 \left( \frac{\chi_{B'}}{\nu(B')} \right) \right\| < \frac{\varepsilon}{4}. \quad (1.5.9)$$

Since $B' \subset A'$, by the analogous property of $A'$, there is a $B'' \in \Sigma_{B'}^+$ with

$$\left\| \frac{\chi_{B''}}{\nu(B''')} - \frac{\chi_{B'}}{\nu(B')} \right\| < \frac{\varepsilon}{4}, \quad (1.5.10)$$

and

$$\left\| \chi_{B''} \cdot T_1 \left( \frac{\chi_{B''}}{\nu(B''')} \right) \right\| < \frac{\varepsilon}{2}. \quad (1.5.11)$$

From (1.5.8) and (1.5.10) we get $\left\| \frac{\chi_{B''}}{\nu(B''')} - \frac{\chi_B}{\nu(B)} \right\| < \varepsilon$. So, if we prove that $\left\| \chi_{B''} \cdot T_2 \left( \frac{\chi_{B''}}{\nu(B''')} \right) \right\| < \frac{\varepsilon}{2}$, then $\left\| \chi_{B''} \cdot (T_1 + T_2) \left( \frac{\chi_{B''}}{\nu(B''')} \right) \right\| < \varepsilon$, and we are done. But this easily follows from (1.5.9), (1.5.10) and the facts that $\|T_2\| \leq 1$ and $B'' \subset B'$.

The proof is completed. \(\square\)

**Remark 1.5.4.** Condition (1.5.2) has been elegantly reformulated by Anton R. Schep in his recent paper [69]. The author recasts (1.5.2) into lattice terms: $T \in \mathcal{M}$ if and only if $T \perp \text{Id}$. Now Theorem 1.5.2 can be understood as an operator analogue of Dor’s theorem we used in the proof.
Furthermore, the author introduces an analogous class in every $L_p$, denoted $M_p$. It is then proved that $T \in M_p$ if and only if it is regular and the following hereditary Daugavet equation holds

$$\| \text{Id}_A \pm T_A \|_r \geq (1 + \|T_A\|_p^p)^{1/p},$$

for all $A \in \Sigma$, where $\|T\|_r = \|T\|$. This, in turn, holds if and only if $T \perp \text{Id}$ in the lattice of all regular operators on $L_p$. Since, in $L_1$ every operator is regular and $\|T\| = \|T\|_r$, we restore our class $\mathcal{M}$ for this particular case.

### 1.5.2 Hereditary Daugavet equation in $C(K)$

In this section we introduce a $C$-version of the hereditary Daugavet equation. As before $K$ denotes a compact Hausdorff space without isolated points.

**Definition 1.5.5.** Let $T: C(K) \to C(K)$ be a bounded operator and $U$ is an open subset of $K$. Denote by $T_U$ the restriction of $T$ onto $C_0(U)$. We say that $T$ satisfies the **hereditary Daugavet equation** if

$$\| \text{Id}_U \pm T_U \| = 1 + \|T_U\|,$$  \hspace{1cm} (1.5.11)

for all open $U \subset K$.

Unlike in the $L_1$ situation, every operator satisfying the hereditary Daugavet equation in $C(K)$ possesses a strong narrowness property described by the following definition.

**Definition 1.5.6.** We say that an bounded operator $T: C(K) \to C(K)$ is **almost diffuse** if for every $\varepsilon > 0$ and open set $U \subset K$ there exists an open subset $V \subset U$ such that $\|T_V\| < \varepsilon$.

This notion was introduced by C. Foias and I. Singer in [29] before the $C$-narrow operators had appeared. Clearly, all almost diffuse operators are $C$-narrow. However many ”small” operators, such as compact or majorable, are already almost diffuse, [29].

As it was pointed out by V. Kadets, there are narrow not almost diffuse operators. Indeed, let $K = [0,1]$ and $Y$ be the closed space spanned on the even numbered vectors of the Schauder system. Denote by $Q: C[0,1] \to C[0,1]/Y$ the canonical factor-map and by $P: C[0,1]/Y \to C[0,1]$ some isometric embedding, which exists due to the universality of $C[0,1]$. Then $T = PQ$ is narrow, but not almost diffuse.
Proposition 1.5.7. If a bounded operator from \( C(K) \) into itself satisfies the hereditary Daugavet equation, then it is almost diffuse.

Proof. Let \( T \) be such an operator. Assume, on the contrary, that there exists an open set \( U_0 \) such that

\[
m = \inf\{\|T_U\| : U \subset U_0\}
\]

is strictly positive. Let us fix a \( U \subset U_0 \) with \( \|T_U\| < m + \delta \), where \( \delta > 0 \) is to be defined later.

We claim that for every \( V \subset U \) there is a non-negative function \( \phi \in B(C_0(V)) \) and a point \( t \in V \) such that

\[
\phi(t) > 1 - \delta, \quad (T\phi)(t) > m - \delta.
\]

Indeed, by (1.5.1) with positive sign, there exists a \( \phi_1 \in B(C_0(V)) \) and open \( V_1 \subset V \) with

\[
\phi_1(v) > 1 - \delta/2, \quad (T\phi_1)(v) > m - \delta/2,
\]

for all \( v \in V_1 \). Again, by (1.5.1) applied to \( V_1 \) there is another function \( \phi_2 \in B(C_0(V_1)) \) and \( V_2 \subset V_1 \) with similar properties:

\[
\phi_2(v) > 1 - \delta/2, \quad (T\phi_2)(v) > m - \delta/2,
\]

for all \( v \in V_2 \), and so on. Put \( \phi = \frac{1}{n} \sum_{i=1}^{n} \phi_i \). Clearly, for every \( v \in V_{n+1} \) we have

\[
\phi(v) > 1 - \delta/2, \quad (T\phi)(v) > m - \delta/2.
\]

Furthermore, from our construction, for all \( v \in V_k \setminus V_{k+1} \) we get

\[
\phi(v) = \frac{1}{n} \left[ \sum_{i=1}^{k} (1 - \delta/2) + \phi_{k+1}(v) \right] \geq -\frac{1}{n}.
\]

So, for \( n \) sufficiently large the function \( \varphi = \max\{\phi, 0\} \) differs from \( \phi \) by less then \( \delta/2 \) and satisfies our requirements.

Analogously, using (1.5.1) with negative sign, one obtains a non-positive \( \psi \in B(C_0(V)) \) with \( \psi(t) < \delta - 1 \) and \( (T\psi)(t) > m - \delta \), for some \( t \in V \).

Now, by our claim, choose a non-negative \( \varphi \in B(C_0(U)) \) and an open set \( V \subset U \) such that

\[
\varphi(v) > 1 - \delta, \quad (T\varphi)(v) > m - \delta.
\]
for all \( v \in V \). Using the claim for \( V \), we get a non-positive function \( \psi \in B(C_0(V)) \) and a point \( t \in V \) with \((T\psi)(t) > m - \delta\). It remains to notice that \( \| \varphi + \psi \| \leq 1 \) and \( m + \delta \geq (T(\varphi + \psi))(t) > 2m - 2d \), which yields a contradiction provided \( \delta < m/3 \).

The converse is not true. Indeed, on \( K = [0,1] \) fix a function \( f_0 \in C[0,1] \), \( \| f_0 \| = 1 \) supported in \( [0,1/2] \) and consider the following rank-1 operator \( Tf = f(2/3)f_0 \). Then \( T \) is almost diffuse, because for every open set \( U \) in \([0,1], T_{U \setminus \{2/3\}} = 0 \). However,

\[
\| \text{Id}_{(1/2,1)} + T_{(1/2,1)} \| = 1 < 2 = 1 + \| T_{(1/2,1)} \| .
\]

\textbf{Problem 1.5.8.} It would be interesting to obtain a characterization of operators satisfying the hereditary Daugavet equation on \( C(K) \).

\textbf{Corollary 1.5.9.} Every operator satisfying the hereditary Daugavet equation is narrow.
Chapter 2

On Riemann-Lebesgue integral sums and limit sets

2.1 Introduction and basic definitions

A well-known result that goes back to M. Krein and V. Šmulian [52] says the following: the closed convex hull of a weakly compact subset of a Banach space is weakly compact.

For what other topologies does this statement remain true?

Recent attention to this question is motivated by its connection with the Boundary Problem posed by G. Godefroy in [31]: let $X$ be a Banach space and $B$ a boundary in the unit ball of $X^*$, i.e. such that $\|x\| = \max_{b \in B} b(x)$ holds for all $x \in X$; is it true that a norm bounded set in $X$ is weakly compact if it merely compact in the topology induced by $B$?

In [74] S. Simons gives a partial positive answer to this question in case of a convex set. This establishes equivalence of the Krein-Šmulian-type theorem for topologies generated by boundaries and the Boundary Problem itself.

Even though these questions remain open by now, a considerable progress towards their solution has been made by J. Bourgain, B. Cascales, G. Godefroy, M. Talagrand, G. Vera and others in a series of papers [12, 13, 14, 15, 16, 17]. In particular, the Boundary Problem has been proved for the spaces of continuous functions, [13]; and for general Banach spaces with boundary given by the set of extreme points on the dual unit ball, [12]. This greatly generalized the classical result of A. Grothendieck, [28]. The problem was also solved positively in all Banach spaces not containing a copy of $\ell_1[0, 1]$, as
the following result due to B. Cascales, G. Manjabacas and G. Vera shows.

**Theorem 2.1.1** ([14]). Suppose $X$ does not contain a copy of $\ell_1[0,1]$ and $B$ is a norming subset of the unit ball of $X^*$. Then the convex hull of every $B$-compact norm bounded set in $X$ is $B$-compact. In particular, the Boundary Problem has positive solution in $X$.

In Section 2.2 we develop a principally different technic producing Krein-Šmulian-type results for general locally convex topologies. It is based on the notion of a Riemann-Lebesgue integral sum introduced by V. Kadets and L. Tseytlin in [46]. Let us formulate the necessary definitions.

Suppose $X$ is a locally convex space endowed with a Hausdorff topology $\tau$ ([65]) and $(\Omega, \Sigma, \mu)$ is a probability space. For a given function $f: \Omega \to X$ (not necessarily measurable in any sense) we define a **Riemann-Lebesgue integral sum** as

$$S(f, \Pi, T) = \sum_{i=1}^{n} f(t_i) \mu(A_i),$$

where $\Pi = \{A_i\}_{i=1}^{n}$ is a partition of $\Omega$ by elements of $\Sigma$ and $T = \{t_i\}_{i=1}^{n}$ is a collection of **sampling points**, i.e. $t_i \in A_i$ for $i = 1, n$.

We endow $\{S(f, \Pi, T)\}_{\Pi, T}$ with a net structure by defining a partial order by the rule: $\Pi_1 \succ \Pi_2$ if and only if $\Pi_1$ is finer than $\Pi_2$ meaning that every element of $\Pi_1$ is contained in some element of $\Pi_2$. Every cluster point of the net $\{S(f, \Pi, T)\}_{\Pi, T}$ is called a **limit point** of the Riemann-Lebesgue integral sums of $f$. The set of all such points is called a **limit set** and denoted by $I_{\tau, \mu}(f)$.

Our main result (Theorem 2.2.3) asserts that the Krein-Šmulian theorem for topology $\tau$ is equivalent to the statement: $I_{\tau, \mu}(f) \neq \emptyset$ for all $\mu$ and all $f$ with relatively $\tau$-compact range. Similarly, in case $X$ is also a Banach space, the Krein-Šmulian theorem for norm bounded sets is equivalent to the same statement with the bounded functions.

It turns out that in many cases it is possible to prove non-emptiness of even "strong" limit lets $I_{\|\cdot\|, \mu}(f)$, producing the Krein-Šmulian-type theorems for all weak topologies. In this context, we recall a result, a part of which was obtained in [41] (see Section 2.3.2 for complete proof).

**Theorem 2.1.2.** $I_{\|\cdot\|, \mu}(f) \neq \emptyset$ for every probability space $(\Omega, \Sigma, \mu)$ and every norm bounded function $f: \Omega \to X$, provided $X$ satisfies one of the following conditions:
1) $X$ is a WCG-space;

2) $X$ has an extended unconditional basis (see Example 2.3.3) and fails to contain a copy of $\ell_1(\Gamma)$ over uncountable $\Gamma$.

Thus, the Krein-Šmulian theorem holds for all weak topologies in $X$.

Remark, that although the first case is already covered by Theorem 2.1.1, the range of topologies is wider in our theorem. In this spirit we generalize an earlier result of B. Cascales and G. Vera’s [16], by showing that a norm-fragmentable set compact in any weak topology obeys the Krein-Šmulian theorem (Theorem 2.2.8). Also, in Section 2.2.3 we give an alternative geometrical proof of quoted Theorem 2.1.1 using the tools developed in Section 2.2.2.

In the next section we address the question of finding a sufficient condition for a space $X$ to have all limit sets $I_{\|\cdot\|,\mu}(f)$ non-empty. We begin with introduction of a so-called $\aleph$-convex subset of $X$, where $\aleph$ is a cardinal number. Basically, this is a set $D$ such that 0 belongs to the closed convex hull of every its subset of cardinality bigger than or equal to $\aleph$. The canonical basis of $c_0(\Gamma)$ is an example of such a set with the countable $\aleph$. More generally, if $X$ contains an extended unconditional basis $D$ (see Example 2.3.3) and fails to contain a copy of $\ell_1(\Gamma)$ with cardinality of $\Gamma$ being equal to $\aleph$, then $D$ is $\aleph$-convex. In Example 2.3.2 we will find out that every WCG-space also possesses a countable-convex subset, which is, in fact, a relatively weakly compact Markushevich basis.

Notice that in these examples $\text{lin}D$ is dense in $X$. Existence of such a set in $X$ is, roughly, the condition we are looking for (Theorem 2.3.6). In particular, it incorporates the cases listed in Theorem 2.1.2.

Further on, we discover some properties of the limit sets which resemble those known for the Bochner integral, such as linearity and the Lebesgue dominated convergence theorem. This shows that a limit set can be viewed as an extension of the conventional integral onto non-measurable functions.

We conclude the chapter by proving that in a Banach space where all $I_{\|\cdot\|,\mu}(f)$ are non-empty, every function with singleton limit set is, in fact, Pettis integrable (the converse being always true). This generalizes the corresponding result by D. R. Lewis proved for weakly measurable functions in WCG-spaces.
2.2 Limit sets in locally convex spaces

2.2.1 Preliminaries

In this section we present some basic properties of limit sets.

The following lemma was shown in [46]. However, we include the proof for convenience.

Lemma 2.2.1. \( I_{\tau,\mu}(f) \) is a \( \tau \)-closed subset of \( \operatorname{conv}^\tau(f(\Omega)) \). If \( \mu \) is atomless, then \( I_{\tau,\mu}(f) \) is convex and \( I_{\text{weak-}\tau,\mu}(f) = I_{\tau,\mu}(f) \).

Proof. For a given partition \( \Pi \), let us denote \( S(\Pi) = \{ S(f,\Pi',T) : \Pi' \succ \Pi \} \).

\[ s_1(\Pi_1) = \sum_{i=1}^{n} f(a_i)\mu(A_i), \quad s_2(\Pi_2) = \sum_{j=1}^{m} f(b_j)\mu(B_j) \]

be two elements of \( S(\Pi) \) and \( \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1 \). Since \( \Pi_1 \prec \Pi \) and \( \Pi_2 \prec \Pi \), without loss of generality we can assume that \( \Pi = \{ \Omega \} \). Now using Lyapunov’s theorem [22] we find measurable sets \( C_{ij}^1 \) and \( C_{ij}^2 \), \( 1 \leq i \leq n, 1 \leq j \leq m \) satisfying the following conditions.

1) \( C_{ij}^1 \cup C_{ij}^2 = A_i \cap B_j, C_{ij}^1 \cap C_{ij}^2 = \emptyset; \)
2) \( \mu(C_{ij}^1) = \lambda_1 \mu(A_i \cap B_j), \mu(C_{ij}^2) = \lambda_2 \mu(A_i \cap B_j); \)
3) if \( a_i \in A_i \cap B_j \), then \( a_i \in C_{ij}^1; \)
4) if \( b_j \in A_i \cap B_j \) and \( b_j \neq a_i \), then \( b_j \in C_{ij}^2. \)

Denoting \( C_i^1 = \bigcup_{j=1}^{n} C_{ij}^1 \) and \( C_j^2 = \bigcup_{i=1}^{m} C_{ij}^2 \) we obtain the partition \( \{ C_i^1, C_j^2 \}_{i,j} \) of \( \Omega \) such that \( \mu(C_i^1) = \lambda_1 \mu(A_i), \mu(C_j^2) = \lambda_2 \mu(B_j); a_i \in C_i^1 \) and \( b_j \in C_j^2 \) unless \( b_j = a_i \) for some \( i \). Finally if \( I = \{ i : a_i = b_j, \text{ for some } j = j(i) \} \) and
\[ J = \{ j : b_j = a_i, \text{ for some } i = i(j) \}, \text{ then} \]
\[
\lambda_1 s_1(\Pi_1) + \lambda_2 s_2(\Pi_2) = \sum_{i=1}^{n} f(a_i)\mu(C_1^i) + \sum_{j=1}^{m} f(b_j)\mu(C_2^j)
\]
\[
= \sum_{i \in I} f(a_i)\mu(C_1^i \cup C_2^{j(i)}) + \sum_{i \notin I} f(a_i)\mu(C_1^i)
\]
\[
+ \sum_{j \notin J} f(b_j)\mu(C_2^j),
\]
which is a well-defined Riemann-Lebesgue integral sum of \( f \).

Motivated by Lemma 2.2.1 we now find all topologies \( \sigma \) in which the equality \( I_{\sigma,\mu}(f) = I_{\tau,\mu}(f) \) holds for all \( f \) and atomless \( \mu \).

**Proposition 2.2.2.** The following assertion are equivalent.

1) \( I_{\sigma,\mu}(f) = I_{\tau,\mu}(f) \) holds for all \( f \) and atomless \( \mu \)

2) \( \sigma \) is not weaker than weak\(-\tau \) and not stronger than the topology of uniform convergence on weak\(^\ast\)\(-\tau\)-compact sets of \((X, \tau)^\ast\).

**Proof.** By the Mackey-Arens Theorem [65] the second condition is equivalent to saying that the \( \tau \)- and \( \sigma \)-closures of an arbitrary convex set coincide.

Suppose that \( \sigma \) is such a topology. Then it yields the same collection of closed convex sets as \( \tau \). Now 1) follows from the proof of Lemma 2.2.1.

Conversely, suppose 1) is true. Fix an arbitrary convex set \( U \subset X \) and denote by \( \alpha \) the cardinality of \( U \). Consider the probability space \((\{0,1\}^\alpha, \mathcal{B}, \mu)\), where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra and \( \mu = \nu^\beta, \nu(\{0\}) = \nu(\{1\}) = \frac{1}{2} \). Now we want to find \( \alpha \) disjoint sets of upper measure 1 (see page 73). For this we consider the collection \( \{F_\gamma\} \) of all closed sets of positive measure \( \mu \). Since every \( F_\gamma \) has cardinality at least \( \alpha \), by the Axiom of Choice we can compose \( \alpha \) sets \( A_\theta, \theta < \alpha \), such that \( A_\theta \cap F_\gamma \neq \emptyset \), for every \( \theta \) and \( \gamma \). Because of the regularity of \( \mu \), this last condition implies that there is no Borel set of positive measure which belongs to the complement of \( A_\theta, \theta < \alpha \). This means that every \( A_\theta \) has upper measure 1.

Now we take any bijection \( \varphi : \{ \theta : \theta < \alpha \} \to U \) and define \( f : \{0,1\}^\alpha \to X \) as follows:
\[
f(t) = \begin{cases} 
\varphi(\theta), & t \in A_\theta \\
\varphi(1), & t \notin \bigcup_{\theta < \alpha} A_\theta
\end{cases}
\]
Clearly, $U \subset I_{\tau,\mu}(f)$ and hence $\overline{U}^\tau \subset I_{\tau,\mu}(f)$. On the other hand, by Lemma 2.2.1, $I_{\tau,\mu}(f) \subset \overline{\text{conv}}^\tau(f(\Omega)) = \overline{U}^\tau$. So, $I_{\tau,\mu}(f) = \overline{U}^\tau$. Analogously, $I_{\tau,\mu}(f) = \overline{U}^\tau$. Since $I_{\tau,\mu}(f) = I_{\sigma,\mu}(f)$, we are done. 

In some cases $I_{\tau,\mu}(f)$ may happen to be empty. For instance, consider $f: [0, 1] \to \ell_1[0, 1]$ that maps $t$ to $e_t$, the canonical basis vector (see [46]). Then the norm of each Riemann-Lebesgue sum with respect to the Lebesgue measure $\lambda$ equals 1; whereas every element of $I_{\|\cdot\|,\lambda}(f)$ must vanish on each $e^*_i$ (see the discussion at the end of Section 2.3.2). Nevertheless, as we explained in the Introduction, it is important to locate those cases when limit sets are non-empty.

2.2.2 Krein-Šmulian-type theorems

Consider the following statements.

(KS) If $H \subset X$ is a $\tau$-compact set in $X$, then $\overline{\text{conv}}(H)$ is also $\tau$-compact.

(KSb) Let, besides, $X$ be a Banach space. If $H \subset X$ is a $\tau$-compact norm bounded set in $X$, then $\overline{\text{conv}}(H)$ is also $\tau$-compact.

Now our nearest goal is to relate (KS) and (KSb) with non-emptiness of the limit sets in $X$.

Let us consider a $\tau$-compact subset $H$ in a locally convex space $X$. Denote by $\mathcal{P}(H)$ the set of all probabilities defined on the algebra of all $\tau$-Borel subsets of $H$. Let also $\text{id}: H \to X$ stand for the identity function.

Theorem 2.2.3. For a locally convex space $X$ with topology $\tau$ the following are equivalent:

1) (KS) holds in $(X, \tau)$ ((KSb) holds in $(X, \tau)$);

2) $I_{\tau,\mu}(f)$ is non-empty for all probability spaces $(\Omega, \Sigma, \mu)$ and $f: \Omega \to X$ with relatively $\tau$-compact (and norm bounded) range;

3) $I_{\tau,\mu}(\text{id})$ is non-empty for all $\tau$-compact (and norm bounded) $H$ and all $\mu \in \mathcal{P}(H)$.

63
As an immediate consequence of Theorems 2.1.2 and 2.2.3 we obtain the following result.

**Corollary 2.2.4.** If \( X \) satisfies either of conditions 1) and 2) of Theorem 2.1.2, then (KSB) holds for any \( \tau \) weaker than the norm topology on \( X \).

The proof of Theorem 2.2.3 consists of the following three lemmas.

**Lemma 2.2.5.** Let \( \mu \in \mathcal{P}(H) \). Then \( I_{\tau,\mu}(id) \) is not empty if and only if \( \mu \) has a barycenter \( x \), in which case \( I_{\tau,\mu}(id) = \{ x \} \).

**Proof.** Suppose \( I_{\tau,\mu}(id) \) is not empty and \( x \in I_{\tau,\mu}(id) \). Since every functional \( x^* \in X^* \) restricted to \( H \) is \( \tau \)-continuous and \( \mu \)-integrable, we have

\[
x^*(x) = \int_H x^*(h) d\mu(h),
\]

which means that \( x \) is the unique barycenter of \( \mu \).

On the other hand, if \( x \) is the barycenter of \( \mu \), then (2.2.1) holds for all \( x^* \in X^* \). Hence \( x \in I_{\text{weak-}\tau,\mu}(id) = I_{\tau,\mu}(id) \) and we are done.

**Lemma 2.2.6.** \( \text{conv}^\tau H = \bigcup_{\mu \in \mathcal{P}(H)} I_{\tau,\mu}(id) \).

**Proof.** It follows from Lemma 2.2.1 that \( I_{\tau,\mu}(id) \subset \text{conv}^\tau H \) for every \( \mu \in \mathcal{P}(H) \).

Now let \( x \in \text{conv}^\tau H \) and \( \{ x_\alpha \}_{\alpha \in A} \subset \text{conv} H \) be a net converging to \( x \). It is clear that \( x_\alpha \in I_{\tau,\mu_\alpha}(id) \) for some purely atomic \( \mu_\alpha \in \mathcal{P}(H) \), \( \alpha \in A \). Since \( \mathcal{P}(H) \) is weak*-compact in \( C^*(H) \), there is a subnet \( \{ \mu_\beta \}_{\beta \in B} \) converging to some \( \mu \in \mathcal{P}(H) \) in the weak*-topology. We claim that \( x \in I_{\tau,\mu}(id) \).

Indeed, for every \( x^* \in X^* \) we have

\[
x^*(x) = \lim_{\beta \in B} x^*(x_\beta) = \lim_{\beta \in B} \int_H x^*(h) d\mu_\beta(h) = \int_H x^*(h) d\mu(h).
\]

So, \( x \) is the barycenter of \( \mu \) and by Lemma 2.2.5, \( x \in I_{\tau,\mu}(id) \). This completes the proof.

Theorem 2.2.3 is a consequence of the following lemma.

**Lemma 2.2.7.** Let \( H \subset X \) be a \( \tau \)-compact set. The following are equivalent:

1) \( \text{conv}^\tau H \) is \( \tau \)-compact;
2) $I_{\tau,\mu}(f)$ is non-empty for all probability spaces $(\Omega, \Sigma, \mu)$ and $f: \Omega \to H$;

3) $I_{\tau,\mu}(id)$ is non-empty for all $\mu \in \mathcal{P}(H)$.

Proof. 1)$\Rightarrow$2). It follows directly from the expression $I_{\tau,\mu}(f) = \bigcap_{\Pi} \overline{S(\Pi)}^{\tau}$, where $S(\Pi)$ as in the proof of Lemma 2.2.1.

2)$\Rightarrow$3). Obvious.

3)$\Rightarrow$1). (This is a reprise of the classical Krein-Šmulian argument) For $\mu \in \mathcal{P}(H)$ we consider $x_\mu \in I_{\tau,\mu}(id)$. The mapping $\varphi: \mathcal{P}(H) \to X$ defined as $\varphi(\mu) = x_\mu$ is weak$^*$-weak-$\tau$ continuous and hence $\varphi(\mathcal{P}(H))$ is compact in the weak-$\tau$-topology. According to Lemmas 2.2.5 and 2.2.6, $\overline{\text{conv}}^\tau H = \varphi(\mathcal{P}(H))$. Moreover, $\overline{\text{conv}}^\tau H$ is $\tau$-totally bounded. This implies that $\overline{\text{conv}}^\tau H$ is also $\tau$-compact, [65].

The proof is over. $\square$

Another case when a $\tau$-compact set may have the $\tau$-compact convex hull is provided by an extra fragmentability assumption. Recall that a subset $H \subset X$ is fragmentable by a norm $\sigma$ if for every subset $C$ of $H$ and every $\varepsilon > 0$ there is a $\tau$-open set $U$ such that $\sigma - \text{diam}(U \cap C) < \varepsilon$.

Theorem 2.2.8. Suppose $X$ is endowed with a locally convex topology $\tau$ and a stronger topology $\sigma$ generated by a Banach norm. If $H \subset X$ is a $\tau$-compact norm bounded set fragmentable by the $\sigma$-norm, then $\overline{\text{conv}}^\tau H = \overline{\text{conv}}^\sigma H$ and $\overline{\text{conv}}^\sigma H$ is $\tau$-compact.

This generalizes an earlier result obtained by B. Cascales and G. Vera in [16].

Proof. We claim it is enough to prove that $I_{\sigma,\mu}(id) \neq \emptyset$ for all $\mu \in \mathcal{P}(H)$. Indeed, since $I_{\sigma,\mu}(id) \subset I_{\tau,\mu}(id)$, Lemma 2.2.7 implies that $\overline{\text{conv}}^\tau H$ is $\tau$-compact. Furthermore, observe that

$$\overline{\text{conv}}^\sigma H = \bigcup_{\mu \in \mathcal{P}(H)} I_{\sigma,\mu}(id)^\sigma,$$

and according to Lemma 2.2.5, $I_{\sigma,\mu}(id) = I_{\tau,\mu}(id)$. Thus, in view of Lemma 2.2.6 we get

$$\overline{\text{conv}}^\tau H = \bigcup_{\mu \in \mathcal{P}(H)} I_{\tau,\mu}(id) = \bigcup_{\mu \in \mathcal{P}(H)} I_{\sigma,\mu}(id) \subset \overline{\text{conv}}^\sigma H,$$

the converse inclusion being true in general.
So, we fix a \( \mu \in \mathcal{P}(H) \) and show that \( I_{\sigma,\mu}(id) \neq \emptyset \). Without loss of
generality we may assume that \( H \) is the support of \( \mu \) and \( H \) lies in the unit
ball of \( X \).

Denote by \( \| \cdot \| \) the norm on \( X \) which generates \( \sigma \).

For any given \( k \in \mathbb{N} \), using the fragmentability of \((H, \tau)\), by the exhaustion
argument, one can find a finite number of open sets \( \{ U_i^k \}_{i=1}^{n_k-1} \) such that
denoting \( V_i^k = U_i^k \), \( V_i^k = U_i^k \setminus \bigcup_{j=1}^{i-1} U_i^k \), \( i = 2, n_k - 1 \) and \( V_i^k = H \setminus V_i^k \), one gets
\[
\| \cdot \| - \operatorname{diam} V_i^k < \frac{1}{2^{k+1}}, \quad i = 2, n_k - 1,
\]  
(2.2.3)

and
\[
\mu(V_i^k) < \frac{1}{2^{k+1}}.
\]
(2.2.4)

Let us now denote \( A_{i_1 \ldots i_k} = \cap_{j=1}^{k} V_j^i \), where \( 1 \leq i_j \leq n_j \), \( 1 \leq j \leq k \), and
define a sequence of partitions of \( H \) as follows:
\[
\Pi_k = \{ A_{i_1 \ldots i_k} : 1 \leq i_j \leq n_j, 1 \leq j \leq k \}.
\]

For each \( k \in \mathbb{N} \) we also fix an arbitrary set of sampling points \( T_k = \{ t_{i_1 \ldots i_k} : t_{i_1 \ldots i_k} \in A_{i_1 \ldots i_k} \} \). We claim that the limit \( \| \cdot \| - \lim_{k \to \infty} S(id, \Pi_k, T_k) \) exists
and belongs to \( I_{\sigma,\mu}(id) \).

Indeed, in view of (2.2.3) and (2.2.4), we have
\[
\left\| S(id, \Pi_k, T_k) - S(id, \Pi_{k+1}, T_{k+1}) \right\|
=
\left\| \sum_{1 \leq j \leq k} \sum_{1 \leq i_j \leq n_j} \sum_{1 \leq j \leq k+1} \sum_{1 \leq i_j \leq n_j} f(t_{1 \ldots i_k}) \mu(A_{1 \ldots i_k}) - f(t_{1 \ldots i_{k+1}}) \mu(A_{1 \ldots i_{k+1}}) \right\|
\leq \sum_{1 \leq j \leq k+1} \sum_{1 \leq i_j \leq n_j} \| f(t_{1 \ldots i_k}) - f(t_{1 \ldots i_{k+1}}) \| \mu(A_{1 \ldots i_{k+1}}) \leq \frac{1}{2^k}.
\]

So, the sequence \( \{ S(id, \Pi_k, T_k) \} \) converges to some vector \( x \in X \). To prove
that \( x \in I_{\sigma,\mu}(id) \), let us fix a partition \( \Pi = \{ B_1, B_2, \ldots, B_N \} \) and \( \varepsilon > 0 \). We
choose a $k \in \mathbb{N}$ so that $\frac{3}{2^k} < \varepsilon$. Then we denote $C_{i_1...i_k} = A_{i_1...i_k} \cap B_p$ and pick arbitrary sampling points $t_{i_1...i_k} \in C_{i_1...i_k}$. Same calculations as above show that

$$\|S(id, \{C_{i_1...i_k}\}, \{t_{i_1...i_k}\}) - S(id, \Pi_k, T_k)\| \leq \frac{1}{2^{k-1}},$$

and hence

$$\|S(id, \{C_{i_1...i_k}\}, \{t_{i_1...i_k}\}) - x\| \leq \frac{1}{2^k} + \frac{1}{2^{k-1}} < \varepsilon.$$

This proves that $x \in I_{\sigma,\mu}(id)$.

A great variety of conditions sufficient for fragmentability is found in [15]. For instance, if $H$ is weakly Lindelöf and $\tau$-compact, where $\tau$ is generated by a norming set of functionals, then $H$ is norm-fragmentable in the topology $\tau$.

### 2.2.3 Another proof of B. Cascales, G. Manjabacas and G. Vera’s result

In this section we give an easier, self-contained and alternative proof of the following theorem obtained by B. Cascales, G. Manjabacas and G. Vera in [14].

**Theorem 2.2.9.** Let $X$ be a Banach space, $B \subset B(X^*)$ a norming set of functionals and $\tau$ the topology on $X$ generated by $B$. If $X$ does not contain a copy of $\ell_1([0,1])$, then (KSb) holds for $(X, \tau)$.

In fact we show the inverse statement, i.e. if there exists a norm bounded $\tau$-compact set $H$ in $X$ such that $\text{conv}^\tau H$ is not $\tau$-compact, then $X$ contains a copy of $\ell_1([0,1])$.

So, from now on we fix a $\tau$-compact norm bounded $H$ with non-$\tau$-compact $\text{conv}^\tau H$. Also, for technical reasons we assume with no loss of generality that $H$ is contained in the unit ball of $X$ and that the norming set $B$ inducing $\tau$ is absolutely convex.

In view of Lemmas 2.2.5 and 2.2.7 there is a measure $\mu \in \mathcal{P}(H)$ without a barycenter. Extracting the purely atomic part from $\mu$, which obviously has a barycenter, we may assume that $\mu$ is atomless. Besides, we can identify $H$ with the support of $\mu$, so every open set in $H$ has positive measure $\mu$. 

67
Our plan is to pick a sequence of functionals \((f_n)_{n\in\mathbb{N}}\) in \(B\) so that
\[
H \cap (\cap_{m\in M}\{f_m > r + \delta\}) \cap (\cap_{n\in N}\{f_n < r\}) \neq \emptyset
\] (2.2.5)
holds for every two disjoint sets of natural numbers \(M\) and \(N\), and some fixed two real numbers \(r\) and \(\delta, \delta > 0\). Such a sequence is called independent over \(H\) (see [67]). Every Banach space, which contains an independent sequence over a compact set, also contains a copy of \(\ell_1[0, 1]\) (see Lemma B in [14]).

Our construction is based on the following lemmas.

**Lemma 2.2.10.** There exists an \(\varepsilon > 0\) and measurable \(A \subset H, \mu(A) > 0\), such that for every measurable \(B \subset A\), \(\mu(B) > 0\), and \(h \in \text{conv} H\) there is an \(f \in B\) satisfying the following inequality:
\[
f(h) > \varepsilon + \frac{1}{\mu(B)} \int_B f(s)d\mu(s).
\] (2.2.6)

**Proof.** Suppose, on the contrary, that for any \(\varepsilon > 0\) and measurable \(A \subset H\) there is a \(B \subset A\) and \(h \in \text{conv} H\) such that
\[
f(h) \leq \varepsilon + \frac{1}{\mu(B)} \int_B f(s)d\mu(s),
\]
whenever \(f \in B\).

Let \(\varepsilon_n = \frac{1}{2^n}, n \in \mathbb{N}\). By the exhaustion argument, using the previous inequality for \(\varepsilon_1 = \frac{1}{2}\), we can find a sequence \((h^1_n)_{n \in \mathbb{N}} \subset \text{conv} H\) and a pairwise disjoint sequence \((A^1_n)_{n \in \mathbb{N}}\) in \(B(H)\) such that \(\mu(H \setminus \bigcup_{n=1}^\infty A^1_n) = 0\) and
\[
f(h^1_n) \leq \varepsilon_1 + \frac{1}{\mu(A^1_n)} \int_{A^1_n} f(s)d\mu(s),
\]
for all \(f \in B\) and \(n \in \mathbb{N}\). Hence, as \(B\) is absolutely convex, we have
\[
\left|\frac{1}{\mu(A^1_n)} \int_{A^1_n} f(s)d\mu(s) - \frac{1}{\mu(A^1_n)} \int_{A^1_n} f(s)d\mu(s)\right| \leq \varepsilon_1,
\]
f \(\in B\), \(n \in \mathbb{N}\). Letting \(h^1 = \sum_{n=1}^\infty \mu(A^1_n)h^1_n\) and summing up the previous inequalities we get
\[
\left|\int_H f(s)d\mu(s) - \int_B f(s)d\mu(s)\right| \leq \varepsilon_1.
\]
In the same manner, for every \( n \in \mathbb{N} \), we can construct an \( h^n \in \text{conv} H \) so that
\[
|f(h^n) - \int_H f(s) d\mu(s)| \leq \varepsilon_n,
\]
for all \( f \in B \). Since \( B \) is norming, it follows that \( \|h^n - h^{n+1}\| \leq \varepsilon_n + \varepsilon_{n+1} \) and hence, the limit \( h = \| \cdot \| - \lim_{n \to \infty} h_n \) exists. Passing to limits in the previous inequality we see that \( h \) is the barycenter of \( \mu \), which contradicts our assumption. \( \square \)

Remark that since \( \mu \) is a regular measure, \( A \) can be chosen closed. Furthermore, restricting \( \mu \) on \( A \) we can and do assume that \( A \) is in fact the whole \( H \).

We say that a set \( K \subset X \) has a finite \( \varepsilon \)-net if there is a finite subset \( F \) of \( K \) such that \( K \subset \bigcup_{x \in F} \{ y \in X : \| y - x \| \leq \varepsilon \} \), [22, p. 201].

**Lemma 2.2.11.** For any norm-compact set \( K \subset \text{conv} H \), any collection of open sets \((U_i)_{i=1}^n \) in \( H \) and non-negative numbers \((\lambda_i)_{i=1}^n \), \( \sum_{i=1}^n \lambda_i = 1 \), there are open sets \((V_i)_{i=1}^n \) satisfying the following conditions:

1) \( V_i \subset U_i, \ i = \overline{1,n} \);
2) \( \text{dist}(K, \sum_{i=1}^n \lambda_i v_i) > \varepsilon/2 \), whenever \( v_i \in V_i, \ i = \overline{1,n} \).

**Proof.** First we find a Borel subset \( W_i \) in every \( U_i \), so that \( \mu(W_i) = \lambda_i \mu(W) > 0 \), where \( W = \bigcup_{i=1}^n W_i \) and \( W_i \cap W_j = \emptyset, \ i \neq j \).

Since \( \mu \) is atomless, in view of the Lyapunov Theorem [22], we can pick disjoint Borel sets \( A_i \subset U_i, \ i = \overline{1,n} \), such that \( \mu(A_i) = \mu(A_j) > 0 \) whenever \( i \neq j \). By the same token, there are sets \( W_i \subset A_i \), such that \( \mu(W_i) = \lambda_i \mu(A_i), \ i = \overline{1,n} \). Clearly, they fulfill our requirement.

Let us fix any finite \( \varepsilon/2 \)-net \((h_k)_{k=1}^N \) in \( K \). In view of Lemma 2.2.10 there is an \( f \in B \) verifying
\[
f(h_1) > \varepsilon + \frac{1}{\mu(W)} \int_W f(s) d\mu(s) \]
\[
= \varepsilon + \sum_{i=1}^n \frac{\lambda_i}{\mu(W_i)} \int_{W_i} f(s) d\mu(s).
\]

69
Then for every $i = \overline{1, n}$ one can find $(w_{ij})_{j=1}^M \subset W_i$ such that

$$f(h_1) > \varepsilon + \sum_{i=1}^n \lambda_i \sum_{j=1}^M \frac{1}{M} f(w_{ij})$$

$$= \varepsilon + \sum_{j=1}^M \frac{1}{M} \sum_{i=1}^n \lambda_i f(w_{ij}).$$

Thus,

$$\sum_{j=1}^M \frac{1}{M} \left| f(h_1) - \sum_{i=1}^n \lambda_i f(w_{ij}) \right| > \varepsilon.$$

So, for some $j_0$ we have

$$\left| f(h_1) - \sum_{i=1}^n \lambda_i f(w_{ij_0}) \right| > \varepsilon.$$

Since $w_{ij_0} \in W_i \subset U_i$, there are open subsets $W_i^1 \subset U_i$ such that the inequality

$$\left| f(h_1) - \sum_{i=1}^n \lambda_i f(w_i) \right| > \varepsilon$$

holds for all $w_i$ in $W_i^1$, $i = \overline{1, n}$. As a consequence we have

$$\left\| h_1 - \sum_{i=1}^n \lambda_i w_i \right\| > \varepsilon,$$

whenever $w_i \in W_i^1$, $i = \overline{1, n}$.

Doing the same for $(W_i^1)_{i=1}^n$ instead of $(U_i)_{i=1}^n$, and $h_2$ instead of $h_1$ we find open sets $W_i^2 \subset W_i^1$ with

$$\left\| h_2 - \sum_{i=1}^n \lambda_i w_i \right\| > \varepsilon,$$

whenever $w_i \in W_i^2$, $i = \overline{1, n}$.

Continuing the process we end up with open sets $V_i = W_i^N$. It is clear from our construction that

$$\left\| h - \sum_{i=1}^n \lambda_i v_i \right\| > \frac{\varepsilon}{2},$$

for all $h \in K$ and $v_i \in V_i$. So, conditions 1) and 2) are satisfied. \qed
Lemma 2.2.12. For any norm-compact set \( K \subset \text{conv}^w H \) and any collection of open sets \((U_i)_{i=1}^n\) in \( H \) there are open sets \((V_i)_{i=1}^n\) satisfying the following conditions:

1) \( V_i \subset U_i, \ i = 1, n; \)

2) \( \text{dist}(K, \sum_{i=1}^n \lambda_i v_i) > \frac{\varepsilon}{4}, \) whenever \( v_i \in V_i, \ i = 1, n, \) and \( \lambda_i \geq 0 \) with \( \sum_{i=1}^n \lambda_i = 1. \)

Proof. To prove this lemma we fix a finite \( \frac{\varepsilon}{4} \)-net in the set

\[ \{(\lambda_1, \lambda_2, \ldots, \lambda_n): \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0\} \]

equipped with the metric \( \rho((\lambda_i), (\nu_i)) = \sum_{i=1}^n |\lambda_i - \nu_i|. \) Then we apply Lemma 2.2.11 successively to all elements of the net. \( \square \)

Lemma 2.2.13. For any collection of open sets \((U_i)_{i=1}^n\) in \( H \) there exist \( f \in B \) and two constants \( a \) and \( b \) with \( b - a > \frac{\varepsilon}{8} \) such that

\[ \{f > b\} \cap U_i \neq \emptyset, \]

\[ \{f < a\} \cap U_i \neq \emptyset, \]

for all \( i = 1, n. \)

Proof. Let us fix arbitrary \( u_i \in U_i, \ i = 1, n \) and denote \( K = \text{conv}(u_i)_{i=1}^n. \) By Lemma 2.2.12, there are vectors \( v_i \in U_i \) such that if \( L = \text{conv}(v_i)_{i=1}^n, \) then \( \text{dist}(K, L) > \frac{\varepsilon}{4}. \)

By the geometric version of the Hahn-Banach Theorem, there exists a \( g \in B(X^*), \|g\| = 1, \) separating \( K - L \) from the ball \( \frac{\varepsilon}{4} B(X), \) i.e.

\[ g(k - l) > \frac{\varepsilon}{4}, \]

for all \( k \in K, \ l \in L. \) Since the weak*-closure of \( B \) is the whole \( B(X^*), \) we can find an \( f \in B, \) for which the inequality

\[ f(k - l) > \frac{\varepsilon}{4} \]

holds, whenever \( k \in K \) and \( l \in L. \)

Now it is easy to see that the constants \( a = \sup_{l \in L} f(l) + \frac{\varepsilon}{10} \) and \( b = \inf_{k \in K} f(k) - \frac{\varepsilon}{10} \) meet the desired conditions. \( \square \)
Construction of independent sequence. First, applying Lemma 2.2.13 to \( U_1 = U_2 = \ldots = U_n = H \) we find \( f_1 \in B \) and constants \( a_1, b_1 \) with \( b_1 - a_1 > \frac{\varepsilon}{8} \) such that

\[
U_1 = \{ f_1 > b_1 \} \cap H \neq \emptyset, \\
U_2 = \{ f_1 < a_1 \} \cap H \neq \emptyset.
\]

Then we apply Lemma 2.2.13 to \( U_1, U_2 \) and get \( f_2 \in B, a_2, b_2 \) with \( b_2 - a_2 > \frac{\varepsilon}{8} \) such that

\[
\{ f_2 > b_2 \} \cap U_i \neq \emptyset, \\
\{ f_2 < a_2 \} \cap U_i \neq \emptyset, \quad i = 1, 2.
\]

It is clear how to continue the process to obtain sequences \( (f_n)_{n \in \mathbb{N}} \subset B \) and \( (b_n, a_n)_{n \in \mathbb{N}}, b_n - a_n > \frac{\varepsilon}{8} \) such that for all finite disjoint sets \( M \) and \( N \) in \( \mathbb{N} \) we have

\[
H \cap (\cap_{m \in M} \{ f_m > b_m \}) \cap (\cap_{n \in N} \{ f_n < a_n \}) \neq \emptyset.
\]

Of course we can assume that \(|a_n - a| < \frac{\varepsilon}{32}\), for some constant \( a \) and every \( n \in \mathbb{N} \). Then letting \( \delta = \frac{\varepsilon}{32}, r = a + \frac{\varepsilon}{32} \) we finally get

\[
H \cap (\cap_{m \in M} \{ f_m > r + \delta \}) \cap (\cap_{n \in N} \{ f_n < r \}) \neq \emptyset,
\]

whenever \( M \) and \( N \) are finite disjoint subsets of \( \mathbb{N} \). The proof is finished.

As a consequence of Theorem 2.2.9 and Simon's result ([74]) we obtain the positive solution to the Boundary Problem in spaces not containing \( \ell_1[0,1] \). Surprisingly, this is also true for any \( \ell_1(\Gamma) \) in the canonical norm. Indeed, every boundary \( B \) over \( \ell_1(\Gamma) \) has to contain an element with arbitrary pre-defined sequence of signs on any countable subset of \( \Gamma \). This implies, in particular, that restricted to every \( \ell_1(\gamma) \) with \( \gamma \subset \Gamma \) countable, \( B \) contains the extreme points of the unit ball of \( \ell_\infty(\gamma) \). Now using the result of J. Bourgain and M. Talagrand [12] and the fact that weak compactness is enough to check only on sequences, we obtain the desired conclusion.

### 2.3 Limit sets in Banach spaces

As we have seen in Section 2.2.2, the Krein-Šmulian theorem works for all weak topologies provided the "strong" limit sets \( I_{\|\cdot\|,\mu}(f) \) are non-empty for all \( \mu \) and bounded \( f \). In this section we investigate those Banach spaces where this condition holds. In particular, we prove Theorem 2.1.2 stated in the Introduction.
2.3.1 ℵ-convex sets

Let us fix some notation.

- $(\Omega, \Sigma, \mu)$ stands for a probability space as usual;
- $\Sigma^+$ and $\Sigma^+_A$ are defined on pages 35 and 50;
- $\mu^*$ denotes the upper measure defined on all subsets of $\Omega$ as follows: $\mu^*(Q) = \inf\{\mu(E) : Q \subset E, E \in \Sigma\}$, where the infimum is, in fact, attained;
- $\aleph$ denotes a cardinal (aleph); $\text{card}(I)$ stands for cardinality of the set $I$; $c$ denotes the continuum, i.e. $\text{card}([0,1])$;
- $\aleph_0$ stands for the first countable cardinal, while $\aleph_1$ denotes the first uncountable one;
- $\aleph_\mu$ is the minimal cardinality of an index set $I$ such that there is an $A \in \Sigma^+$ represented as $A = \bigcup_{i \in I} A_i$, with all $A_i$ being $\mu$-null; it is worthwhile to remark that the equality $\aleph_\mu = c$ for the Lebesgue measure $\mu$ is exactly the statement of Martin’s Axiom (see [70]);
- $\aleph_{\min} = \min_\mu \aleph_\mu$; clearly, $\aleph_1 \leq \aleph_{\min}$;

**Definition 2.3.1.** Let $\aleph$ be a cardinal. A subset $D$ in a Banach space $X$ is called $\aleph$-convex if for every $D' \subset D$ either $\text{card}(D') < \aleph$ or $0 \in \overline{\text{conv}}D'$, where the overline means closure in the norm topology.

A set $D$ is called complete $\aleph$-convex if it is $\aleph$-convex and $\overline{\text{lin}}D = X$.

**Example 2.3.2.** A biorthogonal system $\{x_\alpha, x^*_\alpha\}_{\alpha \in \Gamma}$ is called a Markushevich basis if $\text{lin}\{x_\alpha\}$ is dense in $X$ and $\{x^*_\alpha\}$ separates points in $X$. Clearly, every relatively weakly compact Markushevich basis is a complete $\aleph_0$-convex set.

Recall another notion. A Banach space is said to be weakly compactly generated or a WCG-space if there is a weakly compact subset $K \subset X$ which spans $X$, i.e. $\overline{\text{lin}}K = X$. For instance, all reflexive and separable spaces are WCG.

There is a classical result due to D. Amir and J. Lindenstrauss [4] saying that every WCG-space possesses a relatively weakly compact Markushevich basis. Hence, every WCG-space has a complete $\aleph_0$-convex subset.
On the other hand it is not hard to see that for every ℵ₀-convex set $D$, $D \cup \{0\}$ is weakly compact. So, if $X$ contains a complete ℵ₀-convex subset, then $X$ is WCG.

**Example 2.3.3.** A subset $D = \{x_i\}_{i \in I}$ in a Banach space $X$ is called an extended unconditional basis if $\overline{\text{lin}}D = X$ and every countable subset of $D$ is an unconditional basic sequence. According to Theorem 17.5 (5°) from [75], $D$ is an extended unconditional basis if and only if there is a constant $M > 0$ such that for every finite subset $D' \subset D$, every choice of signs $\varepsilon_d, d \in D'$ the inequality

$$\left\| \sum_{d \in D'} a_d d \right\| \leq M \left\| \sum_{d \in D'} \varepsilon_d a_d d \right\|$$

(2.3.1)

holds for all scalars $a_d \in \mathbb{R}, d \in D'$. Clearly, $D$ becomes a complete ℵ-convex subset, provided $X$ does not contain a copy of $\ell_1(\Gamma)$ with $\text{card}\Gamma = \aleph$.

**Definition 2.3.4.** A subset $LD \in \text{lin}D$ is called disjoint if there are representations of its elements,

$$x = \sum_{d \in D_x} a_d d, \quad d \in D, \quad a_d \in \mathbb{R},$$

such that the finite sets $(D_x)_{x \in LD}$ are disjoint.

**Lemma 2.3.5.** For every disjoint set $LD \subset \text{lin}D$ either $\text{card}LD < \aleph$ or $0 \in \text{conv}LD$.

**Proof.** Suppose $\text{card}LD \geq \aleph$. Without loss of generality, we may assume that there is an $n \in \mathbb{N}$ and $a > 0$ such that $\text{card}D_x = n$ and $\sum_{d \in D_x} |a_d| \leq a$, for every $x \in LD$.

If $0 \notin \text{conv}LD$, then there exists a functional $x^* \in X^*$ such that $x^*(x) > \varepsilon$ for all $x \in LD$ and some $\varepsilon > 0$. For every $x \in LD$ select a $d_x \in D_x$ such that $|x^*(d_x)| > \frac{\varepsilon}{a}$. Then the set $D' = \{d_x : x \in LD\}$ has cardinality not less than $\aleph$. On the other hand, by the ℵ-convexity of $D$ both of the sets

$$D'_1 = \{d \in D : x^*(d) > \frac{\varepsilon}{a}\}$$

$$D'_2 = \{d \in D : x^*(d) < -\frac{\varepsilon}{a}\}$$

must have cardinality less than $\aleph$. This leads to a contradiction. □
2.3.2 The main theorem

Theorem 2.3.6. If a Banach space $X$ contains a complete $\aleph$-convex set, then $I_{\|\cdot\|,\mu}(f) \neq \emptyset$ for every probability space $(\Omega, \Sigma, \mu)$ with $\aleph_\mu \geq \aleph$ and every bounded function $f: \Omega \to X$. In particular, if $X$ contains a complete $\aleph_{\min}$-convex set, then $I_{\|\cdot\|,\mu}(f) \neq \emptyset$ for all $\mu$ and bounded $f$.

In view of Examples 2.3.2 and 2.3.3, Theorem 2.1.2 follows immediately. As another consequence we obtain a generalized Corollary 2.2.4.

Corollary 2.3.7. If a Banach space $X$ contains a complete $\aleph_{\min}$-convex set, then (KSb) holds in $X$ for all $\tau$ weaker than the norm topology.

It was recently proved by B. Cascales and G. Godefroy [13] that (KSb) is valid in every $C(K)$ for topologies induced by boundaries. The following generalizes this result for some special types of $K$.

Corollary 2.3.8. Suppose a Banach space $X$ has an extended unconditional basis and does not have a copy of $\ell_1(\Gamma)$ with $\text{card} \Gamma = \aleph_{\min}$. Let $K \subset X^*$ be any $w^*$-compact set. Then $C(K)$ has a complete $\aleph_{\min}$-convex set. In particular, (KSb) holds in $C(K)$ for all $\tau$ weaker than the norm topology.

Proof. Indeed, let $D$ be an extended unconditional basis in $X$. According to Example 2.3.3, $D$ is a complete $\aleph_{\min}$-convex set in $X$. Consider the canonical mapping $i: X \mapsto C(K)$ defined by

$$i(x)(t) = t(x), \quad x \in X, \ t \in K.$$ 

Fix constant $M$ as in (2.3.1) and $C > 0$ such that

$$\|i(x)\| \leq C\|x\|$$

for all $x \in X$. Put

$$D_K = \{1\} \cup \{i(d_1)...i(d_n) : d_i \in D, n \in \mathbb{N}\}.$$ 

Since $i(D)$ separates points in $K$, $D_K$ is complete. We claim that $D_K$ is $\aleph_{\min}$-convex.

Indeed, let $D_K' \subset D_K$ and $\text{card} D_K' \geq \aleph_{\min}$. If, on the contrary, $0 \notin \text{conv}_D D_K'$, then there is an $\varepsilon > 0$ such that $\varepsilon \leq \|f\|$, for all $f \in \text{conv} D_K'$. Let

$$D' = \{d \in D : i(d)i(d_2)...i(d_n) \in D_K', \text{ for some } d_2, ..., d_n \in D\}.$$
Clearly, \( \text{card} D' \geq \aleph_{\text{min}} \). On the other hand, let \( d_1^{(1)}, \ldots, d_k^{(k)} \in D' \) be arbitrary and \( d_i^{(j)} \in D \) be such that \( i(d_1^{(1)})i(d_2^{(j)})\ldots i(d_n^{(j)}) \in D_K^\prime \), \( j = 1, \ldots, k \). Then whenever \( \alpha_j > 0 \), \( \sum_{j=1}^k \alpha_j = 1 \) we have

\[
\varepsilon \leq \left\| \sum_{j=1}^k \alpha_j i(d_1^{(j)})\ldots i(d_n^{(j)}) \right\| = \sup_{t \in K} \sum_{j=1}^k \alpha_j |i(d_1^{(j)})(t)|\ldots |i(d_n^{(j)})(t)| = C^{n-1} \sup_{t \in K} \sum_{j=1}^k \alpha_j |i(d_1^{(j)})(t)| = C^{n-1} \max_{\theta_j = \pm 1} \left\| \sum_{j=1}^k \alpha_j \theta_j i(d_1^{(j)}) \right\|
\]

This contradicts the \( \aleph_{\text{min}} \)-convexity of \( D \). \( \Box \)

Before we begin the proof of Theorem 2.3.6 let us make some preliminary observations.

From now on we fix a complete \( \aleph \)-convex set \( D \), probability space \( (\Omega, \Sigma, \mu) \) with \( \aleph \mu \geq \aleph \) and a bounded function \( f: \Omega \to X \), as given in the condition of Theorem 2.3.6.

Since \( \text{lin} D = X \), we can represent each value of \( f \) in the form of an absolutely converging series of elements from \( \text{lin} D \):

\[
f(\omega) = \sum_{i=1}^{\infty} x_i(\omega), \quad \omega \in \Omega.
\]

From now on we fix a representation of each \( x_i(\omega) \) as a linear combination of elements from \( D \):

\[
x_i(\omega) = \sum_{d \in D_i(\omega)} a_d d, \quad d \in D, \quad a_d \in \mathbb{R}.
\]

Now for every \( \varepsilon > 0 \) and \( \omega \in \Omega \) choose the smallest \( N(\omega, \varepsilon) \) for which

\[
\left\| \sum_{i=n}^{\infty} x_i(\omega) \right\| < \varepsilon, \quad n \geq N(\omega, \varepsilon) + 1.
\]

Clearly, \( N(\omega, \varepsilon) \) is non-increasing in \( \varepsilon \). Denote \( f_\varepsilon(\omega) = \sum_{i=1}^{N(\omega, \varepsilon)} x_i(\omega) \). Then

\[
\| f(\omega) - f_\varepsilon(\omega) \| < \varepsilon. \quad (2.3.2)
\]

76
We introduce the "support" of \( f_\varepsilon(\omega) \) over \( D \) as
\[
\text{supp} f_\varepsilon(\omega) = \bigcup_{i=1}^{N(\omega,\varepsilon)} D_i(\omega).
\]

Let us formulate a lemma, which will be frequently used in the proof of Theorem 2.3.6.

**Lemma 2.3.9.** Let \( E \in \Sigma^+ \), \( P \subset E \), \( \mu^*(P) = \mu(E) \) and \( \varepsilon > 0 \) be given. Then there exist sets \( F \in \Sigma_E^+ \) and \( Q \subset F \cap P \) with \( \mu^*(Q) = \mu(F) \), and there exists a finite set \( D_{\text{fin}} \subset D \) such that for every \( q \in Q \) we have \( D_{\text{fin}} \subset \text{supp} f_\varepsilon(q) \) and
\[
\mu^*(q \in Q : d \in \text{supp} f_\varepsilon(q)) = 0,
\]
whenever \( d \in D \setminus D_{\text{fin}} \).

**Proof.** Consider the sets \( Q_n = \{ p \in P : \text{card}(\text{supp} f_\varepsilon(p)) = n \}, n \in \mathbb{N} \). Since \( P = \bigcup_n Q_n \), there is some \( n \in \mathbb{N} \) for which \( \mu^*(Q_n) > 0 \).

If \( \mu^*(q \in Q_n : d \in \text{supp} f_\varepsilon(q)) = 0 \) for all \( d \in D \), then put \( Q = Q_n \), \( D_{\text{fin}} = \emptyset \) and let \( F \in \Sigma_E^+ \) be such that \( Q \subset F \) and \( \mu^*(Q) = \mu(F) \). Otherwise, we have \( \mu^*(q \in Q_n : d_1 \in \text{supp} f_\varepsilon(q)) > 0 \), for some \( d_1 \in D \). Denote \( P_1 = \{ q \in Q_n : d_1 \in \text{supp} f_\varepsilon(q) \} \).

If \( \mu^*(q \in P_1 : d \in \text{supp} f_\varepsilon(q)) = 0 \) for all \( d \in D \setminus \{d_1\} \), then we put \( Q = P_1 \), \( D_{\text{fin}} = \{d_1\} \) and let \( F \in \Sigma_E^+ \) be such that \( Q \subset F \) and \( \mu^*(Q) = \mu(F) \). Otherwise, we find a \( d_2 \in D_{\text{fin}} \setminus \{d_1\} \) for which \( \mu^*(q \in P_1 : d_2 \in \text{supp} f_\varepsilon(q)) > 0 \) and denote \( P_2 = \{ q \in P_1 : d_2 \in \text{supp} f_\varepsilon(q) \} \).

Since \( \text{supp} f_\varepsilon(q) \) has cardinality \( n \), for every \( q \in Q_n \), the process terminates no later than after the \( n \)-th step. \( \square \)

**Proof of Theorem 2.3.6.**

**Remark 2.3.10.** In our case when the topology is generated by a complete norm and \( f \) is norm bounded it makes no difference to define a limit set through finite Riemann-Lebesgue integral sums, as we have done in Section 2.1, or through infinite ones. We will tacitely use this observation in the proof.

Let us fix positive decreasing \( \varepsilon_n \) such that \( \sum_n \varepsilon_n < \infty \). We construct a Cauchy sequence \( x_1, x_2, \ldots \) in \( X \) such that for every partition \( \Pi \) and every \( n \in \mathbb{N} \) there is a finer partition \( \Pi_n \) with sampling points \( T_n \) such that
\[
\| S(f, \Pi_n, T_n) - x_n \| < 4\varepsilon_n.
\]

Clearly, \( \lim_{n \to \infty} x_n \) is then a limit point.

77
Construction of the sequence $x_1, x_2, \ldots$. Using the exhaustion principle and Lemma 2.3.9 with $\varepsilon = \varepsilon_1$ we find a partition of $\Omega$ into at most countable number of sets $(E_i)_{i \in \mathbb{N}} \subset \Sigma$, subsets $Q_i \subset E_i$ with $\mu^*(Q_i) = \mu(E_i)$ and finite sets $D_i \subset D$ such that for every $q \in Q_i$, we have $D_i \subset \text{supp} f_{\varepsilon_1}(q)$ and

$$
\mu^*(q \in Q_i : d \in \text{supp} f_{\varepsilon_1}(q)) = 0,
$$

whenever $d \in D \setminus D_i$.

Define a "projection" onto $D_i$ as follows

$$
D_i f_{\varepsilon_1}(q) = \sum_{i=1}^{N(\omega, \varepsilon_1)} \sum_{d \in D_i(\omega) \cap D_i} a_d d, \quad q \in Q_i.
$$

Let us partition each $E_i$ into further subsets $E_{ij}, \ j \in \mathbb{N}$ and find $Q_{ij} \subset Q_i$ such that $Q_{ij} \subset E_{ij}, \mu^*(Q_{ij}) = \mu(E_{ij})$ and

$$
\text{diam}\{D_i f_{\varepsilon_1}(q) : q \in Q_{ij}\} < \varepsilon_1.
$$

Let us reindex $E_{ij}, Q_{ij}$ and $D_i$ into $E_{1p}, Q_{1p}, D_{1p}$ for $p \in \mathbb{N}$. Then by our construction $Q_{1p} \subset E_{1p}, \mu^*(Q_{1p}) = \mu(E_{1p})$; $D_{1p} \subset \text{supp} f_{\varepsilon_1}(q)$ for all $q \in Q_{1p}$, and

$$
\mu^*(q \in Q_{1p} : d \in \text{supp} f_{\varepsilon_1}(q)) = 0,
$$

whenever $d \in D \setminus D_{1p}$. Moreover,

$$
\text{diam}\{D_{1p} f_{\varepsilon_1}(q) : q \in Q_{1p}\} < \varepsilon_1.
$$

Fix arbitrary $q_{1p} \in Q_{1p}$ and put

$$
x_1 = \sum_{p=1}^{\infty} D_{1p} f_{\varepsilon_1}(q_{1p}) \mu(E_{1p}).
$$

To find $x_2$ we carry out similar construction inside each $E_{1p}$ with $\varepsilon = \varepsilon_2$.

Namely, using Lemma 2.3.9 we partition each $E_{1p}$ into sets $E_{pi}$ and find subsets $Q_{pi} \subset E_{pi} \cup Q_{1p}$ with $\mu^*(Q_{pi}) = \mu(E_{pi})$ and finite sets $D_{pi} \subset D$ such that

$$
\mu^*(q \in Q_{pi} : d \in \text{supp} f_{\varepsilon_2}(q)) = 0,
$$

whenever $d \in D \setminus D_{pi}$. This necessarily implies that $D_{1p} \subset D_{pi}$. 78
As before, we split each $E_{pi}$ into further subsets $E_{pij}$ and find $Q_{pij}$ such that
\[ \text{diam}\{D_p f_{\varepsilon_2}(q) : q \in Q_{pij}\} < \varepsilon_2. \]
Again rewrite $E_{pij}, Q_{pij}$ and $D_{pi}$ as $E_{p}^2, Q_{p}^2$ and $D_{p}^2$ so that $Q_{p}^2 \subset E_{p}^2, \mu^*(Q_{p}^2) = \mu(E_{p}^2)$; $D_{p}^2 \subset \text{supp} f_{\varepsilon_2}(q), q \in Q_{p}^2$ and
\[ \mu^*(q \in Q_{p}^2 : d \in \text{supp} f_{\varepsilon_2}(q)) = 0, \]
whenever $d \in D_{p} \setminus D_{p}^2$. Moreover,
\[ \text{diam}\{D_p^2 f_{\varepsilon_2}(q) : q \in Q_{p}^2\} < \varepsilon_2. \]
Pick any $q_{p}^2$ in $Q_{p}^2$ and put
\[ x_2 = \sum_{p=1}^{\infty} D_p^2 f_{\varepsilon_2}(q_{p}^2) \mu(E_{p}^2). \]
Similarly we construct the rest of the sequence. Let us write down the sets and the properties they possess on the $n$-th step.
We have $E_{p}^n, Q_{p}^n$ with $Q_{p}^n \subset E_{p}^n, \mu^*(Q_{p}^n) = \mu(E_{p}^n)$; finite $D_{p}^n \subset D$ such that
\[ D_{p}^n \subset \text{supp} f_{\varepsilon_n}(q), q \in Q_{p}^n \]
\[ \mu^*(q \in Q_{p}^n : d \in \text{supp} f_{\varepsilon_n}(q)) = 0, \quad d \in D \setminus D_{p}^n \]
\[ \text{diam}\{D_p^n f_{\varepsilon_n}(q) : q \in Q_{p}^n\} < \varepsilon_n. \]
We fix $q_{p}^n \in Q_{p}^n$ arbitrary and put
\[ x_n = \sum_{p=1}^{\infty} D_p^n f_{\varepsilon_n}(q_{p}^n) \mu(E_{p}^n). \]
Furthermore, if $i(p)$ denotes the index $p' \in E_{p}^{n-1}$ for which $E_{p}^n \subset E_{p'}^{n-1}$, then $Q_{p}^n \subset Q_{i(p)}^{n-1}$ and $D_{p}^n \subset D_{i(p)}^{n-1}$.

Claim 2.3.11. For every partition $\Pi$ and $n \in \mathbb{N}$ there is a finer partition $\Pi_n \subset \Pi$ with a collection of sampling points $T_n$ such that
\[ ||S(f, \Pi_n, T_n) - x_n|| < 3\varepsilon_n. \]
Proof. Indeed let us fix a partition \( \Pi_{aux} \) finer than \( \Pi \) and \( (E^n_p)_p \). For every \( A \in \Pi_{aux} \) denote by \( p(A) \) the index \( p \) for which \( A \subset E^n_p \). Fix an \( A \in \Pi_{aux} \).

According to (2.3.4) for every \( q \in Q^n_{p(A)} \) we have
\[
\mu^*(\tilde{q} \in Q^n_{p(A)} : (\text{supp} f_{\varepsilon_n}(\tilde{q}) \cap \text{supp} f_{\varepsilon_n}(q)) \setminus D^n_{p(A)} \neq \emptyset) = 0.
\]

Therefore, using the method of transfinite induction and the condition \( \aleph \leq \aleph_\mu \), we can find a set \( Q_{dis} \subset Q^n_{p(A)} \cap A \) of cardinality \( \aleph \) such that the set
\[
\{ f_{\varepsilon_n}(q) - D^n_p f_{\varepsilon_n}(q) : q \in Q_{dis} \}
\]
is disjoint in the sense of Definition 2.3.4. By Lemma 2.3.5, we get
\[
\left\| \sum_k \alpha_k (f_{\varepsilon_n}(q_k) - D^n_{p(A)} f_{\varepsilon_n}(q_k, A)) \right\| < \varepsilon_n,
\]
for some positive \( \alpha_k \) with \( \sum_k \alpha_k = 1 \) and \( q_k, A \in Q_{dis} \). In view of (2.3.5) this inequality implies
\[
\left\| \sum_k \alpha_k f_{\varepsilon_n}(q_k, A) - D^n_{p(A)} f_{\varepsilon_n}(q^n_{p(A)}) \right\| < 2\varepsilon_n,
\]
and by (2.3.2),
\[
\left\| \sum_k \alpha_k f(q_k, A) - D^n_{p(A)} f_{\varepsilon_n}(q^n_{p(A)}) \right\| < 3\varepsilon_n. \tag{2.3.6}
\]

Using the non-atomicity of \( \mu \), let us split \( A \) into subsets \( A_k \in \Sigma \) with \( \mu(A_k) = \alpha_k \mu(A) \) and \( q_k, A \in A_k \). Set \( \Pi_n \) to be \( (A_k)_{k, A \in \Pi_{aux}} \) and \( T_n \) to be \( (q_k, A)_{k, A \in \Pi_{aux}} \). Then
\[
S(f, \Pi_n, T_n) - x_n = \sum_{A \in \Pi_{aux}} \sum_k f(q_k, A) \mu(A_k) - \sum_{p=1}^{\infty} D^n_p f_{\varepsilon_n}(q^n_p) \mu(E^n_p)
\]
\[
= \sum_{A \in \Pi_{aux}} \mu(A) \sum_k \alpha_k f(q_k, A) - \sum_{A \in \Pi_{aux}} D^n_{p(A)} f_{\varepsilon_n}(q^n_{p(A)}) \mu(A)
\]
\[
= \sum_{A \in \Pi_{aux}} \left( \sum_k \alpha_k f(q_k, A) - D^n_{p(A)} f_{\varepsilon_n}(q^n_{p(A)}) \right) \mu(A),
\]
and applying (2.3.6) we conclude the proof of the claim. \( \square \)
(\(x_n\)) is a Cauchy sequence. To prove this statement we show that \(\|x_n - x_{n-1}\| < 6 \varepsilon_{n-1}\) for every \(n \in \mathbb{N}\).

Equation (2.3.4) applied to \(n\) and \(n-1\) yields

\[
\mu^* \left( \tilde{q} \in Q^n_p : \text{or} \supp f_{\varepsilon_n}(q) \cup \supp f_{\varepsilon_n}(\tilde{q}) \backslash D^n_p \neq \emptyset \supp f_{\varepsilon_{n-1}}(q) \cup \supp f_{\varepsilon_{n-1}}(\tilde{q}) \backslash D^{n-1}_{i(p)} \neq \emptyset \right) = 0,
\]

for every \(q \in Q^n_p\). As before, this allows us to find \(Q_{\text{dis}} \subset Q^n_p\) of cardinality \(\aleph\) such that the sets

\[
\{f_{\varepsilon_n}(q) - D^n_p f_{\varepsilon_n}(q) : q \in Q_{\text{dis}}\}
\]

and

\[
\{f_{\varepsilon_{n-1}}(q) - D^{n-1}_{i(p)} f_{\varepsilon_{n-1}}(q) : q \in Q_{\text{dis}}\}
\]

are disjoint in the sense of Definition 2.3.4. Via repeated application of Lemma 2.3.5 we can find a collection of convex combinations of norm less than \(\varepsilon_{n-1}\):

\[
\sum_{q \in Q_\iota} \alpha_q (f_{\varepsilon_n}(q) - D^n_p f_{\varepsilon_n}(q)); \ Q_\iota \subset Q_{\text{dis}}, \ \iota \in I
\]

where \(Q_\iota\) are disjoint finite sets and \(\text{card} I = \aleph\). Then the set of all corresponding combinations

\[
\sum_{q \in Q_\iota} \alpha_q (f_{\varepsilon_{n-1}}(q) - D^{n-1}_{i(p)} f_{\varepsilon_{n-1}}(q)); \ Q_\iota \subset Q_{\text{dis}}, \ \iota \in I
\]

is also disjoint in the sense of Definition 2.3.4. Applying Lemma 2.3.5 again we find another convex combination

\[
\sum_k \beta_k \sum_{q \in Q_{\kappa}} \alpha_q (f_{\varepsilon_{n-1}}(q) - D^{n-1}_{i(p)} f_{\varepsilon_{n-1}}(q))
\]

of norm less than \(\varepsilon_{n-1}\). In other words, for some \(\gamma_k > 0\), \(\sum_k \gamma_k = 1\) and some \(q_k \in Q^n_p\) we have both inequalities:

\[
\left\| \sum_k \gamma_k (f_{\varepsilon_n}(q_k) - D^n_p f_{\varepsilon_n}(q_k)) \right\| < \varepsilon_{n-1}
\]

and

\[
\left\| \sum_k \gamma_k (f_{\varepsilon_{n-1}}(q_k) - D^{n-1}_{i(p)} f_{\varepsilon_{n-1}}(q_k)) \right\| < \varepsilon_{n-1}.
\]
From (2.3.5) we see that
\[
\left\| \sum_k \gamma_k f_{\varepsilon_n} (q_k) - D^n_{p} f_{\varepsilon_n} (q^n_p) \right\| < \varepsilon_{n-1} + \varepsilon_n < 2 \varepsilon_{n-1}
\]
and since by (2.3.2) \( \| f_{\varepsilon_n - 1} (q_k) - f_{\varepsilon_n} (q_k) \| < 2 \varepsilon_{n-1} \), we get from the previous two inequalities
\[
\left\| D^n_{p} f_{\varepsilon_n} (q^n_p) - D^{n-1}_{i(p)} f_{\varepsilon_n - 1} (q^{n-1}_{i(p)}) \right\| < 6 \varepsilon_{n-1}.
\]
This implies the needed inequality:
\[
\| x_n - x_{n-1} \| = \left\| \sum_p D^n_{p} f_{\varepsilon_n} (q^n_p) \mu (E^n_p) - \sum_p D^{n-1}_{i(p)} f_{\varepsilon_n - 1} (q^{n-1}_{i(p)}) \mu (E^n_p) \right\|
\leq \sum_p \| D^n_{p} f_{\varepsilon_n} (q^n_p) - D^{n-1}_{i(p)} f_{\varepsilon_n - 1} (q^{n-1}_{i(p)}) \| \mu (E^n_p) < 6 \varepsilon_{n-1}.
\]

The proof is finished.

\[\square\]

**Lemma 2.3.12.** Suppose \( X \) has a Markushevich basis \( \{ x_\alpha, x^*_\alpha \}_\alpha \). Then \( D = \{ x_\alpha \} \) is \( \aleph_{\text{min}} \)-convex if and only if \( I_{\| \cdot \|, \mu} (f) \neq \emptyset \) for all \( \mu \) and bounded \( f : \Omega \to X \).

**Proof.** Sufficiency is already proved by Theorem 2.3.6.

For the necessity, let us assume that there is a subset \( D' \subset D \) with \( \text{card} D' = \aleph_{\text{min}} \) and \( 0 \notin \operatorname{conv} D' \). Let \( (\Omega, \Sigma, \mu) \) be any non-atomic probability space with \( \aleph''_\mu = \aleph_{\text{min}} \). Then there is an \( A \in \Sigma^+ \) represented as the union \( \bigcup_{i \in I} A_i \), with all \( A_i \) being \( \mu \)-null and \( \text{card} I = \aleph_{\text{min}} \). Fix a bijection \( \beta : I \to D' \) and define \( f \) on \( \Omega \) as follows
\[
f(\omega) = \begin{cases} 0, & \omega \in \Omega \setminus A; \\
\beta(i), & \omega \in A_i. \end{cases}
\]
If there is an \( x \in I_{\| \cdot \|, \mu} (f) \subset \operatorname{conv} D' \), then \( x \neq 0 \). On the other hand, \( x^*_\alpha \) vanishes on \( x \) for every \( \alpha \). This is a contradiction. \[\square\]
Remark 2.3.13. Similar construction shows that there is no a copy of $\ell_1(\Gamma)$ with $\text{card}\Gamma = \aleph_{\text{min}}$ in every Banach space with $I_{\|\cdot\|,\mu}(f) \neq \emptyset$ for all $\mu$ and bounded $f$.

**Proposition 2.3.14.** Suppose $X$ has a Markushevich basis. Then the following assertions are equivalent:

1) $I_{\|\cdot\|,\mu}(f) \neq \emptyset$ for all $\mu$ and bounded $f$: $\Omega \to X$;

2) $X$ has a complete $\aleph_{\text{min}}$-convex set;

3) Every Markushevich basis in $X$ is $\aleph_{\text{min}}$-convex.

For the proof apply Theorem 2.3.6 and Lemma 2.3.12.

Proposition 2.3.14 combined with Example 2.3.3 and Remark 2.3.13 yield the following result.

**Proposition 2.3.15.** Suppose $X$ has an extended unconditional basis in the sense of Example 2.3.3. Then the assertions 1)-3) of Proposition 2.3.14 are equivalent to the following one:

4) $X$ does not contain a copy of $\ell_1(\Gamma)$ with $\text{card}\Gamma = \aleph_{\text{min}}$.

**Problem 2.3.16.** In Theorem 2.3.6 and Remark 2.3.13 we have shown the following chain of implications:

\[ X \text{ has a complete } \aleph_{\text{min}}-\text{convex set} \implies I_{\|\cdot\|,\mu}(f) \neq \emptyset \text{ for all } \mu \text{ and bounded } f \implies X \text{ has no copies of } \ell_1(\Gamma) \text{ with } \text{card}\Gamma = \aleph_{\text{min}} \]

Propositions 2.3.14 and 2.3.15 give a couple of special cases when some or all of these implications are reversible. We do not know if anything can be said in general.

### 2.3.3 Calculus of non-measurable functions

Clearly, $I_{\|\cdot\|,\mu}(f) = \{\int_{\Omega} f \, d\mu\}$, provided $f$ is Bochner integrable. The purpose of this section is to show that in general the limit sets behave in much the same way as the Bochner integral.

We introduce the space $L_1(\mu, X)$ of all functions $f: \Omega \to X$ for which $\|f(\cdot)\|: \Omega \to \mathbb{R}$ has an integrable majorant, i.e. a function $g: \Omega \to \mathbb{R}$ with $\|f(\omega)\| \leq g(\omega)$ a.e.
Remark 2.3.17. \( \overline{L}_1(\mu, X) \) endowed with the norm

\[
\|f\|_{\overline{L}_1(\mu, X)} = \inf \left\{ \int_{\Omega} gd\mu : \|f\| \leq g \text{ a.e.} \right\}
\]

becomes a Banach space with the Daugavet property. The proof traditionally goes via application of Lemma 1.2.4. There are, however, certain complications due to the non-measurability condition.

All the statements that were proved in the previous section for the limit sets of bounded functions hold for the limit sets of functions from \( \overline{L}_1(\mu, X) \) as well. To see this, fix a majorant \( g \in L_1(\mu) \) of \( f \in \overline{L}_1(\mu, X) \) and find a partition \( \{A_i\} \) of \( \Omega \) such that

\[
\sum_{i=1}^{\infty} \mu(A_i) \sup_{\omega \in A_i} g(\omega) < \infty.
\]

Observe that \( f \) is bounded on each \( A_i \) and if \( x_i \in I_{\|\cdot\|, \mu} (f\chi_{A_i}) \), then \( \sum x_i \in I_{\|\cdot\|, \mu}(f) \).

Throughout the rest of the section we assume that \( X \) has a complete \( \aleph_{\text{min}} \)-convex set. Let us now find other spaces built from \( X \), which share this property too.

Lemma 2.3.18. The finite sum \( X \oplus X \oplus ... \oplus X \) has a complete \( \aleph_{\text{min}} \)-convex set.

Proof. Indeed, denote \( D^k = \{(0, \ldots, 0, d, 0, \ldots, 0) : d \in D\} \). Since each \( D^k \) is \( \aleph_{\text{min}} \)-convex, their finite union is also \( \aleph_{\text{min}} \)-convex and clearly, is complete.

Let \( \oplus_c X \) be the space of all converging sequences \( (x_1, x_2, ...) \) in \( X \), endowed with the sup-norm. Denote by \( X_{\text{lim}} \) the subspace of \( (\oplus_c X) \oplus X \) consisting of those elements \( (x_1, x_2, ..., x) \) for which \( \lim_{n \to \infty} x_n = x \). Observe that \( X_{\text{lim}} \) is closed.

Lemma 2.3.19. \( X_{\text{lim}} \) has a complete \( \aleph_{\text{min}} \)-convex set.

Proof. Denote \( \overline{D} = \{(d, d, \ldots, d) : d \in D\} \) and \( D^k = \{(0, \ldots, 0, d, 0, \ldots, 0) : d \in D\} \), \( k \in \mathbb{N} \). Clearly, \( \overline{D} \) and all \( D^k \) are \( \aleph_{\text{min}} \)-convex subsets of \( X_{\text{lim}} \). Put \( D_{\text{lim}} = (\cup_k D^k) \cup \overline{D} \).
Since $D_{\text{lim}}$ is a countable union of $\aleph_{\min}$-convex sets and $\aleph_{\min} > \aleph_0$, $D_{\text{lim}}$ itself is $\aleph_{\min}$-convex.

To prove completeness, pick any $F \in X_{\text{lim}}^*$ that sends to zero all elements of $D_{\text{lim}}$. Then $F$ vanishes on the subspaces $X = \{(x, x, \ldots; x) : x \in X\}$ and $X^k = \{(0, \ldots, 0, x, 0, \ldots; 0) : x \in X\}$. Hence, $F$ vanishes on the algebraic sum of the spaces $X$ and $\bigoplus c_0 X^k$, which is $X$.

Now we are ready to investigate integral properties of limit sets. For simplicity, denote $I(f) = I_{\|\cdot\|, \mu}(f)$.

**Linearity.** If $f_1, \ldots, f_n \in L_1(\mu, X)$ and $a_1, \ldots, a_n \in \mathbb{R}$, then

$$I(a_1 f_1 + \ldots + a_n f_n) \cap [a_1 I(f_1) + \ldots + a_n I(f_n)] \neq \emptyset.$$ 

**Proof.** According to Lemma 2.3.18 and Theorem 2.3.6, $I(a_1 f_1 \oplus \ldots \oplus a_n f_n) \neq \emptyset$, where

$$a_1 f_1 \oplus \ldots \oplus a_n f_n : \Omega \to X \oplus \ldots \oplus X$$

is defined by

$$a_1 f_1 \oplus \ldots \oplus a_n f_n(\omega) = (a_1 f_1(\omega), \ldots, a_n f_n(\omega)).$$

Let $(x_1, \ldots, x_n) \in I(a_1 f_1 \oplus \ldots \oplus a_n f_n)$. Then $x_i \in a_i I(f_i)$, $i = 1, \ldots, n$, and $x_1 + \ldots + x_n \in I(a_1 f_1 + \ldots + a_n f_n)$.

**The Lebesgue Dominated Convergence Theorem.** Let $f_n \in L_1(\mu, X)$, $n \in \mathbb{N}$, and suppose that there is a $g \in L_1(\mu, X)$ such that

$$\sup_n \|f_n(\omega)\| \leq g(\omega), \text{ a.e.}$$

If $f_n \to f$ a.e., then one can select a sequence $x_n \in I(f_n)$ converging to an element of $I(f)$.

**Proof.** By the assumption, the function $f = (f_1, f_2, \ldots; f)$ takes values in $X_{\text{lim}}$ a.e. and moreover $f \in L_1(\mu, X_{\text{lim}})$. According to our Lemma 2.3.19 and Theorem 2.3.6 there is a vector $(x_1, x_2, \ldots; x)$ in $I(f)$. Applying the component projections, we see that $x_n \in I(f_n)$ and $x \in I(f)$. On the other hand, by the very definition of $X_{\text{lim}}$, $\lim_{n \to \infty} x_n = x$.
2.3.4 Pettis integral and limit sets

Recall that a function \( f: \Omega \rightarrow \mathbb{R} \) is called Pettis integrable if for every \( x^* \in X^* \) the real-valued function \( x^* f \) is measurable (we say \( f \) is weakly measurable) and for each \( A \in \Sigma \) there exists an \( x_A \in X \) such that

\[
x^*(x_A) = P - \int_A x^* f \, d\mu, \quad \text{for all } x^* \in X^*.
\]

We denote \( x_A = P - \int_A f \, d\mu \) and call it the Pettis integral of \( f \) over \( A \) (see [22]).

If \( f \) is Pettis integrable, then clearly

\[
P - \int_\Omega f \, d\mu \in I_{\text{weak-}\|\cdot\|,\mu}(f).
\]

Then according to Lemma 2.2.1,

\[
P - \int_\Omega f \, d\mu \in I_{\|\cdot\|,\mu}(f),
\]

and in fact, \( I_{\|\cdot\|,\mu}(f) \) contains no other points.

In the following theorem we establish the converse statement at least in those Banach spaces where all limit sets are non-empty.

**Theorem 2.3.20.** Suppose \( X \) is a Banach space such that \( I_{\|\cdot\|,\mu}(f) \neq \emptyset \) for all \( \mu \) and \( f \in L_1(\mu, X) \). If the limit set \( I_{\|\cdot\|,\mu}(f) \) of a function \( f \in L_1(\mu, X) \) consists of a single point \( x \), then \( f \) is Pettis integrable and \( x = P - \int_\Omega f \, d\mu \).

A moment of reflection reveals that for every bounded weakly measurable function \( f \), \( I_{\|\cdot\|,\mu}(f) \) is a singleton. Hence, according to our theorem, such a function is Pettis integrable. This generalizes an unpublished result by D. R. Lewis stated in WCG-spaces (see [22, p.88]).

As in the previous section, we denote \( I(f) := I_{\|\cdot\|,\mu}(f) \), where the underlying measure space may vary, but is always clear from the context. Also we denote by \( \mu_* \) the lower measure defined by

\[
\mu_*(Q) = \max\{\mu(B) : B \subset Q, \ B \in \Sigma\}, \quad Q \subset \Omega.
\]

We prove some preliminary lemmas first.
Lemma 2.3.21. Let $X$ and $f$ be as in Theorem 2.3.20 and let $A \in \Sigma$. Then $I(f|A)$ is a singleton.

Proof. Due to the assumption on $X$, $I(f|A)$ is not empty. To prove that $I(f|A)$ is a singleton, assume there are two distinct points $x_1, x_2 \in I(f|A)$. Fix an $\varepsilon > 0$. Denote $B = \Omega \setminus A$ and pick any $y \in I(f|B)$.

Now take an arbitrary partition $\Pi$ of $\Omega$. Let $\Pi'$ be a partition that is finer than $\Pi$ and $\{A, B\}$. Then every member set of $\Pi'$ is either a subset of $A$ or a subset of $B$. Therefore we can consider the partition $\Pi^A$ of $A$ formed by those members of $\Pi'$ that lie within $A$, and a partition $\Pi^B$ of $B$ formed by those members of $\Pi'$ that lie within $B$.

Since $x_1 \in I(f|A)$, there exists a partition $\Pi_1^A \succ \Pi^A$ and a set of sampling points $T_1^A$ such that $\|S(f|_{A}, \Pi_1^A, T_1^A) - x_1\| < \varepsilon/2$. Since also $x_2 \in I(f|A)$, there exists a partition $\Pi_2^A \succ \Pi^A$ and a set of sampling points $T_2^A$ such that $\|S(f|_{A}, \Pi_2^A, T_2^A) - x_2\| < \varepsilon/2$. And since $y \in I(f|B)$, there exists a partition $\Pi_1^B \succ \Pi^B$ and a set of sampling points $T_1^B$ such that $\|S(f|_{B}, \Pi_1^B, T_1^B) - y\| < \varepsilon/2$. Let us combine the partitions $\Pi_1^A$ and $\Pi_1^B$ into one partition $\Pi_1$ of the entire $\Omega$, and put $T_1 = T_1^A \cup T_1^B$. Then $\|S(f, \Pi_1, T_1) - (x_1 + y)\| < \varepsilon$. At the same time combine the partitions $\Pi_1^A$ and $\Pi_1^B$ into a partition $\Pi_2$ of the entire $\Omega$, and put $T_2 = T_2^A \cup T_2^B$. Then $\|S(f, \Pi_2, T_2) - (x_2 + y)\| < \varepsilon$. Since $\Pi_1 \succ \Pi$ and $\Pi_2 \succ \Pi$, both $x_1 + y$ and $x_2 + y$ belong to $I(f)$, which is impossible. Hence, $I(f|A)$ consists of a single point. \hfill $\square$

Lemma 2.3.22. Let $X$ and $f$ be as in Theorem 2.3.20. Then $f$ is weakly measurable.

Proof. Assume the contrary. Then there exists a functional $x^* \in X^*$ such that $x^*f$ is a non-measurable function. Since $f \in \overline{L_1(\mu, X)}$, $x^*f$ must have an integrable (and hence measurable) majorant. Therefore, there exists the smallest measurable majorant $f_2$ of $x^*f$ and the largest measurable minorant $f_1$ of $x^*f$. In other words, if $g: \Omega \to \mathbb{R}$ is measurable and $x^*f \leq g$ a.e., then $f_2 \leq g$ a.e., and similarly, if $g: \Omega \to \mathbb{R}$ is measurable and $x^*f \geq g$ a.e., then $f_1 \geq g$ a.e. Note that $f_1$ and $f_2$ cannot coincide almost everywhere, since that would mean $f_1 = x^*f = f_2$ a.e. and $x^*f$ would be measurable. Note that $\{t: f_1(t) \neq f_2(t)\} = \bigcup_{n=1}^{\infty}\{t: f_2(t) - f_1(t) > 1/n\}$. Since the set at the left-hand side is non-negligible, one of the sets on the right-hand side must be non-negligible too. Therefore, there exists a non-negligible measurable set $A$ and an $\varepsilon > 0$ such that $f_1(t) < f_2(t) - \varepsilon$ for any $t \in A$. 87
Consider $f|_A$. Due to Lemma 2.3.21, $I(f|_A)$ consists of a single point. Consider the following two sets:

$$A_1 = \{ t \in A : x^* f(t) > \frac{2}{3} f_2(t) + \frac{1}{3} f_1(t) \}$$

$$A_2 = \{ t \in A : x^* f(t) < \frac{2}{3} f_1(t) + \frac{1}{3} f_2(t) \}$$

and let $B_1 = A \setminus A_1$, $B_2 = A \setminus A_2$. Note that $B_1$ cannot contain any measurable non-negligible set. Indeed, assume that $C$ is a measurable set, $\mu(C) > 0$ and $C \subset B_1$. This means that for any $t \in C$, $x^* f(t) \leq \frac{2}{3} f_2(t) + \frac{1}{3} f_1(t) < f_2(t)$.

Now consider function $g$, which is equal to $\frac{2}{3} f_2(t) + \frac{1}{3} f_1(t)$ for $t \in C$ and coincides with $f_2$ outside of $C$. This function is measurable; it is a majorant of $x^* f$, but $g < f$ on a non-negligible set $C$. This contradicts the definition of $f_2$ as the smallest measurable majorant of $x^* f$. Thus we have shown that $B_1$ contains no measurable non-negligible subset, which means that $\mu_*(B_1) = 0$ and $\mu^*(A_1) = \mu(A)$. It is easy to apply the same argument to $B_2$ and $A_2$ to show that $\mu^*(A_2) = \mu(A)$.

Consider a $\sigma$-field $\Sigma_{A_1}$ of subsets of $A_1$ of the form $C \cap A_1$, where $C \in \Sigma$. Define $\mu|_{A_1}(C \cap A_1) = \mu(C \cap A)$. So, we obtain a measure space $(A_1, \Sigma_{A_1}, \mu|_{A_1})$. It is easy to verify that this space is correctly defined, since $\mu^*(A_1) = \mu(A)$. The restriction $f|_{A_1}$ is a function from this measure space to the Banach space $X$. Due to the properties of $X$, there exists an $x_1 \in I(f|_{A_1})$. By an analogous argument we can construct the measure space $(A_2, \Sigma_{A_2}, \mu|_{A_2})$ and find a point $x_2 \in I(f|_{A_2})$.

Let us show that $x_1 \neq x_2$. Indeed, note that $x^*(x_1) \in I(x^* f|_{A_1})$ and $x^*(x_2) \in I(x^* f|_{A_2})$. Consider an integral sum of $x^* f|_{A_1}$. It has the form $\sum x^* f(t_i) \mu(\Delta_i \cap A_1) = \sum x^* f(t_i) \mu(\Delta_i)$, where $\{\Delta_i\}$ is a partition of $A$ and $t_i \in \Delta_i \cap A_1$. Thus all integral sums of $x^* f|_{A_1}$ dominate the integral sums of the function $\frac{2}{3} f_2(t) + \frac{1}{3} f_1(t)$ over $A$. On the other hand, the same argument shows that all integral sums of $x^* f|_{A_2}$ are dominated by the integral sums of the function $\frac{2}{3} f_1(t) + \frac{1}{3} f_2(t)$ over $A$. Since the values if these two functions differ by at least $\varepsilon/3$ at all points of $A$, this implies that $x^*(x_1) > x^*(x_2)$, which means that $x_1 \neq x_2$.

Let us show that $x_1, x_2 \in I(f|_A)$. Indeed, since $\mu^*(A_1) = \mu^*(A_2) = \mu(A)$, any integral sum over $A_1$ of the form $\sum f(t_i) \mu(\Delta_i \cap A_1)$ is equal to the integral sum $\sum f(t_i) \mu(\Delta_i)$ over $A$, and the same is true for $A_2$. So, we have found two different points $x_1$ and $x_2$ in $I(f|_A)$, which is impossible. This contradiction
proves that $f$ is weakly measurable. \qed

Proof of Theorem 2.3.20.

The previous lemma shows that the function $f$ is weakly measurable. Take any measurable $A \subset \Omega$. Take an $x^* \in X^*$. The real-valued function $x^* f|_A$ is measurable. Since $f$ has an integrable majorant, so does $x^* f|_A$. Therefore, $x^* f|_A$ is Lebesgue-integrable and hence, $I(x^* f|_A)$ consists of a single point, $\int_A x^* f d\mu$. Let $x_A$ be the only point of $I(f|_A)$ whose uniqueness is shown in Lemma 2.3.21. Since $x^*(x_A) \in I(x^* f|_A)$, we have $x^*(x_A) = \int_A x^* f d\mu$, and this holds for any functional $x^* \in X^*$ and any subset $A \in \Sigma$. So, $f$ is Pettis integrable and $I(f) = \{P - \int_\Omega f d\mu\}$. \qed

89
Chapter 3

Hyperbolic semigroups and Fourier multipliers

A parametrized collection of bounded operators $\mathbf{T} = (T_t)_{t \geq 0}$ on a Banach space is called a strongly continuous semigroup if the following three conditions are satisfied:

1) $T_0 = I$, the identity operator;

2) $T_s \cdot T_t = T_{s+t}$;

3) $\lim_{t \to 0} T_t x = x$, for all $x \in X$.

For every strongly continuous semigroup $\mathbf{T}$ there is a dense subspace $X' \subset X$ such that $\lim_{t \to 0} \frac{T_t x - x}{t}$ exists for all $x \in X'$. This limit defines a closed linear operator $A$ called the (infinitesimal) generator of $\mathbf{T}$. We denote $D(A) := X'$ and sometimes $e^{tA} := T_t$.

We refer the reader to textbooks [27, 32, 60] for the proofs of all basic facts we will use in this chapter.

3.1 Introduction

Suppose $X$ is a complex Banach space and $\mathbf{T} = (T_t)_{t \geq 0}$ is a strongly continuous semigroup of operators on $X$. Let $A$ denote its infinitesimal generator.

An autonomous version of a well-known result that goes back to O. Perron says the following: a homogeneous differential equation $\dot{u} = Au$ admits exponential dichotomy on $\mathbb{R}$ if and only if the inhomogeneous equation $\dot{u} = Au + f$
has a unique mild solution \( u \in F(\mathbb{R}; X) \) for each \( f \in F(\mathbb{R}; X) \), see [20] or [56], and the literature therein. Here \( F(\mathbb{R}; X) \) is a space of \( X \)-valued functions, for instance, \( F(\mathbb{R}; X) = L^p(\mathbb{R}; X) \), \( 1 \leq p < \infty \). The exponential dichotomy for \( \dot{u} = Au \) means that the semigroup generated by \( A \) is hyperbolic, that is, condition \( \sigma(T_t) \cap \{|z| = 1\} = \emptyset \), \( t \neq 0 \), holds for the spectrum \( \sigma(\cdot) \).

Passing, formally, to the Fourier transforms in the equation \( \dot{u} = Au + f \) we have that the solution \( u \) is given by \( u = Mf \), where \( M: f \mapsto [R(i\cdot, A)f]^\vee \), \( R(\lambda, A) \) is the resolvent operator, and \( \land, \lor \) are the Fourier transforms. Thus, heuristically, the above-mentioned Perron-type theorem could be reformulated to state that the hyperbolicity of the semigroup is equivalent to the fact that the function \( s \mapsto R(is, A) \) is a Fourier multiplier on \( L^p(\mathbb{R}; X) \), \( 1 \leq p < \infty \), see, e.g., [3, 36] for the definition of Fourier multipliers. One of the objectives of the current work is to systematically study the connections of hyperbolicity and \( L^p \)-Fourier multiplier properties of the resolvent.

The use of Fourier multipliers for stability and hyperbolicity for strongly continuous semigroups has a fairly long history. To put our work in this context, we briefly review relevant results. Probably, the first Fourier multiplier type result was obtained in the important paper [40] by M. Kaashoek and S. Verduyn Lunel. These authors used scalar functions ("matrix elements" of the resolvent) defined by

\[
r_\rho(s, x, x^*) = \langle x^*, R(\rho + is, A)x \rangle, \quad \rho \in \mathbb{R}, \quad s \in \mathbb{R}, \quad x \in X, \quad x \in X^*.
\]

They proved that \( T \) is hyperbolic if and only if the following two conditions hold:

(i) \( |\langle r_\rho, \Phi \rangle| \leq K \|x\|\|x^*\|\|\Phi\|_{L^1} \) for some \( K > 0 \), \( \rho_0 > 0 \) and all \( \rho \) with \( |\rho| < \rho_0 \) and all \( \Phi \in S \), the Schwartz class of scalar functions on \( \mathbb{R} \);

(ii) the Césaro integral

\[
G_0x = \frac{1}{2\pi} (C, 1) \int_\mathbb{R} R(is, A)x ds = \frac{1}{2\pi} \lim_{N \to \infty} \frac{1}{N} \int_0^N \int_{-l}^l R(is, A)x ds dl
\]

converges for all \( x \in X \).
Remark, that one of the results of the current work (Theorem 3.2.7) shows that condition (ii), in fact, follows from (i).

L. Weis in [82] used Fourier multiplier properties of the resolvent on Besov spaces to give an alternative proof of the fact that the supremum $\omega_\alpha(T)$ of the growth bounds of “$\alpha$–smooth” solutions $T_t x$ are majorated by the boundedness abscissa $s_0(A)$ of the resolvent. Remark, that in Section 3.2 of the present work we derive a formula (Theorem 3.3.6) for $\omega_\alpha(T)$ in terms of Fourier multipliers on $L_p$. Moreover, in subsequent sections we use Fourier multipliers to study an analogue of dichotomy (hyperbolicity) for the smooth solutions.

A similar formula for $\omega_0(T)$ in terms of the resolvent of the generator was obtained in [19], see also [54] and formula (5.20) in [18]. Formally, Fourier multipliers have not been used in [54] and [19]. The hyperbolicity of $T$ was characterized in [54] and [19], see also [18], in terms of the invertibility of generator $\Gamma$ of the evolution semigroup $\{E^t\}$ defined on $L_p(\mathbb{R}; X)$ as $(E^t f)(\tau) = e^{tA} f(\tau - t)$. However, a simple calculation (see Remark 2.2 below) shows that $\Gamma^{-1} = -M$. Thus, formula (5.20) in [18] for the growth bound of $T$ is, in fact, a Fourier multiplier result that is generalized in Theorem 3.3.6 below.

Via completely different approach based on an explicit use of Fourier multipliers, M. Hieber [37] gave a characterization of uniform stability for $T$ in terms of Fourier multiplier properties of the resolvent. Also, he proved a formula for $\omega_0(T)$ that is contained in Theorem 3.3.6 when $\alpha = 0$. An important ingredient of his proof was the use of well-known Datko-van Neerven Theorem saying that $T$ is uniformly stable if and only if the convolution with $T$ is a bounded operator on $L_p(\mathbb{R}; X)$, [81]. Since the resolvent is the Fourier transform of $T$, the Fourier multipliers characterization of uniform stability follows.

Among other things, this result with a different proof was given in [53], where Datko-van Neerven Theorem was also used. In fact, Theorem 3.3.6 was proved in [53] for $\alpha = 0$ or $\alpha = 1$. Also, a spectral mapping theorem from [54] was explained in [53] using Fourier multipliers instead of evolution semigroups. In addition, a particular case of Theorem 3.4.1 of the current paper (with a different proof) was established in [53]. Thus, in the present work we use new technique to “tie the ends”, and give a universal treatment for the results in [40, 19, 37, 53] in a more general context.

We also mention that Theorem 3.2.7 has been recently used by A. Bátkai, E. Fašanga and the author, [6], to characterize the hyperbolicity condition
in the context of delay equations.

## 3.2 Characterization of hyperbolicity

Let us fix some notation:

- \( T = (T_t)_{t \geq 0} \) is a strongly continuous semigroup on a Banach space \( X \) with the generator \( A \);
- \( \mathcal{L}(X) \) – the set of bounded linear operators on \( X \);
- \( \omega_0 = \omega_0(T) \) denotes the growth bound of \( T \), i.e. \( \omega_0(T) = \inf \{ \omega \in \mathbb{R} : \| T_t \| \leq M \omega e^{\omega t} \} \);
- \( s_0(A) \) denotes the abscissa of uniform boundedness of the resolvent, i.e. \( s_0(A) = \inf \{ s \in \mathbb{R} : \sup \{ \| R(\lambda) \| : \Re \lambda > s \} < \infty \} \);
- \( \hat{f}(t) = \int f(s) e^{-ist}ds \); \( \check{f}(t) = \frac{1}{2\pi} \int f(s) e^{ist}ds \); then \( (\hat{f})^\vee = f \) and \( \langle \hat{f}, g \rangle = \langle f, \check{g} \rangle \);
- \( S \) stands for the class of Schwartz functions;
- \( \langle r, \Phi \rangle \) denotes the value of a distribution \( r \) on \( \Phi \in S \).

\( \text{mes} \) stands for the Lebesgue measure on the line \( \mathbb{R} \); \( L_p(\mathbb{R}, X) \) denotes the space of \( X \)-valued strongly measurable \( L_p \)-integrable functions;

\( \| \cdot \|_p \) will denote the norm in the scalar \( L_p \) over \( \mathbb{R} \).

**Definition 3.2.1.** We say that the semigroup \( T \) is *hyperbolic* if there is a bounded projection \( P \) on \( X \), called splitting, such that \( PT_t = T_tP \) for all \( t > 0 \) and there exist positive numbers \( \omega \) and \( M \) such that

1. \( \| T_t x \| \leq Ke^{-\omega t}\| x \| \), for all \( t > 0 \) and \( x \in \text{Im} P \),
2. \( \| T_t x \| \leq Ke^{\omega t}\| x \| \), for all \( t < 0 \) and \( x \in \text{Ker} P \).

The semigroup \( T \) is called uniformly exponentially stable if \( P = I \).
In other words, conditions 1 and 2 say that \((T_t)_{t \geq 0}\) is uniformly exponentially stable on \(\text{Im } P\); all the \(T_t\)'s are invertible on \(\text{Ker } P\) and the semigroup \((T_{-t})_{t \geq 0}\) is uniformly exponentially stable there.

**Definition 3.2.2.** The function
\[
G(t) = \begin{cases} 
T_t P, & t > 0 \\
-T_t (I - P), & t < 0 
\end{cases}
\]
is called the Green’s function corresponding to the hyperbolic semigroup \(T\).

Definition 3.2.1 allows an equivalent reformulation in terms of spectral properties of \(T\). Namely, \(T\) is hyperbolic if and only if the unit circle \(\mathbb{T}\) lies in the resolvent set of \(T_t\) for one/all \(t\) (see [27, Proposition V.1.15]).

Let us recall the following inversion result.

**Lemma 3.2.3.** Suppose \(\rho > s_0(A)\) and \(x \in X\), then
\[
F_t(x) = \frac{1}{2\pi i} (C, 1) \int_{\text{Re } \lambda = \rho} e^{\lambda t} R(\lambda) x d\lambda, \quad t \in \mathbb{R},
\]
where \(F_t\) is defined as
\[
F_t(x) = \begin{cases} 
T_t x, & t > 0 \\
\frac{1}{2} x, & t = 0 \\
0, & t < 0 
\end{cases}
\]
In particular, \(\tilde{r}_\rho(t, x, x^*) = e^{-\rho t} (x^*, F_t(x))\).

The proof can be found in [81, Theorem 1.3.3]. See also Corollary 3.3.5.

Below we establish some algebraic properties of the distributions \(\tilde{r}_\rho\) for small \(|\rho|\) without any additional assumptions on \(s_0(A)\). They trivialize in the case \(s_0(A) < 0\), when according to Lemma 3.2.3 \(\tilde{r}_\rho\) becomes the scalar element of the semigroup.

In order to be able to threat \(r_\rho\)'s as distributions and to justify some computations, we assume that the function \(s \mapsto \|R(is)\|\) is bounded on \(\mathbb{R}\), although the proofs below merely require that this function grows not faster than a power of \(|s|\).

**Lemma 3.2.4.** If \(\tau > 0\), then
\[
\tilde{r}_0(t - \tau, T_\tau x, x^*) = \tilde{r}_0(t, x, x^*) - \langle x^*, T_t x \rangle \chi_{[0, \tau]}(t), \quad t \in \mathbb{R}.
\]
Proof. Let us take arbitrary \( \Phi \in \mathcal{S} \). Then

\[
\langle \tilde{r}_0(\cdot - \tau, T_\tau x, x^*) \rangle \Phi = \langle \tilde{r}_0(\cdot, T_\tau x, x^*), \Phi(\cdot + \tau) \rangle \\
= \langle \tilde{r}_0(\cdot, T_\tau x, x^*), e^{-is\tau} \Phi \rangle = \int_{-\infty}^{+\infty} \langle x^*, R(is)e^{-is\tau}T_\tau x \rangle \tilde{\Phi}(s) ds.
\]

Note that

\[
e^{-is\tau}R(is)T_\tau x = R(is)x - \int_0^\tau T_\tau x \cdot e^{-isr} dr.
\]

(3.2.1)

Continuing the previous line of equalities, we obtain

\[
\langle \tilde{r}_0(\cdot - \tau, T_\tau x, x^*) \rangle \Phi = \int_{-\infty}^{+\infty} \langle x^*, R(is)x \rangle \tilde{\Phi}(s) ds \\
- \int_{-\infty}^{+\infty} \left\langle x^*, \int_0^\tau T_\tau e^{-isr} x dr \right\rangle \Phi(s) ds = \langle \tilde{r}_0, \Phi \rangle \\
- \int_0^\tau \langle x^*, T_\tau x \rangle \Phi(r) dr = \langle \tilde{r}_0 - \langle x^*, T_\tau x \rangle \chi_{[0,\tau]}(\cdot), \Phi \rangle.
\]

Lemma 3.2.5. \( \tilde{r}_0(t, T_\tau x, x^*) = \tilde{r}_0(t, x, T_\tau^* x^*) \), \( \tau > 0 \), \( t \in \mathbb{R} \).

The proof is obvious.

Lemma 3.2.6. \( \tilde{r}_0(t) = e^{\rho t} \tilde{r}_\rho(t) \) for all \( \rho \) with \( |\rho| < \rho_0 \) and \( t \in \mathbb{R} \).

Proof. We choose \( \rho_0 \) such that \( \sup \{ \| R(is + \rho) \| : s \in \mathbb{R}, |\rho| < \rho_0 \} \) is finite. Suppose \( \Phi \in \mathcal{S} \) has compact support. Then \( \tilde{\Phi} \) is an entire function. Moreover,

\[
\lim_{\alpha \to -\infty} \tilde{\Phi}(\alpha + i\beta) = 0,
\]

(3.2.2)

uniformly for all \( \beta \) from some finite interval \([a, b]\). It is an immediate consequence of the following equality:

\[
\tilde{\Phi}(\alpha + i\beta) = \int_{\mathbb{R}} e^{i\beta x} \Phi(x)e^{i\alpha x} dx = -\frac{1}{i\alpha} \int_{\mathbb{R}} [\beta e^{i\beta x} \Phi(x) + e^{i\beta x} \Phi'(x)] e^{i\alpha x} dx.
\]

95
Now using Cauchy’s theorem and (3.2.2) we get

\[ \langle \tilde{r}_\rho, \Phi \rangle = \langle r_\rho, \Phi \rangle = \int_{\mathbb{R}} \langle x^*, R(is + \rho)x \rangle \Phi(s)ds = \int_{\mathbb{R}} \langle x^*, R(i\lambda)x \rangle \Phi(\lambda + i\rho)d\lambda \]

\[ = \int_{\mathbb{R}} \langle x^*, R(is)x \rangle \Phi(s + i\rho)ds + 2i \lim_{\alpha \to \pm\infty} \int_{0}^{\rho} \langle x^*, R(i(\alpha + i\beta))x \rangle \Phi(\alpha + i(\beta + \rho))d\beta \]

\[ = \int_{\mathbb{R}} \langle x^*, R(is)x \rangle \Phi(s + i\rho)ds = \langle r_0, \Phi(\cdot + i\rho) \rangle = \langle \tilde{r}_0, e^{-\rho \cdot} \Phi \rangle, \]

and the result follows. \( \square \)

Now we are in a position to prove our main theorem. Let us denote by \( M_\rho \) the operator acting by the rule

\[ M_\rho: f \mapsto [R(i \cdot + \rho)f]^\vee. \]

Recall that a function \( m \in L_\infty(\mathbb{R}; L(X)) \) is called a Fourier multiplier on \( L_p(\mathbb{R}; X) \) if the operator \( M: f \mapsto [m(\cdot)f]^\vee \) is bounded on \( L_p(\mathbb{R}; X) \). Let \( L_{1,\infty}(\mathbb{R}; X) \) denote the weak-\( L_1 \) space with values in \( X \) (see, e.g., [79, 1.18.6]), that is, the set of all \( X \)-valued strongly continuous functions \( f \) with the finite norm

\[ \|f\|_{1,\infty} := \sup_{\sigma > 0} \left\{ \text{mes} \left( \{ s \in \mathbb{R} : \|f(s)\| \geq \sigma \} \right) < \infty \right\}. \]

Note that \( L_1(\mathbb{R}; X) \subset L_{1,\infty}(\mathbb{R}; X) \).

**Theorem 3.2.7.** For a strongly continuous semigroup \( T \) on \( X \) the following conditions are equivalent:

1) \( T \) is hyperbolic;

2) \( R(i \cdot) \) is a Fourier multiplier on \( L_p(\mathbb{R}, X) \) for some/all \( p, 1 \leq p < \infty \);

3) There exists a \( \rho_0 > 0 \) such that for all \( \rho \) with \( |\rho| < \rho_0 \), \( M_\rho \) maps \( L_1(\mathbb{R}, X) \) into \( L_{1,\infty}(\mathbb{R}, X) \);
4) There exists a \( \rho_0 > 0 \) such that for all \( \rho \) with \(|\rho| < \rho_0\) and all \( \Phi \in \mathcal{S} \) we have \( |\langle r_\rho, \Phi \rangle| \leq K_\rho \|x\|\|x^*\|\|\tilde{\Phi}\|_1 \).

Furthermore, if one of these properties holds, then for every \( t \in \mathbb{R} \) and \( x \in X \) the integral 

\[
G(t)x = \frac{1}{2\pi}(C, 1) \int_{\mathbb{R}} R(is)xe^{ist}ds
\]

converges and represents the Green’s function of \( T \). Moreover, \( M_0 f = G \ast f \) for \( f \in L_1(\mathbb{R}, X) \), and the splitting projection is given by the formula

\[
P = \frac{1}{2}I + G(0).
\]

**Proof.** 1) \( \Rightarrow \) 4). This is a part of Theorem 0.2 from [40], however we reproduce the proof for convenience of the reader.

By rescaling it is enough to prove 4) only for \( \rho = 0 \). To this end, let us denote \( X^s = \text{Im} P \) and \( X^u = \text{Ker} P \), where \( P \) as in Definition 3.2.1. If \( x \in X^s \), then by Lemma 3.2.3, \( \tilde{r}_0(t, x, x^*) \) becomes \( \langle F_t(x), x^* \rangle \), and therefore, \( \|\tilde{r}_0\|_1 \leq K \|x\|\|x^*\| \). Thus, we get

\[
|\langle r_\rho, \Phi \rangle| = |\langle \tilde{r}_0, \tilde{\Phi} \rangle| \leq K \|x\|\|x^*\|\|\tilde{\Phi}\|_1.
\]

Similarly, same inequality holds for \( x \in X^u \), and we are done.

4) \( \Rightarrow \) 2). It follows from 4) that \( \tilde{r}_\rho \in L_\infty, |\rho| < \rho_0 \) and \( \|\tilde{r}_\rho\|_\infty \leq K_\rho \|x\|\|x^*\| \).

By Lemma 3.2.6, \( \tilde{r}_0(t) = e^{-\rho t} \tilde{\tau}_\rho(t) \) a.e. and \( \tilde{r}_0(t) = e^{\rho t} \tilde{\tau}_\rho(t) \) a.e. for some \( \rho > 0 \). So, \( |\tilde{r}_0(t)| \leq e^{-\rho |t|} K \|x\|\|x^*\| \) a.e. for every \( x \in X \) and \( x^* \in X^* \), where \( K = \max \{K_\rho, K_{-\rho}\} \). Now let us fix \( p, 1 \leq p < \infty \), and consider a function \( \Phi = \sum_{k=1}^n x_k \otimes \Phi_k \), where \( \Phi_k \in \mathcal{S} \) and \( \{\Phi_k\} \) have disjoint supports. Then \( \|\Phi\|_p = \sum_{k=1}^n \|x_k\|^p \|\Phi_k\|_p \). So, we get the following estimates:

\[
\|M_0(\Phi)\|_p^p = \left\| \int_{\mathbb{R}} M_0(\Phi)(t) e^{ist}ds \right\|_p^p = \frac{1}{2\pi} \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} R(is)\tilde{\Phi}(s)e^{ist}ds \right\|_p^p dt
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \sup_{\|x^*\| \leq 1} \left\| \sum_{k=1}^n \int_{\mathbb{R}} r_0(s, x_k, x^*) \Phi_k(s)e^{ist}ds \right\|_p^p dt
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \sup_{\|x^*\| \leq 1} \left\| \sum_{k=1}^n \int_{\mathbb{R}} \tilde{r}_0(t, x_k, x^*) \Phi_k(t - \tau)d\tau \right\|_p^p dt
\]

\[
\leq K^p \int_{\mathbb{R}} \left( \sum_{k=1}^n \int_{\mathbb{R}} e^{-\rho |\tau|} \|x_k\|\|\Phi_k(t - \tau)\|d\tau \right)^p dt
\]

97
\begin{align*}
  &= K^p \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-\rho|\tau|} \left( \sum_{k=1}^{n} \|x_k\| |\Phi_k(t-\tau)| \right) d\tau \right)^p dt \\
  &\leq C_\rho K^p \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\rho|\tau|} \left( \sum_{k=1}^{n} \|x_k\| |\Phi_k(t-\tau)| \right)^p d\tau dt \\
  &= C_\rho K^p \int_{\mathbb{R}} \left( \sum_{k=1}^{n} \|x_k\| |\Phi_k(t-\tau)| \right)^p d\tau dt \\
  &= C_\rho K^p \int_{\mathbb{R}} \left( \sum_{k=1}^{n} \|x_k\|^p \|\Phi_k\|^p \right) d\tau \\
  = C_\rho K^p \int_{\mathbb{R}} \left( \sum_{k=1}^{n} \|x_k\|^p \|\Phi_k\|^p \right) d\tau = C_\rho K^p \|\Phi\|^p.
\end{align*}

Since the functions \( \Phi \) are dense in \( L_p(\mathbb{R}, X) \), the proof of \( 4) \Rightarrow 2) \) is finished.

\( 2) \Rightarrow 1) \). Suppose \( 2) \) holds for some \( p, 1 \leq p < \infty \). Then, by the
transference principle \( \) (see, for example, [77, Thm. VII.3.8]), \( \{ R(ik + i\xi) \}_{k \in \mathbb{Z}} \)
\( \) is a multiplier in \( L_p(T, X) \) for all \( \xi \in \mathbb{R} \), where \( T \) is the unit circle. So, using
\[ [54, Theorem 2.3] \text{ or } [53, Theorem 1], \] we conclude that \( e^{2\pi i \xi} \in \rho(T_{2\pi}) \) for all
\( \xi \in \mathbb{R}. \) Thus, \( T \subset \rho(T_{2\pi}) \) and hence, \( T \) is hyperbolic.

This completes the proof of \( 1) \Leftrightarrow 2) \Leftrightarrow 4). \)

\( 2) \Rightarrow 3). \) It is easy to see using the resolvent identity, that there exists a
\( \rho_0 > 0 \) such that \( R(i \cdot + \rho) \) is a \( L_1(\mathbb{R}, X) \)-multiplier for all \( \rho \) such that \( |\rho| < \rho_0. \)

\( 3) \Rightarrow 4). \) Without loss of generality, assume \( \rho = 0. \) Denote
\[ \mu = \sup_{0 \leq \tau \leq 1} \|T_\tau\| \]
and fix \( x \in X, x^* \in X^* \), \( \|x\| = \|x^*\| = 1 \). Let us take a function \( \Phi \in S. \) By
condition \( 3) \) \( \) we have
\[ \|M_0(\Phi \otimes x)\|_{1, \infty} \leq K\|\hat{\Phi}\|_1. \]

So, \( \text{mes}\{ \tau : \|M_0(\Phi \otimes x)(\tau)\| > 2K\|\hat{\Phi}\|_1 \} \leq \frac{1}{2}. \) This implies that there is a \( \tau, \)
\( -1 < \tau < 0, \) such that
\[ \|M_0(\Phi \otimes x)(\tau)\| \leq 2K\|\hat{\Phi}\|_1. \]

Let us apply the functional \( T^* x^* \) to the left-hand side of the inequality.
Then we have:
\[ \left| \frac{1}{2\pi} < T^* x^*, [R \cdot \Phi \otimes x]^{\vee}(\tau) > \right| \leq 2\mu KC\|\hat{\Phi}\|_1. \]
By Lemma 3.2.4 and 3.2.5 the expression under the absolute value sign is equal to
\[
\tilde{r}_0(\cdot, x, T^* x^*) \ast \Phi(\tau) = \langle \tilde{r}_0(\cdot, T^* x^* \cdot), \Phi(\tau - \cdot) \rangle
\]
\[
= \langle \tilde{r}_0(\cdot + \tau, T^* x, x^*), \Phi \rangle = \langle \tilde{r}_0(\cdot, x, x^*), \Phi \rangle - \int_0^\tau \langle x^*, T_t x \rangle \Phi(t) dt.
\]
By the triangle inequality, we have
\[
|\langle \tilde{r}_0(\cdot, x, x^*), \Phi \rangle| \leq 2K\mu \|\hat{\Phi}\|_1 + \mu \|\hat{\Phi}\|_1 \leq 3K\mu \|\hat{\Phi}\|_1,
\]
which is what we wanted.

Now we turn to the second part of the theorem. First, we prove an auxiliary well-known Fejér-type lemma.

**Lemma 3.2.8.** If \( f \in L_1(\mathbb{R}, X) \), then the integral
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(s) e^{ist} ds
\]
converges to \( f(t) \) a.e. Moreover,
\[
f = \frac{1}{2\pi} L_1 - \lim_{N \to \infty} \frac{1}{N} \int_0^N \int_{-\ell}^{\ell} \hat{f}(s) e^{isr} ds dr.
\]

**Proof.**
\[
\frac{1}{2\pi} \frac{1}{N} \int_{-\ell}^{\ell} \int_0^N \hat{f}(s) e^{ist} ds = \frac{1}{2\pi} \int_{-\ell}^{\ell} \hat{f}(s) e^{ist} \left( 1 - \frac{|s|}{N} \right) ds
\]
\[
= \frac{1}{2\pi} \int_{-\ell}^{\ell} \int_{-\infty}^{+\infty} f(r) e^{isr} dr \cdot e^{ist} \left( 1 - \frac{|s|}{N} \right) ds
\]
\[
= \int_{-\infty}^{+\infty} f(r) \frac{1}{2\pi} \int_{-\infty}^{N} e^{is(t-r)} \left( 1 - \frac{|s|}{N} \right) ds dr.
\]
The inner integral is equal to \( K_N(t-r) = \frac{1}{\pi N(t-r)^2} \left[ 1 - \cos N(t-r) \right] \). One can easily check that \( K_N \) is a positive kernel in \( L_1 \), that is, \((K_N \ast f)(\cdot)\) tends to \( f(\cdot) \) a.e. and in \( L_1 \) as \( N \to \infty \).
Suppose $f \in L_1(\mathbb{R})$. Then by 2) we have that $M_0(f \otimes x) \in L_1(\mathbb{R}, X)$. By Lemma 3.2.8, there is a $\tau \in (-1, 0)$ such that

$$M_0(f \otimes x)(\tau) = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)x \hat{f}(s)e^{irs} ds.$$ 

Let us apply the operator $T_{-\tau}$. Then using (3.2.1) we obtain:

$$T_{-\tau}([R\hat{f} \otimes x]^\vee(\tau)) = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)T_{-\tau}xe^{irs} \hat{f}(s) ds$$

$$= \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} \left[ R(is)x \hat{f}(s) - \int_{-\tau}^{\tau} T_{-r}xe^{-isr} dr \cdot \hat{f}(s) \right] ds.$$

Since $f(-r) = L_1 - \lim_{N \to \infty} \frac{1}{2\pi} \frac{1}{N} \int_{-N}^{N} \int_{-\ell}^{\ell} \hat{f}(s)e^{-isr} ds d\ell$
and $V \varphi = \int_{0}^{\tau} T_{-r} \varphi(r) dr$ is a bounded linear operator from $L_1(\mathbb{R})$ to $X$, we conclude that the $(C, 1)$-integral of the second summand converges and equals $\frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)x \hat{f}(s) ds$ converges. Let us denote it by $G(0, f)$. Also let

$$G(t, f) = G(0, f(-t)) = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)x \hat{f}(s)e^{ist} ds$$
for $f \in L_1$, $t \in \mathbb{R}$, $x \in X$. Now we introduce the following operators:

$$S_{N}^{t}(f, x) = \frac{1}{2\pi N} \int_{0}^{N} \int_{-\ell}^{\ell} R(is)x \hat{f}(s)e^{ist} ds d\ell;$$

$$I_{N}^{t}(x) = \frac{1}{2\pi N} \int_{0}^{N} \int_{-\ell}^{\ell} R(is)x e^{ist} ds d\ell. \quad \text{(3.2.4)}$$

It is easy to see that $\|S_{N}^{t}(f, x)\| \leq C_N \|f\|_1 \|x\|$. On the other hand, we have just proved that $G(t, f)x = \lim_{N \to \infty} S_{N}^{t}(f, x)$ exists for all $f \in L_1$, $x \in X$. So, by the boundedness principle for bilinear operators, $\|S_{N}^{t}\| \leq C$, where $C$ does not depend on $N$ and $t$.

Let $f_\epsilon$, $\epsilon > 0$, be a kernel in $L_1(\mathbb{R})$, that is, $f_\epsilon \ast \Phi \to \Phi$ as $\epsilon \to 0$ for each $\Phi \in L_1(\mathbb{R})$. Then $I_{N}^{t}(x) = \lim_{\epsilon \to 0} S_{N}^{t}(f_\epsilon, x)$ and hence, $\|I_{N}^{t}\| \leq C$. 

100
Let us show that \( G(t)x = \lim_{N \to \infty} I_N^t(x) \) exists for all \( x \in D(A^2) \). This will be enough to prove that \( G(t)x \) exists for all \( x \in X \). Fix \( x \in D(A^2) \) and notice that

\[
I_N^t(x) = \frac{1}{2\pi} \int_0^\ell R(is)x e^{ist} \, ds \, d\ell
\]

\[
= \frac{1}{2\pi} \int_0^\ell R(is)x e^{ist} \, ds \, d\ell + \frac{1}{2\pi} \int_{1 \leq |s| \leq \ell} R(is)x e^{ist} \, ds \\
+ \frac{1}{2\pi} \int_{1 \leq |s| \leq \ell} \left[ -\frac{R(is)A^2x}{s^2} + \frac{x}{is} - \frac{Ax}{s^2} \right] e^{ist} \, ds \, d\ell,
\]

So,

\[
\lim_{N \to \infty} I_N^t(x) = \frac{1}{2\pi} \int_{|s| \leq 1} R(is)x e^{ist} \, ds \\
- \frac{1}{2\pi} \int_{|s| \geq 1} \left[ \frac{Ax}{s^2} + \frac{R(is)A^2x}{s^2} \right] e^{ist} \, ds \\
+ x \int_{t}^{+\infty} \frac{\sin s}{s} ds \cdot \chi_{\mathbb{R}\setminus\{0\}}(t).
\]

Finally, it is only left to verify that \( G(t) \) is indeed the Green’s function.

Let us prove the first equality in Definition 3.2.2, the second one being anal-
ogous. We have
\[
T_{\tau}P_x = \frac{1}{2} T_{\tau}x + \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)xT_{\tau}ds
\]
\[
= \frac{1}{2} T_{\tau}x + \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)xe^{ist}ds
\]
\[
- \frac{1}{2\pi} (C, 1) \int_{0}^{\tau} T_{\tau}xe^{-istr}dr \cdot e^{istr}ds
\]
\[
= G(\tau)x + \frac{1}{2} T_{\tau}x - \frac{1}{2} T_{\tau}x = G(\tau)x,
\]
where we use the ordinary Fejér’s theorem.

It follows from the above that \(G(\tau)\) is an exponentially decaying function. So, \(f \mapsto G*f\) is a bounded operator on \(L_1\). On the other hand, \(M_0\Phi = G*\Phi\) for all \(\Phi \in S\). Hence, \(M_0f = G*f\) for all \(f \in L_1\).

The proof of (3.2.3) can be found in [40].

**Corollary 3.2.9** ([30, 35]). Suppose \(X\) is a Hilbert space. Then the semigroup \(T\) is hyperbolic if and only if the resolvent \(R(\lambda, A)\) is bounded on some strip containing the imaginary axes.

**Remark 3.2.10.** Condition 3) can be considerably weakened in the following way. Suppose \(F\) is a space of functions on \(\mathbb{R}\) with the following property: for any \(f \in F\) there is a \(t \in [-1, 0]\) such that \(|f(t)| \leq c\|f\|_F\). Many quasi-normed function spaces have this property, for example, \(L_{q,r}(\mathbb{R})\), \(H_{p}(\mathbb{R})\), \(C_0(\mathbb{R})\), or any function lattice with \(\|\chi_{[-1,0]}\| \neq 0\). Denote by \(F(X)\) the space of all strongly measurable functions \(f\) with values in \(X\) such that \(\|f(\cdot)\| \in F\). Our proof shows that it is enough to require that \(M_\rho\) maps \(L_1(\mathbb{R}, X)\) into \(F(X)\) (see also the proof of Theorem 3.3.1).

**Remark 3.2.11.** Recall that the generator \(\Gamma\) of the evolution semigroup \((E_t)_{t\geq 0}\), defined on \(L_p(\mathbb{R}, X)\) by \((E_t f)(s) = T_t f(s - t)\), is the closure of the operator \(-d/dt + A\) on the domain \(D(-d/dt) \cap D(A)\). It is known, see [18, Thm.2.39], that \(T\) is hyperbolic if and only if the operator \(\Gamma\) is invertible on one/all \(L_p(\mathbb{R}, X)\), \(1 \leq p < \infty\). This result immediately implies that conditions 1) and 2) in Theorem 3.2.7 are equivalent. Indeed, if \(x \in D(A)\) and \(\Phi \in S\), then \(\Gamma(\Phi \otimes x) = -\Phi' \otimes x + \Phi \otimes Ax\). Using elementary properties of Fourier transform, we have that
\[
M_0 \Gamma(\Phi \otimes x) = -\Phi \otimes x, \quad x \in D(A)
\]
and 
\[ \Gamma M_0(\Phi \otimes x) = -\Phi \otimes x, \quad x \in X, \]
and the result follows.

It is worth noting that in the special case \( s_0(A) < 0 \), by Lemma 3.2.3, the splitting projection turns into the identity and our theorem gives the characterization of uniform exponential stability observed in [53].

There is a Mikhlin-type sufficient condition due to M. Hieber [36] for an operator-valued symbol to be \( L_1 \)-multiplier. Applied to the resolvent it yields the following: if there exists a \( \delta > \frac{3}{4} \) such that \( \sup \{ |s|^{\delta} \| R(is) \| \} < \infty \), then \( R(i \cdot) \) is a multiplier.

Yet another condition for operator-valued symbol to be a multiplier has been recently found in [83]. It works if \( X \) is a UMD-space requires the families \( \{ R(is) \}_{s \in \mathbb{R}} \) and \( \{ sR^2(is) \}_{s \in \mathbb{R}} \) to be \( R \)-bounded.

A recent application of Theorem 3.2.7 to delay equations was obtained in [6].

3.3 Extension to the case \( \alpha > 0 \)

It turns out that many arguments from Section 3.2 work in a more general situation, when the resolvent multiplier is restricted to \( L_p(\mathbb{R}, X_\alpha) \), where \( X_\alpha \) is the domain of the fractional power \( (A - \omega)\alpha \), endowed with the norm \( \| x \|_\alpha = \| (A - \omega)^\alpha x \| \). In this section we show that \( R(i \cdot + \rho) \) is a multiplier from \( L_p(\mathbb{R}, X_\alpha) \) to \( L_p(\mathbb{R}, X) \) for small values of \( \rho \) if and only if the following modified Kaashoek - Verduyn Lunel inequality holds: 
\[ |\langle r_\rho, \Phi \rangle| \leq K \| x \|_\alpha \| x^* \| \| \tilde{\Phi} \|_1. \]
Also in this case \( G(t)x \) exists for all \( x \in X_\alpha \) and is exponentially decaying as \( |t| \to \infty \). As a by-product of this results we obtain the following relationship between the fractional growth bound \( \omega_\alpha(T) \) and its spectral analogue \( s_\alpha(A) \) (see (3.3.5) for the definitions): \( \omega_\alpha(T) \) is the infimum of all \( \omega > s_\alpha(A) \) such that \( R(i \cdot + \omega) \) is a multiplier from \( L_p(\mathbb{R}, X_\alpha) \) to \( L_p(\mathbb{R}, X) \). In the particular case, when \( X \) is a Hilbert space, the latter condition will be shown to hold for all \( \omega > s_\alpha(A) \). So, \( s_\alpha(A) = \omega_\alpha \), which gives a different proof of G. Weiss’s [86] result for arbitrary \( \alpha \geq 0 \), also obtained by L. Weis and V. Wrobel in [85]. The main result in this section is an extension of Theorem 3.2.7 to the case of arbitrary \( \alpha > 0 \). To be more precise, we treat only conditions 2)-4), as hyperbolicity is ambiguous in this situation and therefore it is postponed to the next section.
One can notice that most of the proof of Theorem 3.2.7 work for all $\alpha > 0$ if one replaces all $X$-norms by $X_\alpha$-norms. However, the “some/all” part of condition 2), being an easy consequence of results in [54] and the spectral characterization of hyperbolicity in case $\alpha = 0$, requires some additional duality argument.

Before we state our main theorem, let us recall the notion of a fractional power of $A$. Suppose $\omega > \max\{\omega_0 + 3, 3\}$. Denote $A - \omega$ by $A_\omega$. Let $\gamma$ be the path consisting of two rays $\Gamma_1 = \{-1 + te^{i\theta} : t \in [0, +\infty]\}$ and $\Gamma_2 = \{-1 - te^{i\theta} : t \in [0, +\infty]\}$ directed upwards. We assume that $\theta, \theta < \frac{\pi}{6}$, is small enough to ensure the inequality $\| R(\mu + \omega) \| \leq C \frac{1}{|\mu|^2}$ in the smaller sector bordered by $\gamma$. For any $\alpha > 0$ we define $A_\omega^\alpha$ as the inverse to the operator $A_\omega^{-\alpha}$ acting on $X$ by the rule

$$A_\omega^{-\alpha}(x) = \frac{1}{2\pi i} \int_\gamma \mu^{-\alpha} R(\mu + \omega) xd\mu.$$ 

Let us denote by $X_\alpha$ the domain of $A_\omega^\alpha$ endowed with the norm $\| x \|_\alpha = \|(A - \omega)^{\alpha}x\|$. Then $X_\alpha$ is a Banach space and it does not depend on the particular choice of $\omega, \omega > \omega_0$ (see [27] for more about fractional powers of operators).

**Theorem 3.3.1.** Assume that there exists a $\rho_0 > 0$ such that

$$\sup \left\{ \frac{\| R(\lambda) \|}{1 + |\lambda|^\alpha} : |\text{Re} \lambda| < \rho_0 \right\} < \infty. \quad (3.3.1)$$

Then the following conditions are equivalent:

1) $R(i \cdot + \rho)$ is a multiplier from $L_p(\mathbb{R}, X_\alpha)$ to $L_p(\mathbb{R}, X)$, for some/all $p, 1 \leq p < \infty$, and all $\rho, |\rho| < \rho_0$;

2) $R(i \cdot + \rho)$ is a multiplier from $L_p(\mathbb{R}, X_\alpha)$ to $E(X)$, for some $p, 1 \leq p < \infty$, and all $\rho, |\rho| < \rho_0$, where $E$ is a rearrangement invariant quasi-Banach lattice;

3) $R(i \cdot + \rho)$ is a multiplier from $L_1(\mathbb{R}, X_\alpha)$ to $F(X)$ for all $\rho, |\rho| < \rho_0$, where $F$ is some rearrangement invariant quasi-Banach lattice;

4) $|\langle r_\rho, \Phi \rangle| \leq K \| x \|_\alpha \| x^* \| \| \hat{\Phi} \|_1$ for all $\Phi \in S, x \in X_\alpha, x^* \in X^*$ and $|\rho| < \rho_0$.  

104
If one of these conditions holds, then the integral

\[ G(t)x = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)xe^{ist}ds \]

converges for all \( x \in X_\alpha \), and \( \|G(t)x\| \leq K\|x\|_\alpha e^{-\rho|t|} \) for all \( \rho \), \( 0 < \rho < \rho_0 \). Moreover, \( M_0f = G * f \) for \( f \in L_1(\mathbb{R}, X_\alpha) \).

**Proof.** 1) \( \Rightarrow \) 2) is evident.

2) \( \Rightarrow \) 3). Assume for simplicity that \( \rho = 0 \). First we claim that \( M_0 \) maps \( L^p(\mathbb{R}, X_\alpha) \) into \( L^\infty(\mathbb{R}, X) \). To prove this, let us take an arbitrary function \( \Phi \) of the form \( \sum_{i=1}^n \Phi_i x_i \), where \( x_i \in X_\alpha \) and \( \Phi_i \in \mathcal{S} \). Then, by condition 2),

\[ \|M_0(\Phi)\|_{E(X)} \leq K\|\Phi\|_{L^p(\mathbb{R}, X_\alpha)}. \]

It implies that for every \( n \in \mathbb{Z} \) there exists a \( t \in [n, n+1] \) such that

\[ \|\left[R \hat{\Phi}\right]^\vee(t)\|_X \leq \frac{2K}{\varphi(1)}\|\Phi\|_{L^p(\mathbb{R}, X_\alpha)}, \]

where \( \varphi \) is the characteristic function of \( E \). For any fixed \( \tau \in [0, 2] \), let us apply the operator \( T_\tau \) to the right-hand side of this inequality. Then we get

\[ \left\| \frac{1}{2\pi} \int_{\mathbb{R}} T_\tau R(is) \hat{\Phi}(s)e^{ist}ds \right\| \leq C\|\Phi\|_{L^p(\mathbb{R}, X_\alpha)}. \]

Now using equality (3.2.1) we obtain the following

\[ \frac{1}{2\pi} \int_{\mathbb{R}} T_\tau R(is) \hat{\Phi}(s)e^{ist}ds = \frac{1}{2\pi} \int_{\mathbb{R}} R(is) \hat{\Phi}(s)e^{is(t+\tau)}ds - \frac{1}{2\pi} \int_{\mathbb{R}} \int_0^\tau T_r \hat{\Phi}(s)e^{is(\tau-r)}drds = M_0(\Phi)(t + \tau) - \frac{1}{2\pi} \int_0^\tau T_r \Phi(\tau - r)dr. \]

Thus, \( \|M_0(\Phi)(t + \tau)\|_X \leq \tilde{C}\|\Phi\|_{L^p(\mathbb{R}, X_\alpha)} \). By the choice of \( \tau \) and \( t \) we have the same inequality on the whole real line. Since \( \tau \) was chosen arbitrary, the claim is proved.

Let us observe that the boundedness of \( M_0 \) is equivalent to the fact that \( R(is)A^{-\alpha}_w \) is an \( L_p(\mathbb{R}, X) - L^\infty(\mathbb{R}, X) \) multiplier.
Denote by $X^\ominus$ the sun dual to $X$ on which the dual semigroup is strongly continuous (see [27]). One can easily check, by duality, that for a test function $\Phi = \sum_{i=1}^{n} \Phi_i x^\ominus_i$ one has

$$\left\| \left[ R^\ominus(A^\ominus)^{-\alpha} \Phi \right] \right\|_{L_q(\mathbb{R}, X^\ominus)} \leq \hat{C} \|\Phi\|_{L_1(\mathbb{R}, X^\ominus)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $A^\ominus$ is the generator of the sun dual semigroup. In other words, $M^\ominus_0$ maps $L_1(\mathbb{R}, X^\ominus)$ into $L_q(\mathbb{R}, X^\ominus)$.

By what we just proved, $M^\ominus_0$ is bounded from $L_1(\mathbb{R}, X^\ominus)$ to $L_\infty(\mathbb{R}, X^\ominus)$, and again by duality, $M_0$ maps $L_1(\mathbb{R}, X^\alpha)$ into $L_\infty(\mathbb{R}, X)$, which proves condition 3) with $F = L_\infty$.

The proofs of all other implications are completely analogous to those of Theorem 3.2.7.

Let us now turn to the second part of our theorem. Although its proof is also essentially the same, some comments will be in order. By Lemma 3.3.2, proved below, assumption (3.3.1) is equivalent to

$$\sup \left\{ \|R(\lambda)\|_{X^\alpha \rightarrow X} : |\Re \lambda| < \rho_0 \right\} < \infty.$$

So, the operators $S^t_N$, introduced in (3.2.4), are bounded from $X^\alpha \times L_1(\mathbb{R})$ to $X$. Uniform boundedness follows from the fact that $\lim_{N \rightarrow \infty} S^t_N(x, f)$ exists for all $x \in X^\alpha$ and $f \in L_1(\mathbb{R})$ by Lemma 3.2.8. Consequently, $\|I^t_N\|_{X^\alpha \rightarrow X} \leq C$.

Formula (3.2.5) still makes sense for all $x \in X^{\alpha+2}$, because then $A^2x \in X^\alpha$ and all the integrals converge absolutely. So, $G(t)x$ exists for all $x \in X^\alpha$, and it is continuous in $t$, $t \neq 0$.

Since $\langle x^*, G(t)x \rangle = \tilde{r}_0(t, x, x^*)$, by condition 3) and Lemma 3.2.6 we have that $|\langle x^*, G(t)x \rangle| \leq Ke^{-\rho|t|}\|x\|_{X^\alpha}\|x^*\|$ almost everywhere and hence, by the continuity of $G(t)x$, for all $t \in \mathbb{R}$. Thus, $\|G(t)x\|_X \leq Ke^{-\rho|t|}\|x\|_{X^\alpha}$ and the proof is finished.

Lemma 3.3.2. Let $S = \{ \lambda \in \mathbb{C} : a < \Re \lambda < b \}$, $a, b \in \mathbb{R}$, be a subset of $\rho(A)$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$. Then conditions

$$\sup \left\{ \frac{\|R(\lambda)\|}{1 + |\lambda|^2} : \lambda \in S \right\} < \infty \quad \text{and} \quad \sup \left\{ \|R(\lambda)A^{-\alpha}\| : \lambda \in S \right\} < \infty$$

are equivalent.
Proof. Since \( b \) is finite, there are constants \( c > 0 \) and \( \varphi_0, 0 < \varphi_0 < \pi \) such that \( |\mu - e^{i\varphi}| > |\mu| + c \) for all \( \mu \in \gamma \) and \( \varphi_0 < |\varphi| < \pi - \varphi_0 \). Pick \( N > 1 \) large enough to satisfy \( \frac{\pi}{N} < \frac{\pi}{2} \) and such that whenever \( \lambda \in S \) and \( |\lambda| > N \), then \( \varphi_0 < |\arg \lambda| < \pi - \varphi_0 \) and \( \lambda \) does not belong to the sector bounded by the contour \( |\lambda| \gamma \). For all such \( \lambda \) we have

\[
|\mu + \frac{\omega}{|\lambda|} - e^{i\arg \lambda}| > |\mu| + \frac{c}{2}.
\]

(3.3.2)

Let us consider the following integral:

\[
I_\lambda = \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} d\mu, \quad \lambda \in S, \quad |\lambda| > N.
\]

By the choice of \( N \), the integrand does not have singular points between \( \gamma \) and \( |\lambda| \gamma \). By the Cauchy Theorem, we have

\[
I_\lambda = \int_{|\lambda| \gamma} \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} d\mu = \frac{1}{|\lambda|^\alpha} \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} d\mu.
\]

Inequality (3.3.2) implies that the absolute value of the last integral is bounded from above by a constant that does not depend on \( \lambda \), whenever \( \lambda \in S \), \( |\lambda| > N \). The analogous estimate from below follows from geometric considerations. Thus,

\[
\frac{d_1}{|\lambda|^\alpha} \leq |I_\lambda| \leq \frac{d_2}{|\lambda|^\alpha},
\]

(3.3.3)

for some positive \( d_1 \) and \( d_2 \).

Suppose \( x \in X \). Then

\[
R(\lambda) A_\omega^{-\alpha} x = \frac{1}{2\pi i} \int_\gamma \mu^{-\alpha} R(\lambda) R(\mu + \omega) x d\mu
\]

\[
= \frac{1}{2\pi i} I_\lambda R(\lambda) x - \frac{1}{2\pi i} \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} R(\mu + \omega) x d\mu.
\]

Let us notice that \( |\mu + \omega - \lambda| \geq K(|\mu| + 1) \) for some \( K > 0 \) and all \( \mu \in \gamma \), \( \lambda \in S \), \( |\lambda| > N \), whereas \( \|R(\mu + \omega)\| \leq C \frac{1}{1+|\mu|} \). Consequently,

\[
\left\| \int_\gamma \frac{\mu^{-\alpha}}{\mu + \omega - \lambda} R(\mu + \omega) x d\mu \right\| \leq K_\alpha \|x\|.
\]
In combination with (3.3.3) this gives the following estimates:

\[ \| R(\lambda)A^{-\alpha}x \| \geq d_1 \frac{\| R(\lambda)x \|}{|\lambda|^\alpha} - K_\alpha \| x \|, \]

\[ \| R(\lambda)A^{-\alpha}x \| \leq d_2 \frac{\| R(\lambda)x \|}{|\lambda|^\alpha} + K_\alpha \| x \|, \]

for all \( \lambda \in \mathbb{S}, |\lambda| > N \) and \( x \in X \), which proves the lemma.

**Remark 3.3.3.** In view of Lemma 3.3.2, assumption (3.3.1) in Theorem 3.3.1, in fact, follows from condition 1) or 3).

**Remark 3.3.4.** Just as in the proof of Theorem 3.2.7 one can show the following identities:

\[ G(t) = T_t P, \quad t > 0 \]  
\[ G(t)T_{-t} = -(I - P), \quad t < 0 \]  

on \( X_\alpha \), where \( P \) is defined as \( \frac{1}{2}I + G(0) \).

Let us now recall the definition of the fractional growth bound \( \omega_\alpha(T) \) and its spectral counterpart \( s_\alpha(A) \): \( \omega_\alpha(T) \) is the infimum of all \( \omega \in \mathbb{R} \) such that \( \| T_t x \| \leq M_\omega e^{\omega t} \| x \|_\alpha \), for some \( M_\omega > 0 \) and all \( x \in X_\alpha \) and \( t \geq 0 \), and

\[ s_\alpha(A) = \inf \left\{ s : \sup \left\{ \frac{\| R(\lambda) \|}{1 + |\text{Im} \lambda|^\alpha} : \text{Re} \lambda > s \right\} < \infty \right\}. \]

(3.3.5)

As another consequence of Lemma 3.3.2 we get the following inversion formula (see [81] for the case \( \alpha = 0 \)).

**Corollary 3.3.5.** Let \( x \in X_\alpha \) and \( h > s_\alpha(A) \). If \( F_t \) is defined as in Lemma 3.2.3, then

\[ F_t(x) = \frac{1}{2\pi i} (C, 1) \int_{\text{Re} \lambda = h} e^{\lambda t} R(\lambda) x d\lambda \]

for all \( t \in \mathbb{R} \).

**Proof.** If \( h \geq \omega \), then our statement is the ordinary inversion formula (see Lemma 3.2.3). Otherwise, by the resolvent identity, we have

\[ R(u + iv)x = (1 - (u - \omega)R(u + iv))A^{-\alpha}_u R(\omega + iv)A^\alpha_\omega x, \]
for all \( u, h \leq u \leq \omega \). So, in view of Lemma 3.3.2, \( \lim_{v \to \infty} R(u + iv)x = 0 \) uniformly in \( u \in [h, \omega] \). Then, by the Cauchy Theorem, we get

\[
\frac{1}{2\pi i} (C, 1) \int_{\Re \lambda = h} e^{\lambda u} R(\lambda)x d\lambda = \frac{1}{2\pi i} (C, 1) \int_{\Re \lambda = \omega} e^{\lambda u} R(\lambda)x d\lambda = F_t(x).
\]

Let us recall the inequality \( \omega_\alpha(T) \geq s_\alpha(A) \) (see [86]). Now suppose \( \omega > s_\alpha(A) \) and \( R(i \cdot + \omega) \) is a multiplier from \( L_p(\mathbb{R}, X_\alpha) \) to \( L_p(\mathbb{R}, X) \). One can easily notice that the implication \( 1) \Rightarrow 4) \) of Theorem 3.3.1 was proved individually for every \( \rho \). Thus,

\[
|\langle r_\omega, \Phi \rangle| \leq |\langle \hat{r}_\omega, \hat{\Phi} \rangle| \leq K \|x\|_\alpha \|x^*\| \|\hat{\Phi}\|_1.
\]

However, by Corollary 3.3.5, \( \hat{r}_\omega(t) = e^{-\omega t} \langle x^*, T_t x \rangle \), \( t > 0 \), which implies \( \|T_t x\| \leq e^{\omega t} \|x\|_\alpha \). So, \( \omega_\alpha(T) \leq \omega \).

On the other hand, if \( \omega > \omega_\alpha(T) \), then \( \|e^{-\omega t} T_t\|_{X_\alpha \to X} \) is exponentially decaying. Consequently, the operator \( M_\omega \), being a convolution with the kernel \( e^{-\omega t} T_t \), maps \( L_p(\mathbb{R}, X_\alpha) \) into \( L_p(\mathbb{R}, X) \) as a bounded operator. Thus, we have proved the following result.

**Theorem 3.3.6.** For any \( C_0 \)-semigroup \( T \) on a Banach space \( X \), \( \omega_\alpha(T) \) is the infimum over all \( \omega > s_\alpha(A) \) such that \( R(i \cdot + \omega) \) is a multiplier from \( L_p(\mathbb{R}, X_\alpha) \) to \( L_p(\mathbb{R}, X) \), for some \( p \), \( 1 \leq p < \infty \).

**Corollary 3.3.7 ([86, 85])**. If \( X \) is a Hilbert space, then \( \omega_\alpha(T) = s_\alpha(A) \) for any strongly continuous semigroup \( T \) and \( \alpha \geq 0 \).

There are many results about properties of the constants \( \omega_\alpha(T), s_\alpha(A) \) and relations between them. We refer the reader to paper [85] for a detailed exposition of the subject.

We conclude this section by proving an \( \alpha \)-analogue of Perron’s Theorem, cf. [53]. Let us recall the classical result: a \( C_0 \)-semigroup \( T \) with generator \( A \) is hyperbolic if and only if for every \( g \in L_p(\mathbb{R}, X) \) the following integral equation

\[
u(\theta) = T_{\theta-\tau} u(\tau) + \int_\tau^\theta T_{\theta-s} g(s) ds, \quad \theta \geq \tau,
\]

has unique solution in \( L_p(\mathbb{R}, X) \) (see, e.g. [18, Theorem 4.33]).
In case of arbitrary $\alpha \geq 0$, we are looking for a necessary and sufficient condition on $T$, which provides existence and uniqueness of solution to (3.3.6) in $L_p(\mathbb{R}, X)$ for any given $g \in L_p(\mathbb{R}, X_\alpha)$. It turns out that the multiplier property of $R(is)$ is the condition we need.

**Theorem 3.3.8.** Suppose $i\mathbb{R} \subset \rho(A)$. Then the following assertions are equivalent:

1) $R(i\cdot)$ is a multiplier from $L_p(\mathbb{R}, X_\alpha)$ to $L_p(\mathbb{R}, X)$;

2) for every $g \in L_p(\mathbb{R}, X_\alpha)$ there exists a unique solution of (3.3.6) belonging to $L_p(\mathbb{R}, X)$.

Before we prove the theorem let us state one auxiliary fact, see [80] or [18, Prop.4.32].

**Lemma 3.3.9.** A function $u$ is a solution of (3.3.6) if and only if $u \in D(\Gamma)$ and $\Gamma u = -g$, where $\Gamma$ is the generator of the associated evolution semigroup.

**Proof.** 2) $\Rightarrow$ 1). Denote by $L$ the linear operator that maps $g \in L_p(\mathbb{R}, X_\alpha)$ to the corresponding solution of (3.3.6). By the Closed Graph Theorem, $L$ is bounded. We prove that actually $L = M_0$. Indeed, by Lemma 3.3.9, $Lg \in D(\Gamma)$ and $\Gamma Lg = -g$, for every $g \in L_p(\mathbb{R}, X_\alpha)$. On the other hand, a straightforward computation shows that if $g$ is a $C^\infty$-function with compact support, then $M_0g \in D(\Gamma)$ and $\Gamma M_0g = -g$. Thus, $\Gamma(M_0g - Lg) = 0$. However, if $\Gamma u = 0$ for some $u \in D(\Gamma)$, then again by Lemma 3.3.9, $u$ is a solution of (3.3.6) corresponding to $g = 0$. By the uniqueness, we get $u = 0$. So, $M_0g = Lg$ on a dense subspace of $L_p(\mathbb{R}, X_\alpha)$ and boundedness of $M_0$ is proved.

1) $\Rightarrow$ 2). Suppose $M_0$ is bounded from $L_p(\mathbb{R}, X_\alpha)$ to $L_p(\mathbb{R}, X)$. For a fixed $C^\infty$-function $g$ having compact support, we show that $u = M_0g$ solves (3.3.6). Indeed, using (3.2.1), we get

$$
u(\theta) - T_{\theta-\tau} u(\tau) = \int_\mathbb{R} R(is)\hat{g}(s)e^{is\theta} \, ds - \int_\mathbb{R} R(is)T_{\theta-\tau} \hat{g}(s)e^{is\tau} \, ds$$

$$= \int_0^{\theta-\tau} T_r \int_\mathbb{R} e^{is(\theta-r)} \hat{g}(s) \, ds \, dr$$

$$= \int_0^{\theta-\tau} T_r g(\theta-r) \, dr = \int_\tau^\theta T_{\theta-r} g(r) \, dr,$$
which is precisely (3.3.6).

Now suppose \( g \) is an arbitrary function from \( L^p(\mathbb{R}, X_\alpha) \). Let us approximate \( g \) by functions \((g_n)\) of considered type. Then \( u_n = M_0g_n \) converge to \( u = M_0g \) in \( L^p(\mathbb{R}, X) \) and, without loss of generality, pointwise on a set \( E \subset \mathbb{R} \) with \( m\text{es}\{\mathbb{R}\setminus E\} = 0 \). Thus, (3.3.6) is true for \( u, g \) and all \( \theta \) and \( \tau \) from \( E \). To get (3.3.6) for all \( \theta \) and \( \tau \), we will modify \( u \) on the set \( \mathbb{R} \setminus E \). To this end, let us take a decreasing sequence \((\tau_n)\subset E\) such that \( \lim \tau_n = -\infty \). Observe that the functions \( f_n(\theta) = T_{\theta-\tau_n}u(\tau_n) + \int_{\tau_n}^{\theta} T_{\theta-s}g(s)ds \) defined for \( \theta \geq \tau_n \) are continuous. Since \( u = f_n = f_m \) on \((+\infty, \max(\tau_n, \tau_m)] \cap E\), we get \( f_n = f_m \) everywhere in the half-line \((+\infty, \max(\tau_n, \tau_m)]\). Put \( \tilde{u} \) to be \( f_n \) on \((+\infty, \tau_n\]}. By the above, \( \tilde{u} \) is a well-defined function on all \( \mathbb{R} \). Obviously, \( u = \tilde{u} \) on \( E \). Let us show that \( \tilde{u} \) satisfies (3.3.6). Indeed, for any \( \theta \geq \tau \) and \( \tau > \tau_n \) we have

\[
T_{\theta-\tau}\tilde{u}(\tau) + \int_{\tau}^{\theta} T_{\theta-s}g(s)ds = T_{\theta-\tau}[T_{\tau-\tau_n}u(\tau_n) + \int_{\tau_n}^{\tau} T_{\tau-s}g(s)ds]
+ \int_{\tau}^{\theta} T_{\theta-s}g(s)ds
= T_{\theta-\tau_n}u(\tau_n) + \int_{\tau_n}^{\theta} T_{\tau-s}g(s)ds = \tilde{u}(\theta).
\]

Clearly, assertion 1) in Theorem 3.3.8 is weaker than condition 1) in Theorem 3.3.1. We do not know if they are equivalent. In case \( \alpha = 0 \), though, we can apply the resolvent identity to argue that if \( R(i\cdot) \) is a multiplier, then \( R(i\cdot+\rho) \) is also a multiplier for small values of \( \rho \). So, by Theorem 3.2.7, this is equivalent to hyperbolicity of the semigroup \( T \), and our statement turns into classical Perron’s Theorem.

### 3.4 An \( \alpha \)-analogue of hyperbolicity

We begin with a discrete version of Theorem 3.3.1 in the spirit of [53, Theorem 5]. Denote by \( \text{Rg} \, T \) the range of an operator \( T \).

**Theorem 3.4.1.** Suppose \( i\mathbb{Z} \subset \rho(A) \). Then the following conditions are equivalent:
1) \( X_\alpha \subset \text{Rg}(I - T_{2\pi}) \);

2) The sum \( (C, 1) \sum_{k \in \mathbb{Z}} R(ik)x \) exists in \( X \)-norm for all \( x \in X_\alpha \);

3) \( \{R(ik)\}_{k \in \mathbb{Z}} \) is a multiplier from \( L_p(\mathbb{T}, X_\alpha) \) to \( L_p(\mathbb{T}, X) \) for some/all \( 1 \leq p < \infty \);

4) \( \{R(ik)\}_{k \in \mathbb{Z}} \) is a multiplier from \( L_1(\mathbb{T}, X_\alpha) \) to \( F(\mathbb{T}, X) \), where \( F \) is some quasi-normed function lattice;

5) There exists a constant \( K > 0 \) such that

\[
|\langle r_0, \Phi \rangle| = \left| \sum_{k \in \mathbb{Z}} r_0(k, x, x^*)\Phi(k) \right| \leq K \|x\|_\alpha \|x^*\| \|\hat{\Phi}\|_{L_1(\mathbb{T})}
\]

holds for all \( x \in X_\alpha \), \( x^* \in X^* \), and \( \Phi \in C_\infty(\mathbb{T}) \).

Proof. 1) \( \Leftrightarrow \) 2). Note that

\[
\frac{1}{2\pi} R(ik)(I - T_{2\pi})x = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt}T_t x dt,
\]

for all \( x \in X \). So,

\[
\frac{1}{2\pi} (C, 1) \sum_{k \in \mathbb{Z}} R(ik)(I - T_{2\pi})x = \frac{1}{2}(I + T_{2\pi})x.
\]

Thus, 1) implies 2).

Now assume 1). Denote \( S = \frac{1}{2\pi} (C, 1) \sum_{k \in \mathbb{Z}} R(ik)x \). Then

\[
(\frac{1}{2} I + S)(I - T_{2\pi})x = (I - T_{2\pi})(\frac{1}{2} I + S)x = x, \tag{3.4.1}
\]

for all \( x \in X_\alpha \), and 1) follows.

Following [53] we denote by \( K \) the operator of convolution with the semi-group, i.e.

\[
K f(t) = \int_0^{2\pi} T_s f((t - s) \mod 2\pi) ds.
\]

Clearly, \( K \) is bounded on \( L_p(\mathbb{T}, X) \) for all \( 1 \leq p < \infty \) and \( \alpha > 0 \). Now we define the discrete multiplier operator \( L \) by the rule

\[
L f(\theta) = \sum_{k \in \mathbb{Z}} R(ik)\hat{f}(k)e^{ik\theta}, \quad \theta \in [0, 2\pi],
\]

112
where \( f \) is a trigonometric polynomial. One can check the identity

\[
K = L(I - T_{2\pi}) = (I - T_{2\pi})L. \tag{3.4.2}
\]

By the assumption and the spectral mapping theorem for the point spectrum the operator \((I - T_{2\pi})\) is one-to-one. Suppose 1) holds. Then \((I - T_{2\pi})\) has the left inverse \( U_1 := (I - T_{2\pi})^{-1} \) defined on \( X_\alpha \). By the Closed Graph Theorem \( U_1 \) is bounded as an operator from \( X_\alpha \) to \( X_\alpha \). Then (3.4.2) says that \( KU_1 \) maps \( L_p(\mathbb{T}, X_\alpha) \) into \( L_p(\mathbb{T}, X_\alpha) \) for all \( 1 \leq p < \infty \), which is what is stated in 3).

If the assertion in 3) is true only for some \( p \), then as in the proof of Theorem 3.3.1, \( LA^{-\alpha}_w \) maps \( L_p(\mathbb{T}, X) \) into \( L_\infty(\mathbb{T}, X) \). By duality, \( (LA^{-\alpha}_w)^* \) maps \( L_1(\mathbb{T}, X^\circ) \) into \( L_\infty(\mathbb{T}, X^\circ) \) and hence into \( L_\infty(\mathbb{T}, X^\circ) \). So, \( LA^{-\alpha}_w \) is a bounded operator from \( L_1(\mathbb{T}, X) \) to \( L_\infty(\mathbb{T}, X) \), which proves 4) with \( F = L_\infty(\mathbb{T}, X) \).

Assume 4). Then for every \( f \in C_\infty(\mathbb{T}, X) \) there is a \( \theta \in [0, 2\pi] \) such that

\[
\left\| \sum_{k \in \mathbb{Z}} R(ik) \hat{f}(k)e^{ik\theta} \right\| \leq K\|f\|_1.
\]

Applying \( T_\theta \) in the above inequality and using (3.2.1) we have:

\[
\left\| \sum_{k \in \mathbb{Z}} R(ik) \hat{f}(k) \right\| \leq K'\|f\|_1.
\]

In particular, for \( f = \Phi \otimes x \) the last inequality yields 5).

If 5) holds, then taking \( \Phi = \sum_{n=1}^N \Phi_n \otimes x_n \), with \( \Phi_n \in C_\infty(\mathbb{T}) \) having disjoint supports we get the following estimates

\[
\| L\Phi \|_1 = \frac{1}{2\pi} \int_0^{2\pi} \sup_{\| x^* \|=1} \left| \sum_{n=1}^N \sum_{k \in \mathbb{Z}} r_0(k, x_n, x^*) \hat{\Phi}_n(k) e^{ik\theta} \right| d\theta
\]

\[
\leq \sum_{n=1}^N \frac{1}{2\pi} \int_0^{2\pi} \sup_{\| x^* \|=1} |r_0, \Phi(\cdot + \theta)| d\theta
\]

\[
\leq \sum_{n=1}^N \| x_n \|_\alpha \| \Phi_n \|_1 = \| \Phi \|_1.
\]

So, we proved 3).
Finally, similarly to the proof of convergence of the \((C,1)\)-integral in Theorem 3.3.1, we can show that 2) follows from \(L_1\)-boundedness of \(LA_{-\alpha}\).

In Theorem 3.2.7 we have proved that conditions 2) through 4), involving multipliers, are equivalent to the hyperbolicity of the semigroup, that is, to a spectral property of \(T\). A natural question is to see if the multipliers-type conditions 2) through 4) in Theorem 3.3.1 are equivalent to a spectral property that could be formulated in terms of \(T\) acting on the Banach space \(X\), and not in terms of a space of \(X\)-valued functions. Theorem 3.4.1 suggests that each of the conditions 2) through 4) in Theorem 3.3.1 implies the inclusion \(X_\alpha \subset \text{Rg}(zI - T_{2\pi})\) for all \(z\) from some annulus \(A\) containing \(T\). It turns out that this inclusion alone is not equivalent to any of the conditions in Theorem 3.3.1. So, let us find the needed complement.

Assume that \(S\) is some strip containing the imaginary axes and \(S \subset \rho(A)\). Suppose also that \(X_\alpha \subset \text{Rg}(zI - T_{2\pi})\) for all \(z\) from some annulus \(A\) containing \(T\). By the Point Spectrum Mapping Theorem \((zI - T_{2\pi})\) is one-to-one. Thus, the left inverse operator \(U_z = (zI - T_{2\pi})_{\text{left}}^{-1} : X_\alpha \rightarrow X\) exists and is bounded by the Closed Graph Theorem. The family \(U = \{U_z\}_{z \in \mathbb{A}}\) obeys the resolvent identity on vectors from \(X_{2\alpha}\). However, to prove analyticity, first of all one needs uniform boundedness of \(U\). And that is the condition we are looking for.

**Theorem 3.4.2.** Suppose there is a strip \(S\) such that \(i\mathbb{R} \subset S \subset \rho(A)\). Then any of the equivalent conditions of Theorem 3.3.1 holds if and only if there exists an annulus \(A\) containing \(T\) such that \(X_\alpha \subset \text{Rg}(zI - T_{2\pi})\) for all \(z \in A\), and \(\sup\{\|U_z\|_{X_\alpha \rightarrow X} : z \in \mathbb{A}\} < \infty\).

**Proof.** Suppose that \(M_\rho\) maps \(L_p(\mathbb{R}, X_\alpha)\) into \(L_p(\mathbb{R}, X)\) for \(|\rho| < 2\rho_0\). By the Uniform Boundedness Principle \(M_\rho\) are uniformly bounded for all \(|\rho| < \rho_0\). Then, by transference, \(\{R(i(k + \xi) + \rho)\}_{k \in \mathbb{Z}}\) is a multiplier uniformly in \(\xi \in [0,1)\) and \(|\rho| < \rho_0\). In view of just proved Theorem 3.4.1 we get \(X_\alpha \subset \text{Rg}(zI - T_{2\pi})\) for all \(z\) from some open annulus \(A\) containing \(T\). In order to show uniform boundedness of \(U\), let us look at identity (3.4.1) first. It shows, in particular, that \(U_1 = \frac{1}{2} + S\). Just like in the second part of the proof of Theorem 3.2.7, one can estimate \(\|S\|_{X_\alpha \rightarrow X}\) by the multiplier norm of \(\{R(ik)\}\). Rescaling gives the same conclusion for all \(U_z\). Since norms of the corresponding multipliers are uniformly bounded, the desired result is proved.
Now let us prove the converse statement.

Clearly, the family $\mathcal{U} = \{U_z\}_{z \in \mathbb{A}}$ obeys the resolvent identity on vectors from $X_{2\alpha}$. Since, in addition, it is bounded, the mapping $z \to U_z$ is strongly continuous on vectors from $X_{2\alpha}$ and, hence, on all $X_{\alpha}$. Again by the resolvent identity, $U_z$ is strongly analytic on $X_{2\alpha}$. Since for any $x \in X_{\alpha}$, $U_zx$ is the uniform limit of a sequence $U_zx_n$ with $x_n \in X_{2\alpha}$, $U_zx$ is analytic.

It suffices to show that the integral $G(t)x = \frac{1}{2\pi} \int_{\mathbb{R}} R(is)xe^{ist}ds$ converges for all $x \in X_{\alpha}$ and there exists a $\beta > 0$ such that $\|G(t)x\| \leq Ke^{-\beta|t|}\|x\|_{\alpha}$.

So, let us fix $x \in X_{\alpha}$ and $t \in \mathbb{R}$. Then for any $s \in \mathbb{R}$, $x = (e^{2\pi is} - T_{2\pi})U_{e^{2\pi is}}x$. Thus

$$R(is)xe^{ist} = R(is)(I - e^{-2\pi is}T_{2\pi})U_{e^{2\pi is}}xe^{ist}e^{2\pi is}$$

From this we get

$$G(t)x = \frac{1}{2\pi} \lim_{N \to \infty} \int_0^{2\pi} \int_{-N}^N T_r \int_0^{2\pi} U_{e^{2\pi is}}xe^{(2\pi t + r)i}s(1 - \frac{|s|}{N})dsdr$$

$$= \frac{1}{2\pi} \lim_{N \to \infty} \int_0^{2\pi} \int_0^{2\pi} U_{e^{2\pi is}}xe^{(2\pi t + r)i}s \sum_{n = -N}^N e^{(t-r)in}(1 - \frac{|s + n|}{N})dsdr$$

$$= \frac{1}{2\pi} \lim_{N \to \infty} \int_0^{2\pi} \int_0^{2\pi} U_{e^{2\pi is}}xe^{(2\pi t + r)i}s \sum_{n = -N}^N e^{(t-r)in}(1 + \frac{|n|}{N})dsdr$$

$$= \lim_{N \to \infty} \int_0^{2\pi} \int_0^{2\pi} U_{e^{2\pi is}}xe^{(2\pi t + r)i}s F_N((t-r) \mod 2\pi)dsdr,$$

where $F_N$ is the Fejér kernel. Passing to limit inside the integral we get

$$G(t) = \frac{1}{2} \left[ T_{2\pi} \int_0^1 U_{e^{2\pi is}}xe^{ist}ds + \int_0^1 U_{e^{2\pi is}}xe^{is(2\pi t + t)}ds \right]$$

$$= \frac{1}{2} \left[ T_{2\pi} \frac{1}{2\pi i} \int_{\mathbb{T}} z^{t/2\pi} - 1 U_zxdz + \frac{1}{2\pi i} \int_{\mathbb{T}} z^{t/2\pi} U_zxdz \right],$$

if $t = 0 \mod 2\pi$. And

$$G(t) = T_{t \mod 2\pi} \int_0^1 U_{e^{2\pi is}}xe^{(1 + (t - t \mod 2\pi)/2\pi)i}ds$$

$$= T_{t \mod 2\pi} \frac{1}{2\pi i} \int_{\mathbb{T}} z^{(t - t \mod 2\pi)/2\pi} U_zxdz,$$
otherwise. In either case, the integrand is an analytic function on $A$. Replacing $T$ by $(1 + \epsilon)T$, if $t < 0$; or by $(1 - \epsilon)T$, if $t > 0$, we get the desired exponential decay.

3.5 Strong $\alpha$-hyperbolicity

In this section we introduce yet another notion of $\alpha$-hyperbolicity for strongly continuous semigroups. The spectral property we considered in the previous section, though strong enough, fails to produce any splitting projection, which is so natural in the case $\alpha > 0$. Therefore, we investigate a notion of strong $\alpha$-hyperbolicity, in which we force such a projection to exist.

**Definition 3.5.1.** A $C_0$-semigroup $T = (T_t)_{t \geq 0}$ is said to be strongly $\alpha$-hyperbolic if there exists a projection $P$ on $X$, called splitting, such that $P T_t = T_t P$, $t \geq 0$ and the following two conditions hold:

1. $\omega_\alpha(T|_{\text{im}P}) < 0$;
2. the restriction of $T$ on $\text{Ker}P$ is a group, and $\omega_\alpha(T^{-1}|_{\text{Ker}P}) < 0$, where $T^{-1} = (T^{-1})_{t \geq 0}$.

The function $G(t)$ defined as in Definition 3.2.2 is called the Green’s function corresponding to the $\alpha$-hyperbolic semigroup $T$.

It is an immediate consequence of the definition that Green’s function exponentially decays at infinity on vectors from $X_\alpha$.

Now we prove an analogue of Theorem 3.2.7 for $\alpha$-hyperbolic semigroups.

**Theorem 3.5.2.** A semigroup $T$ is $\alpha$-hyperbolic if and only if one of the equivalent conditions of Theorem 3.3.1 is satisfied and the operator

$$G(t)x = \frac{1}{2\pi} (C, 1) \int_{\mathbb{R}} R(is)x e^{ist} ds$$

has a continuous extension to all of $X$ for each $t \in \mathbb{R}$.

If this is the case, then $G(t)$ represents the Green’s function. Furthermore, the splitting projection is unique and given by

$$P = \frac{1}{2} I + G(0).$$  \hfill (3.5.1)
Proof. Let us prove necessity.

If $T$ is $\alpha$-hyperbolic, then there is a splitting projection $P$. Suppose $x \in (\text{Im } P)_{\alpha}$. Then by Corollary 3.3.5 applied to the semigroup $T|_{\text{Im } P}$, we have $F_t(x) = G(t)x$, where $F_t$ is defined in Lemma 3.2.3. In particular, $Px = x = \frac{1}{2}x + G(0)x$. On the other hand, if $x \in (\text{Ker } P)_{\alpha}$, then by the same reason, $\tilde{F}_t(x) = \frac{1}{2}\pi(C,1) \int_{\mathbb{R}} R(is,-A)x e^{ist} ds = -G(-t)x$, where $\tilde{F}_t(x)$ is defined by

\[ \tilde{F}_t(x) = \begin{cases} T_{-t}x, & t > 0 \\ \frac{1}{2}x, & t = 0 \\ 0, & t < 0 \end{cases} \]

So, $Px = 0 = \frac{1}{2}x + G(0)x$. Since $X_{\alpha} = (\text{Im } P)_{\alpha} + (\text{Ker } P)_{\alpha}$ is dense in $X$, this shows that $G(t)$ continuously extends to all of $X$ and equality (3.5.1) is true. The uniqueness of $P$ follows automatically from (3.5.1).

Since the $(X_{\alpha} \to X)$-norm of $G(t)$ is exponentially decaying and $M_0(\Phi) = G * \Phi$ for all $\Phi \in S$, $M_0$ is bounded from $L_1(\mathbb{R}, X_{\alpha})$ to $L_1(\mathbb{R}, X)$. To show boundedness of $M_\rho$, it is enough to notice that if $T$ is $\alpha$-hyperbolic, then the scaled semigroup $e^{\rho T}$ is also $\alpha$-hyperbolic, for small values of $\rho$.

Now we prove sufficiency.

Let us introduce the operator $P = \frac{1}{2}I + G(0)$. Since Theorem 3.2.7 is valid, and hence formulas (3.3.4) in Remark 3.3.4 are true, the norm of $T_t$ on $P(X_{\alpha})$ is exponentially decaying. Consequently, by the ordinary inversion formula for Laplace transform, we get $G(0)x = \frac{1}{2}x$, for all $x \in P(X_{\alpha})$. This implies $P^2 = P$ on all $X$, in view of the continuity of $P$. So, $P$ is a projection.

Obviously, $PT_t = T_t P$. On the other hand, since $P(X_{\alpha}) = (\text{Im } P)_{\alpha}$, we have $\omega_\alpha(T|_{\text{Im } P}) < 0$ and condition 1 of Definition 3.5.1 is proved.

To show invertibility of $T_t$ on $\text{Im }(I - P)$, we apply formula (3.3.4). It implies that $\|G(t)\|\|T_{-t}x\| \geq \|x\|$, for $x$ in $\text{Im }(I - P)$, and hence, $T_{-t}|_{\text{Im }(I - P)}$ is invertible. Another application of (3.3.4) and the second part of Theorem 3.2.7 proves condition 2 in Definition 3.5.1. \qed
Bibliography


