

THE UNSTABLE SPECTRUM OF THE NAVIER-STOKES OPERATOR IN THE LIMIT OF VANISHING VISCOSITY

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ABSTRACT. A general class of linear advective PDEs, whose leading order term is of viscous dissipative type, is considered. It is proved that beyond the limit of the essential spectrum of the underlying inviscid operator, the eigenvalues of the viscous operator, in the limit of vanishing viscosity, converge precisely to those of the inviscid operator. The general class of PDEs includes the equations of incompressible fluid dynamics. Hence eigenvalues of the Navier-Stokes operator converge in the inviscid limit to the eigenvalues of the Euler operator beyond the essential spectrum.

1. INTRODUCTION

The equations of motion governing an incompressible fluid with viscosity ε are the Navier-Stokes equations

$$(1.1a) \quad \frac{\partial q_\varepsilon}{\partial t} = -(q_\varepsilon \cdot \nabla)q_\varepsilon - \nabla p_\varepsilon + \varepsilon \Delta q_\varepsilon + F_\varepsilon,$$

$$(1.1b) \quad \nabla \cdot q_\varepsilon = 0,$$

where q_ε denotes the n -dimensional velocity vector, p_ε denotes the pressure and F_ε is an external force vector. Here n can be any integer with $n \geq 2$, but the case $n = 3$ is of the most interest.

The same equations with zero viscosity are the Euler equations

$$(1.2a) \quad \frac{\partial q}{\partial t} = -(q \cdot \nabla)q - \nabla p,$$

$$(1.2b) \quad \nabla \cdot q = 0.$$

An important connection between the Euler and the Navier-Stokes systems is the behavior of (1.1) in the limit of vanishing viscosity (i.e. $\varepsilon \rightarrow 0$). This limit is likely to be crucial in the understanding of many physical problems of fluid flow, such as the transition to turbulence.

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It is clear, since the types of the two systems are very different ((1.1) is parabolic and (1.2) is degenerate hyperbolic), that the limit of vanishing viscosity is a subtle and singular limit. There are a number of partial results for the nonlinear system as $\varepsilon \rightarrow 0$. The history of such results is briefly surveyed in the appendix of the book of Temam [8].

Open questions remain even for the linearized problem. In this present paper we address the connections between the spectra of the linearized Navier-Stokes operators in the inviscid limit and the spectrum of the linearized Euler operator. The results are closely tied to issues of linear, and even nonlinear, instabilities for fluid flows (c.f. Yudovich [12]).

Let $u(x, \varepsilon)$ be an arbitrary steady solution of (1.1):

$$(1.3a) \quad 0 = -(u \cdot \nabla)u - \nabla P + \varepsilon \Delta u + F_\varepsilon,$$

$$(1.3b) \quad \nabla \cdot u = 0.$$

We assume that $u(x, \varepsilon)$ and F_ε are infinitely smooth vector valued functions on the torus \mathbb{T}^n with regular dependence on $\varepsilon \in [0, \varepsilon_0)$ and that $\lim_{\varepsilon \rightarrow 0} F_\varepsilon = 0$. For the sake of simplicity we will present the proof of the theorems only for the case where $u(x)$ has no dependence on ε . The more general results follow from similar arguments.

The linearized Navier-Stokes equations for the evolution of a small perturbation velocity $v(x, t)$ are

$$(1.4a) \quad \frac{\partial v}{\partial t} = -(u \cdot \nabla)v - (v \cdot \nabla)u - \nabla p + \varepsilon \Delta v,$$

$$(1.4b) \quad \nabla \cdot v = 0.$$

The corresponding linearized Euler equations are

$$(1.5a) \quad \frac{\partial v}{\partial t} = -(u \cdot \nabla)v - (v \cdot \nabla)u - \nabla p,$$

$$(1.5b) \quad \nabla \cdot v = 0.$$

We will study general classes of differential operators on \mathbb{T}^n which include the operators of the fluid equations defined by (1.4) and (1.5). We will investigate the relationship between the unstable point spectrum of the inviscid operator and the eigenvalues of the viscous operator in the limit of vanishing viscosity.

For a general equilibrium $u(x)$ the Euler operator defined in (1.5) is non self-adjoint, non elliptic and degenerate. Hence, contrary to the case of the elliptic Navier-Stokes operator given by (1.4), standard spectral results for elliptic operators do not apply to the Euler operator. However in the past decade considerable progress has been made in understanding the structure of the spectrum of the Euler operator using

techniques of geometric optics. A survey of these results is given in Friedlander and Lipton-Lifschitz [3]. In particular, Vishik [9] obtained an explicit, and often computable, expression for the essential spectral radius of the Euler evolution operator in terms of a geometric quantity that can be considered as a "fluid" Lyapunov exponent. Recently Shvydkoy [7] has extended these results to a general class of advective PDEs with pseudodifferential bounded perturbation. The approach in which the evolution operator is partitioned into high frequency and low frequency parts will also be used in Section 3 of this present paper. We make use of the decomposition of the inviscid operator proved in [7] to obtain the analogous decomposition of the viscous operator (see Theorem 3.1). This result requires certain explicit estimates on the symbols of PDOs on the torus that we state in the appendix.

In Section 4 we prove the main result. We first prove a result for spectral convergence in the inviscid limit for the semigroups. We then prove that beyond the limit of the essential spectrum of the inviscid operator the eigenvalues of the viscous operator converge precisely to those of the inviscid operator. A key step is to use the decomposition established in Section 3 to split off a finite dimensional subspace corresponding to growing modes. An analogous argument was used by Lyashenko and Friedlander [5] to obtain a sufficient condition for instability in the limit of vanishing viscosity for a class of operators satisfying certain compactness and accretive properties. The properties required in [5] do not hold in general for the Euler operator (1.5) (although they do hold for the coupled rotating fluid/body system as noted in [5]). The goal of this present paper is to adapt the arguments of [5] to a wider class of operators that include the generic Euler and Navier-Stokes operators themselves.

2. FORMULATION OF THE RESULT

We consider the following class of differential operators on \mathbb{T}^n :

$$(2.1) \quad \mathbf{L}^\varepsilon f = -(u \cdot \nabla) f + \mathbf{A}f + \varepsilon \Delta f.$$

Here $u \in C^\infty(\mathbb{T}^n)$ is a divergence-free time independent vector field, f takes values in \mathbb{C}^d , $d \in \mathbb{N}$, and \mathbf{A} is a global pseudodifferential operator (PDO) on \mathbb{T}^n given by

$$(2.2) \quad \mathbf{A}f(x) = \text{Op}[\mathbf{a}]f(x) = \sum_{k \in \mathbb{Z}^n} e^{ik \cdot x} \mathbf{a}(x, k) \hat{f}(k),$$

where $\dot{\mathbb{Z}}^n = \mathbb{Z}^n \setminus \{0\}$. We assume that $\mathbf{a} \in \mathcal{S}^0$ (see the appendix) is a $d \times d$ -matrix valued symbol, which allows decomposition

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1,$$

where \mathbf{a}_0 is 0-homogenous in ξ , and $\mathbf{a}_1 \in \mathcal{S}^{-1}$. For instance, the linearized Navier-Stokes equation (1.4) has the right hand side of the form (2.1), where \mathbf{A} has principal symbol

$$(2.3) \quad \mathbf{a}_0(x, \xi) = \left(2 \frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id} \right) \partial u(x)$$

(see [7] for derivation).

We further consider a smooth linear bundle \mathcal{F} over $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$. We assume that \mathcal{F} is 0-homogenous. A function $f \in L^2(\mathbb{T}^n) = L^2$ is said to satisfy the frequency constraints determined by \mathcal{F} if $\hat{f}(k) \in F(k)$, for all $k \in \mathbb{Z}^n$, where $F(k)$ denotes the fiber over k (we consider the fiber $F(0)$ separately). The space of all such functions is denoted $L^2_{\mathcal{F}}$.

Let $\{\mathbf{G}_t^\varepsilon\}_{t \geq 0}$ be a C_0 -semigroup generated by \mathbf{L}^ε over L^2 . We assume that \mathbf{G}_t^ε leaves $L^2_{\mathcal{F}}$ invariant so that the equation

$$(2.4) \quad f_t = \mathbf{L}^\varepsilon f$$

is well-posed on $L^2_{\mathcal{F}}$.

The first order advective operator \mathbf{L}^0 was treated in [7]. It was shown there (and previously in [9] for the Euler equation) that the action of \mathbf{G}_t^0 on shortwave localized envelopes of the form

$$f_\delta(x) = b_0(x) e^{i\xi_0 \cdot x / \delta}, \quad \delta \ll 1$$

is described by the asymptotic formula

$$(2.5) \quad \mathbf{G}_t^0 f_\delta(x) = \mathbf{B}_t(\varphi_{-t}(x), \xi_0) f_\delta(\varphi_{-t}(x)) + O(\delta),$$

as $\delta \rightarrow 0$. In this formula φ_t is the integral flow of the steady field u , and \mathbf{B}_t is the fundamental matrix solution of the amplitude equation

$$(2.6) \quad b_t = \mathbf{a}_0(\chi_t(x_0, \xi_0)) b,$$

over the phase flow χ_t on $\Theta = \mathbb{T}^n \times \dot{\mathbb{R}}^n$ determined by the bicharacteristic system

$$(2.7) \quad \begin{cases} \dot{x}_t = u(x) \\ \dot{\xi}_t = -\partial u^\top(x) \xi \end{cases}$$

subject to the frequency constraint $b_0 \in F(\xi_0)$. One can modify the symbol \mathbf{a}_0 in such a way that the action of \mathbf{L}^0 on functions from $L^2_{\mathcal{F}}$ is the same, while (2.6) leaves \mathcal{F} invariant, i.e. $b(t) \in F(\xi(t))$ (see [7]). Thus one can consider (2.6) as a dynamical system over the bundle \mathcal{F} .

It was proved that the exponential instabilities of the amplitude equation (2.6) not only cause exponential instability of the semigroup \mathbf{G}^0 via (2.5), but also create the essential spectrum of the semigroup operator \mathbf{G}_t^0 in the unstable region. More precisely, the following formula for the essential spectral radius holds:

$$(2.8) \quad r_{\text{ess}}(\mathbf{G}_t^0) = e^{t\mu},$$

where μ is the maximal Lyapunov exponent of the dynamical system (2.6) (see [1]). The main result of this present article states that beyond this limit of the essential spectrum the eigenvalues of \mathbf{L}^ε converge precisely to the eigenvalues of \mathbf{L}^0 (and, of course, by spectral mapping the same is true for the semigroups). Even stronger, we show convergence of the corresponding spectral subspaces.

For a closed operator \mathbf{L} we use the following notation:

$$\sigma_a^+(\mathbf{L}) = \{\lambda \in \sigma(\mathbf{L}) : \text{Re } \lambda > a\},$$

and we denote by $m_a(\lambda, \mathbf{L})$ the algebraic multiplicity of λ .

Theorem 2.1. *Suppose that $\sigma_\mu^+(\mathbf{L}^0) \neq \emptyset$. Then*

- (i) *there exists $\varepsilon_0 > 0$ such that $\sigma_\mu^+(\mathbf{L}^\varepsilon) \neq \emptyset$ for all $0 \leq \varepsilon < \varepsilon_0$,*
- (ii) *for any $\lambda \in \sigma_\mu^+(\mathbf{L}^0)$ and any sufficiently small $r > 0$ there is $\varepsilon_r > 0$ such that for all $\varepsilon < \varepsilon_r$ one has*

$$(2.9) \quad m_a(\lambda, \mathbf{L}^0) = \sum_{\substack{\lambda' \in \sigma_\mu^+(\mathbf{L}^\varepsilon) \\ |\lambda - \lambda'| < r}} m_a(\lambda', \mathbf{L}^\varepsilon),$$

- (iii) *we have the limit*

$$(2.10) \quad \lim_{\varepsilon \rightarrow 0} \sum_{\substack{\lambda' \in \sigma_\mu^+(\mathbf{L}^\varepsilon) \\ |\lambda - \lambda'| < r}} \mathbf{P}_{\lambda'}^\varepsilon = \mathbf{P}_\lambda^0,$$

where $\mathbf{P}_\lambda^\varepsilon$ denotes the Riesz projection onto the spectral subspace corresponding to λ .

We note that (iii) \Rightarrow (ii) \Rightarrow (i). So, it suffices to prove only part (iii). The proof heavily relies on the results of the next section, and will be finished in Section 4. In the appendix we state some of the general facts on PDO's in the way that is convenient to use in the subsequent arguments.

3. HIGH FREQUENCY DECOMPOSITION

In this section we identify the high frequency part of the semigroup operator \mathbf{G}_t^ε . It is a PDO shifted by the flow φ_t , while the rest is

a sum of an operator of order $O(\sqrt{\varepsilon})$ and a compact operator that behaves like a PDO of order -1 uniformly in ε . We introduce the following notation. As before, we let φ_t denote the flow generated by u on \mathbb{T}^n , and χ_t the phase flow of the bicharacteristic equations (2.7) on $\mathbb{T}^n \times \dot{\mathbb{R}}^n$. The fundamental matrix solution of the amplitude equation (2.6), which we denoted $\mathbf{B}_t(x, \xi)$, is a smooth linear cocycle over the flow χ_t . We call it the b -cocycle. Clearly, the b -cocycle is a 0-homogenous in ξ symbol of class \mathcal{S}^0 .

We consider the operator of composition with the inverse flow φ_{-t} :

$$(3.1) \quad \Phi_t f = f \circ \varphi_{-t},$$

and the orthogonal projector:

$$(3.2) \quad \Pi : L^2 \rightarrow L^2_{\mathcal{F}},$$

which, as one can easily see, is a Fourier multiplier with the symbol given by the orthogonal frequency projector onto the fiber $F(\xi)$.

The following decomposition was proved in [7] in the case of $\varepsilon = 0$:

$$(3.3) \quad \mathbf{G}_t^0 = \mathbf{H}_t^0 + \mathbf{U}_t^0,$$

where

$$(3.4) \quad \mathbf{H}_t^0 = \Pi \Phi_t \text{Op}[\mathbf{B}_t]$$

and \mathbf{U}_t^0 is a compact operator, which behaves like a PDO of order -1 (hence the asymptotic formula (2.5)). For any positive ε formula (3.3) can be generalized as follows.

Theorem 3.1. *For any $0 \leq t < T$ and $0 \leq \varepsilon < \varepsilon_0$ the following decomposition holds:*

$$(3.5) \quad \mathbf{G}_t^\varepsilon = \mathbf{H}_t^\varepsilon + \sqrt{\varepsilon} \mathbf{T}_t^\varepsilon + \mathbf{U}_t^\varepsilon,$$

where

$$(3.6) \quad \mathbf{H}_t^\varepsilon = \Pi \Phi_t \text{Op}[\tau_t^\varepsilon],$$

$$(3.7) \quad \tau_t^\varepsilon(x, \xi) = \mathbf{B}_t(x, \xi) \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\},$$

the family $\{\mathbf{T}_t^\varepsilon\}_{0 \leq \varepsilon < \varepsilon_0, 0 \leq t < T}$ is uniformly bounded, and $\{\mathbf{U}_t^\varepsilon\}_{0 \leq \varepsilon < \varepsilon_0, 0 \leq t < T}$ is uniformly compact.

By a uniform compact family we mean the following.

Definition 3.2. Let $\psi_N(\xi)$ be the characteristic function of the ball $\{|\xi| < N\}$. Define the projection multiplier $\mathbf{P}_N f = (\psi_N \hat{f})^\vee$. We say

that a family of operators $\{\mathbf{U}_l\}_{l \in I}$ on L^2 , or its subspace invariant with respect to \mathbf{P}_N , is uniformly compact if

$$(3.8) \quad \lim_{N \rightarrow \infty} \sup_{l \in I} \|\mathbf{U}_l - \mathbf{U}_l \mathbf{P}_N\| = 0.$$

The rest of the section is devoted to the proof of Theorem 3.1.

First, we notice that the theorem easily reduces to the case when $\mathbf{\Pi} = \mathbf{I}$. Indeed, consider the semigroup \mathbf{G}_t^ε defined on the whole L^2 . If (3.5) holds on all L^2 , then by applying $\mathbf{\Pi}$ and restricting to $L^2_{\mathcal{F}}$, we see that (3.5) holds on $L^2_{\mathcal{F}}$ too.

Using the fact that the b -cocycle solves the amplitude equations (2.6) we find the evolution equation for \mathbf{H}_t^ε by straightforward differentiation:

$$(3.9) \quad \begin{aligned} \frac{d}{dt} \mathbf{H}_t^\varepsilon &= -(u \cdot \nabla) \mathbf{H}_t^\varepsilon + \mathbf{\Phi}_t \text{Op}[(\mathbf{a}_0 \circ \chi_t) \boldsymbol{\tau}_t^\varepsilon] - \\ &\quad - \varepsilon \mathbf{\Phi}_t \text{Op}[|\partial \varphi_t^{-\top}(x) \xi|^2 \boldsymbol{\tau}_t^\varepsilon]. \end{aligned}$$

We compare the second term on the right hand side with $\mathbf{A} \mathbf{H}_t^\varepsilon$. First, the change of variables rule implies

$$\mathbf{A} \mathbf{\Phi}_t = \mathbf{\Phi}_t \mathbf{A}'$$

where $\mathbf{A}' = \text{Op}[\mathbf{a}_0 \circ \chi_t] + \text{Op}[\mathbf{a}'_t]$ with $\mathbf{a}'_t \in \mathcal{S}^{-1}$ uniformly in $t < T$. The latter follows from the fact that $\varphi_t \in C^\infty(\mathbb{T}^n)$ uniformly in $-T < t < T$. Hence we obtain

$$(3.10) \quad \mathbf{A} \mathbf{H}_t^\varepsilon = \mathbf{A} \mathbf{\Phi}_t \mathbf{S}_t^\varepsilon = \mathbf{\Phi}_t \text{Op}[\mathbf{a}_0 \circ \chi_t] \text{Op}[\boldsymbol{\tau}_t^\varepsilon] + \mathbf{\Phi}_t \text{Op}[\mathbf{a}'_t] \text{Op}[\boldsymbol{\tau}_t^\varepsilon].$$

Let us show that the symbols $\boldsymbol{\tau}_t^\varepsilon$ satisfy a uniformity condition in the x -variable.

Lemma 3.3. *For any multi-index α there exists a constant B_α independent of $0 \leq \varepsilon < \varepsilon_0$ and $t < T$ such that*

$$(3.11) \quad \sup_{x \in \mathbb{T}^n, \xi \neq 0} |\partial_x^\alpha \boldsymbol{\tau}_t^\varepsilon(x, \xi)| \leq B_\alpha.$$

Proof. By the Leibnitz rule,

$$\partial_x^\alpha \boldsymbol{\tau}_t^\varepsilon = \sum_{\alpha' \leq \alpha} \partial_x^{\alpha - \alpha'} \mathbf{B}_t(x, \xi) \partial_x^{\alpha'} \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\}.$$

Hence, by the uniform boundedness of the b -cocycle,

$$|\partial_x^\alpha \boldsymbol{\tau}_t^\varepsilon| \leq C_{t, \alpha} \sup_{\alpha' \leq \alpha, x, \xi} \left| \partial_x^{\alpha'} \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\} \right|.$$

One can check by induction that if $g = g(x_1, \dots, x_n)$ is a smooth function, then

$$\partial_x^\alpha (\exp(g)) = \exp(g) \sum (\partial_x^{\gamma_1} g)^{l_1} \dots (\partial_x^{\gamma_r} g)^{l_r}$$

where the sum is taken over a subset of indexes satisfying

$$|\gamma_1|l_1 + \dots + |\gamma_r|l_r = |\alpha|.$$

In our case $g = -\varepsilon \int_0^t |\partial\varphi_s^{-\top}(x)\xi|^2 ds$. Then,

$$|(\partial_x^{\gamma_1} g)^{l_1} \dots (\partial_x^{\gamma_r} g)^{l_r}| \leq C_{t,\alpha} \varepsilon^{l_1+\dots+l_r} |\xi|^{2(l_1+\dots+l_r)}.$$

Using that $|\partial\varphi_t^{-\top}(x)\xi| \geq c_t|\xi|$, we obtain

$$\left| \partial_x^{\alpha'} \exp \left\{ -\varepsilon \int_0^t |\partial\varphi_s^{-\top}(x)\xi|^2 ds \right\} \right| \leq C_{t,\alpha} e^{-c_t \varepsilon |\xi|^2} \sum (\varepsilon |\xi|^2)^{l_1+\dots+l_r} \leq C'_{t,\alpha}$$

uniformly in ε . \square

Using Lemmas 3.3 and 5.4 we immediately conclude that the family $\mathbf{U}_{t,\varepsilon}^{(1)} = \Phi_t \text{Op}[\mathbf{a}'_t] \text{Op}[\boldsymbol{\tau}_t^\varepsilon]$ is uniformly compact in $0 \leq t < T$ and $0 \leq \varepsilon < \varepsilon_0$. By Theorem 5.1, with $m_1 = m_2 = 0$ and $N = 4$,

$$\text{Op}[\mathbf{a}_0 \circ \chi_t] \text{Op}[\boldsymbol{\tau}_t^\varepsilon] = \text{Op}[\boldsymbol{\lambda}_t^\varepsilon],$$

where

$$\boldsymbol{\lambda}_t^\varepsilon = (\mathbf{a}_0 \circ \chi_t) \boldsymbol{\tau}_t^\varepsilon + \sum_{1 \leq |\gamma| < 4} \frac{(-1)^{|\gamma|}}{\gamma!} (\partial_\xi^\gamma \mathbf{a}_0 \circ \chi_t) (\partial_x^\gamma \boldsymbol{\tau}_t^\varepsilon) + \mathbf{r}_4^{t,\varepsilon}.$$

From the estimate on the remainder (5.5) and the $\boldsymbol{\tau}_t^\varepsilon$ given in (3.11), we see that the families of symbols $\{\mathbf{r}_4^{t,\varepsilon}\}$ and $\{(\partial_\xi^\gamma \mathbf{a}_0 \circ \chi_t) (\partial_x^\gamma \boldsymbol{\tau}_t^\varepsilon)\}$, with $|\gamma| \geq 1$, satisfy the assumption of Lemma 5.3. Hence, they contribute a uniformly compact family $\{\mathbf{U}_{t,\varepsilon}^{(2)}\}$.

Summarizing the above, we have shown the identity

$$(3.12) \quad \mathbf{A}\mathbf{H}_t^\varepsilon = \Phi_t \text{Op}[(\mathbf{a}_0 \circ \chi_t) \boldsymbol{\tau}_t^\varepsilon] + \mathbf{U}_{t,\varepsilon}^{(3)},$$

where $\{\mathbf{U}_{t,\varepsilon}^{(3)}\}$ is uniformly compact.

Let us now consider the term

$$(3.13) \quad -\varepsilon \Phi_t \text{Op}[|\partial\varphi_t^{-\top}(x)\xi|^2 \boldsymbol{\tau}_t^\varepsilon]$$

and compare it to

$$(3.14) \quad \varepsilon \Delta \mathbf{H}_t^\varepsilon = -\varepsilon \text{Op}[|\xi|^2] \Phi_t \text{Op}[\boldsymbol{\tau}_t^\varepsilon].$$

By the change of variables rule, we obtain

$$\text{Op}[|\xi|^2] \Phi_t = \Phi_t \text{Op}[|\partial\varphi_t^{-\top}(x)\xi|^2].$$

By Theorem 5.1 with $m_1 = 2$, $m_2 = 0$, $N = 6$

$$\varepsilon \text{Op}[|\partial\varphi_t^{-\top}(x)\xi|^2] \text{Op}[\boldsymbol{\tau}_t^\varepsilon] = \text{Op}[\boldsymbol{\lambda}_t^\varepsilon],$$

where

$$(3.15) \quad \boldsymbol{\lambda}_t^\varepsilon = \varepsilon |\partial \varphi_t^{-\top}(x) \xi|^2 \boldsymbol{\tau}_t^\varepsilon + \varepsilon \sum_{1 \leq |\gamma| < 6} \frac{(-1)^{|\gamma|}}{\gamma!} (\partial_\xi^\gamma |\partial \varphi_t^{-\top}(x) \xi|^2) (\partial_x^\gamma \boldsymbol{\tau}_t^\varepsilon) + \varepsilon \mathbf{r}_6^{t,\varepsilon}.$$

Substitution of the first term on the right hand side of (3.15) into (3.14) gives us precisely (3.13). From (5.5), (3.11), and Theorem 5.2 we see that $\varepsilon \Phi_t \text{Op}[\mathbf{r}_6^{t,\varepsilon}] = \varepsilon \mathbf{T}_{t,\varepsilon}^{(1)}$, with $\{\mathbf{T}_{t,\varepsilon}^{(1)}\}$ being uniformly bounded.

Now, for all $|\gamma| = 1$ and any α one has

$$|\partial_x^\alpha (\varepsilon \partial_\xi^\gamma |\partial \varphi_t^{-\top}(x) \xi|^2 \partial_x^\gamma \boldsymbol{\tau}_t^\varepsilon)| \leq C_{t,\alpha} \varepsilon |\xi| e^{-c_t \varepsilon |\xi|^2} \leq \sqrt{\varepsilon} C'_{t,\alpha}$$

uniformly for all $0 \leq \varepsilon < \varepsilon_0$, $0 \leq t < T$, $\xi \in \mathbb{R}^n$, $x \in \mathbb{T}^n$. Hence, the terms with $|\gamma| = 1$ add up to a term of the form $\sqrt{\varepsilon} \mathbf{T}_{t,\varepsilon}^{(2)}$, where $\{\mathbf{T}_{t,\varepsilon}^{(2)}\}$ is uniformly bounded.

The terms with $|\gamma| = 2$ can be estimated as follows

$$|\partial_x^\alpha (\varepsilon \partial_\xi^\gamma |\partial \varphi_t^{-\top}(x) \xi|^2 \partial_x^\gamma \boldsymbol{\tau}_t^\varepsilon)| \leq C_{t,\alpha} \varepsilon e^{-c_t \varepsilon |\xi|^2}.$$

So, they contribute a term $\varepsilon \mathbf{T}_{t,\varepsilon}^{(3)}$. And finally, all the terms with $|\gamma| > 2$ vanish.

Thus, the evolution equation (3.9) has the form

$$(3.16) \quad \frac{d}{dt} \mathbf{H}_t^\varepsilon = -(u \cdot \nabla) \mathbf{H}_t^\varepsilon + \mathbf{A} \mathbf{H}_t^\varepsilon + \varepsilon \Delta \mathbf{H}_t^\varepsilon + \sqrt{\varepsilon} \mathbf{T}_{t,\varepsilon}^{(4)} + \mathbf{U}_{t,\varepsilon}^{(4)},$$

where $\{\mathbf{T}_{t,\varepsilon}^{(4)}\}$ is uniformly bounded, and $\{\mathbf{U}_{t,\varepsilon}^{(4)}\}$ is uniformly compact. By the Duhamel principle one gets

$$(3.17) \quad \mathbf{H}_t^\varepsilon = \mathbf{G}_t^\varepsilon + \sqrt{\varepsilon} \int_0^t \mathbf{H}_{t-s}^\varepsilon \mathbf{T}_{s,\varepsilon}^{(4)} ds + \int_0^t \mathbf{H}_{t-s}^\varepsilon \mathbf{U}_{s,\varepsilon}^{(4)} ds.$$

It remains to observe that by Lemma 3.3 and Theorem 5.2 the family $\{\mathbf{H}_t^\varepsilon\}$ itself is uniformly bounded, and hence, the integrals define operators \mathbf{T}_t^ε and \mathbf{U}_t^ε with the desired properties.

This finishes the proof of Theorem 3.1.

4. PROOF OF THE MAIN THEOREM

Let us recall that μ is the maximal Lyapunov exponent of the b -cocycle, which determines the essential spectral radius for the semi-group operator \mathbf{G}_t^0 through formula (2.8). Let us fix a $\delta > 0$. We can find a large t such that

$$\sup_{x,\xi} |\mathbf{B}_t(x, \xi)| < \frac{1}{4} e^{t(\mu+\delta)}.$$

Then by the sharp Gårding inequality, for N large enough, we get

$$(4.1) \quad \|\mathbf{H}_t^\varepsilon - \mathbf{H}_t^\varepsilon \mathbf{P}_N\| \leq 2 \sup_{x, \xi} |\tau_t^\varepsilon(x, \xi)| < \frac{1}{2} e^{t(\mu+\delta)},$$

for all $0 \leq \varepsilon < \varepsilon_0$. By the uniform compactness we also have

$$(4.2) \quad \|\mathbf{U}_t^\varepsilon - \mathbf{U}_t^\varepsilon \mathbf{P}_N\| < \frac{1}{3} e^{t(\mu+\delta)}.$$

Let us fix N for which both (4.1) and (4.2) hold, and split the semi-group \mathbf{G}_t^ε into the sum

$$(4.3) \quad \mathbf{G}_t^\varepsilon = \mathbf{G}_{t,\varepsilon}^- + \mathbf{G}_{t,\varepsilon}^+,$$

where we denote

$$(4.4) \quad \mathbf{G}_{t,\varepsilon}^- = \mathbf{H}_t^\varepsilon (\mathbf{I} - \mathbf{P}_N) + \sqrt{\varepsilon} \mathbf{T}_t^\varepsilon + \mathbf{U}_t^\varepsilon (\mathbf{I} - \mathbf{P}_N),$$

$$(4.5) \quad \mathbf{G}_{t,\varepsilon}^+ = \mathbf{H}_t^\varepsilon \mathbf{P}_N + \mathbf{U}_t^\varepsilon \mathbf{P}_N.$$

So, $\mathbf{G}_{t,\varepsilon}^+$ is non-zero only on the finite-dimensional subspace $\text{Im } \mathbf{P}_N$, and in view of (4.1) and (4.2), we have the estimate $\|\mathbf{G}_{t,\varepsilon}^-\| < \frac{5}{6} e^{t(\mu+\delta)}$ for all sufficiently small ε . This implies that the resolvent $(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}$ exists and has the power series expansion whenever $|z| > e^{t(\mu+\delta)}$.

Lemma 4.1. *The convergence*

$$(4.6) \quad \lim_{\varepsilon \rightarrow 0} (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} = (\mathbf{G}_{t,0}^- - z\mathbf{I})^{-1}$$

holds in the strong operator topology uniformly on compact subsets of $\{|z| > e^{t(\mu+\delta)}\}$.

Proof. Observe that $\mathbf{L}^\varepsilon f \rightarrow \mathbf{L}^0 f$ for all $f \in C^\infty(\mathbb{T}^n)$, and $C^\infty(\mathbb{T}^n)$ is a core of the operator \mathbf{L}^0 . Hence, by [4, Theorem 7.2], $\mathbf{G}_t^\varepsilon \rightarrow \mathbf{G}_t^0$ strongly. It is straightforward to prove that $\mathbf{H}_t^\varepsilon \rightarrow \mathbf{H}_t^0$ strongly, which by virtue of the decomposition (3.5) also implies that $\mathbf{U}_t^\varepsilon \rightarrow \mathbf{U}_t^0$. We therefore obtain convergence $\mathbf{G}_{t,\varepsilon}^- \rightarrow \mathbf{G}_{t,0}^-$ in the strong operator topology.

Thus, the conclusion of the lemma follows from the preceding remarks. \square

Observe that for any $|z| > e^{t(\mu+\delta)}$ and $0 \leq \varepsilon < \varepsilon_0$ the identity

$$(4.7) \quad \mathbf{G}_t^\varepsilon f = z f$$

can be written as

$$(4.8) \quad f + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ f = 0.$$

This is equivalent to the system of equations

$$(4.9) \quad [\mathbf{P}_N + \mathbf{P}_N(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N]f'_N = 0,$$

$$(4.10) \quad f''_N = -(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N f'_N,$$

where $f'_N = \mathbf{P}_N f$ and $f''_N = (\mathbf{I} - \mathbf{P}_N)f$. We see that f''_N can be found from (4.10) if f'_N is known. So, the original eigenvalue problem (4.7) is equivalent to the finite-dimensional equation (4.9), which in turn has a solution at $z = z_0$ if and only if z_0 is a root of the analytic function

$$(4.11) \quad g(z, \varepsilon) = \det \left\| \left\| (e_j, e_k) + (\mathbf{P}_N(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N e_j, e_k) \right\| \right\|_{j,k=1}^K,$$

where $\{e_1, \dots, e_K\}$ is an orthonormal basis of $\text{Im } \mathbf{P}_N$. By Lemma 4.1, we have

$$(4.12) \quad \lim_{\varepsilon \rightarrow 0} g(z, \varepsilon) = g(z, 0)$$

uniformly on compact sets in $\{|z| > e^{t(\mu+\delta)}\}$.

Lemma 4.2. *The resolvents $(\mathbf{G}_t^\varepsilon - z\mathbf{I})^{-1}$ exist and are uniformly bounded on compact subsets of $\{|z| > e^{t(\mu+\delta)}\} \setminus \sigma(\mathbf{G}_t^0)$, for $0 \leq \varepsilon < \varepsilon_0$, and the limit*

$$(4.13) \quad \lim_{\varepsilon \rightarrow 0} (\mathbf{G}_t^\varepsilon - z\mathbf{I})^{-1} = (\mathbf{G}_t^0 - z\mathbf{I})^{-1}$$

holds in the strong operator topology.

Proof. The existence of the resolvents follows readily from the convergence (4.12).

Let us fix $z \notin \sigma(\mathbf{G}_t^0)$ and observe that

$$(4.14) \quad (\mathbf{G}_t^\varepsilon - z\mathbf{I})^{-1} = [\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+]^{-1}(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}.$$

In view of Lemma 4.1 it suffices to show the convergence

$$(4.15) \quad \lim_{\varepsilon \rightarrow 0} [\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+]^{-1} = [\mathbf{I} + (\mathbf{G}_{t,0}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,0}^+]^{-1}.$$

In the direct sum $L^2 = \text{Im } \mathbf{P}_N \oplus \text{Ker } \mathbf{P}_N$ we have the following block-representation

$$\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+ = \begin{bmatrix} \mathbf{I} + \mathbf{P}_N(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N & 0 \\ (\mathbf{I} - \mathbf{P}_N)(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N & \mathbf{I} \end{bmatrix}.$$

So,

$$[\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+]^{-1} = \begin{bmatrix} [\mathbf{I} + \mathbf{P}_N(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N]^{-1} & 0 \\ \mathbf{F}_{t,\varepsilon} & \mathbf{I} \end{bmatrix},$$

where

$$\mathbf{F}_{t,\varepsilon} = -(\mathbf{I} - \mathbf{P}_N)(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N[\mathbf{I} + \mathbf{P}_N(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}\mathbf{G}_{t,\varepsilon}^+\mathbf{P}_N]^{-1}.$$

Since $g(z, \varepsilon)$ is uniformly bounded away from 0 for small ε , $\mathbf{F}_{t, \varepsilon}$ is uniformly bounded from above, and hence, so is the resolvent (4.14). The limit (4.15) now follows from the above formulas and Lemma 4.1. \square

Lemma 4.2 already proves the spectral convergence result for the semigroups. In order to prove it for the generators as stated in Theorem 2.1 we argue as follows.

Let $\lambda \in \sigma(\mathbf{L}^0)$ be arbitrary. Find a $\delta > 0$ such that $\operatorname{Re} \lambda > \mu + \delta$, and let $t > 0$ be chosen as above to satisfy (4.1) and (4.2). Observe the following identity:

$$(4.16) \quad (\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1} = (\mathbf{G}_t^\varepsilon - e^{\zeta t} \mathbf{I})^{-1} \int_0^t e^{\zeta(t-s)} \mathbf{G}_s^\varepsilon ds.$$

It follows from Lemma 4.2, that the resolvents $(\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1}$ are uniformly bounded on a circle Γ of small radius r centered at λ that does not contain other points of the spectrum of \mathbf{L}^0 . Moreover,

$$(4.17) \quad \lim_{\varepsilon \rightarrow 0} (\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1} = (\mathbf{L}^0 - \zeta \mathbf{I})^{-1}.$$

The Riesz projection on the spectral subspace corresponding to the part of the spectrum of \mathbf{L}^ε inside Γ is given by

$$(4.18) \quad \mathbf{P}^\varepsilon = \sum_{\substack{\lambda' \in \sigma(\mathbf{L}^\varepsilon) \\ |\lambda - \lambda'| < r}} \mathbf{P}_{\lambda'}^\varepsilon = \frac{1}{2\pi i} \int_{\Gamma} (\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1} d\zeta.$$

Using (4.17) the limit $\mathbf{P}^\varepsilon \rightarrow \mathbf{P}_\lambda^0$ follows from the dominated convergence theorem. This proves statement (iii) of our theorem, and hence, (ii) and (i).

4.1. Discussion. As we noted before, the Navier-Stokes operator given by the right hand side of (1.4) is a particular case of the general operator \mathbf{L}^ε . Thus the results of Theorem 2.1 apply to give the convergence of the unstable eigenvalues of the Navier-Stokes operator to eigenvalues of the Euler operator outside the essential spectrum of the latter. Our results therefore extend the theorem of Vishik and Friedlander [10] proving that a necessary condition for instability in the Navier-Stokes equations as $\varepsilon \rightarrow 0$ is an instability in the underlying Euler equations.

When the function space $L^2(\mathbb{T}^n)$ is replaced by the function space $H^m(\mathbb{T}^n)$ it is possible to obtain in place of μ an analogous quantity μ_m which determines the essential spectral radius of \mathbf{G}_t^0 . The role of the b -cocycle defined by (2.6) is replaced by a new so-called $b\xi^m$ -cocycle (see [6, 7]). All the arguments in this present paper remain valid for the convergence of the spectrum of the viscous operator as $\varepsilon \rightarrow 0$ and

the spectrum of the inviscid operator in $H^m(\mathbb{T}^n)$ with $\operatorname{Re} \lambda > \mu_m$. In the particular case of the two dimensional Euler equation in $H^1(\mathbb{T}^n)$ it can be shown that $\mu_1 = 0$, hence Theorem 2.1 implies that in this example there is precise convergence of all the points of the unstable spectra of the Navier-Stokes operators to that of the Euler operator in the inviscid limit.

The results of Theorem 2.1 also apply to other fluid systems such as the equations of geophysical fluid dynamics describing rotating, stratified incompressible flows where the evolution operator is an advective operator of the type \mathbf{L}^0 . We refer to [7] for an extended list of examples.

5. APPENDIX

In this section we recall a few facts about global pseudo-differential operators (PDO) on the torus defined by

$$(5.1) \quad \operatorname{Op}[\boldsymbol{\sigma}]f(x) = \sum_{k \in \dot{\mathbb{Z}}^n} e^{ik \cdot x} \boldsymbol{\sigma}(x, k) \hat{f}(k),$$

where $\dot{\mathbb{Z}}^n = \mathbb{Z}^n \setminus \{0\}$, $f(x) \in \mathbb{C}^d$ and $\boldsymbol{\sigma}$ is a $d \times d$ -matrix valued symbol of class \mathcal{S}^m . We write $\boldsymbol{\sigma} \in \mathcal{S}^m$ if $\boldsymbol{\sigma} \in C^\infty(\mathbb{T}^n \times \dot{\mathbb{R}}^n)$, where $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$, and

$$(5.2) \quad |\partial_x^\alpha \partial_\xi^\beta \boldsymbol{\sigma}(x, \xi)| \leq A_{\alpha, \beta} |\xi|^{m-|\beta|},$$

for all $|\xi| \geq 1$, $x \in \mathbb{T}^n$, and all multi-indexes α, β . Even though in the formula (5.1) we do not need to require symbols to be defined outside the integer lattice, we do assume that they are smooth in $\xi \in \dot{\mathbb{R}}^n$. For such symbols the standard theorems of pseudo-differential calculus hold as in the case of \mathbb{R}^n (see [2]). Below we state the composition rule with an estimate on the remainder term, which can be deduced from a careful examination of the proof given in [11].

For $a, b \geq 0$, and $\{A_{\alpha, \beta}\}$ given in (5.2), let us define

$$\tilde{A}_{a, b} = \sum_{|\alpha| \leq a, |\beta| \leq b} A_{\alpha, \beta}$$

Theorem 5.1. *Suppose $\boldsymbol{\sigma} \in \mathcal{S}^{m_1}$ and $\boldsymbol{\tau} \in \mathcal{S}^{m_2}$ with the corresponding norms $\{A_{\alpha, \beta}\}$ and $\{B_{\alpha, \beta}\}$. Then*

$$(5.3) \quad \operatorname{Op}[\boldsymbol{\sigma}] \circ \operatorname{Op}[\boldsymbol{\tau}] = \operatorname{Op}[\boldsymbol{\lambda}],$$

with $\boldsymbol{\lambda} \in \mathcal{S}^{m_1+m_2}$. Moreover, for all $N \in \mathbb{N}$, $\boldsymbol{\lambda}$ has the following representation

$$(5.4) \quad \boldsymbol{\lambda} = \sum_{|\gamma| < N} \frac{(-1)^{|\gamma|}}{\gamma!} (\partial_\xi^\gamma \boldsymbol{\sigma})(\partial_x^\gamma \boldsymbol{\tau}) + \mathbf{r}_N,$$

where $\mathbf{r}_N \in \mathcal{S}^{m_1+m_2-N}$, and for $N > m_1 + 3$ satisfies the estimate

$$(5.5) \quad |\partial_x^\alpha \mathbf{r}_N(x, \xi)| \leq c \tilde{A}_{|\alpha|, N+n} \tilde{B}_{2N+|\alpha|-m_1-1, 0} |\xi|^{m_1+m_2+1-N},$$

for $|\xi| \geq 1$, where $c = c(\alpha, N, n, m_1, m_2)$ is independent of the symbols.

In (5.5) the restriction $N > m_1 + 3$ and one extra power of $|\xi|$ is needed in order to obtain the explicit bound in terms of the norms of σ and τ . We also emphasize that the estimate (5.5) uses only the x -smoothness constant of τ , and not its ξ -smoothness.

We note that in the case of the torus the L^2 -norm of a PDO is bounded by the norm of only x -derivatives.

Theorem 5.2. *Suppose $\sigma \in \mathcal{S}^0$ satisfies (5.2). Then $\text{Op}[\sigma]$ is bounded on L^2 and*

$$(5.6) \quad \|\text{Op}[\sigma]\| \leq c \tilde{A}_{n+1, 0}$$

where c is independent of the symbol.

The proof uses Minkowski's inequality, and is similar to that of the next lemma.

Lemma 5.3. *Let $\mathbf{U}_\iota = \text{Op}[\sigma_\iota]$, $\iota \in I$. Suppose there exists a constant $A > 0$ independent of ι such that*

$$(5.7) \quad |\partial_x^\alpha \sigma_\iota(x, \xi)| \leq A |\xi|^{-1}, \quad |\xi| \geq 1, \quad |\alpha| \leq n + 1,$$

holds for all $\iota \in I$. Then the family $\{\mathbf{U}_\iota\}_{\iota \in I}$ is uniformly compact.

Proof. Let $f \in L^2$, $\|f\| = 1$, and $\text{supp } \hat{f} \subset \{|k| \geq N\}$. We obtain

$$\begin{aligned} \|\mathbf{U}_\iota f\|^2 &= \sum_{q \in \mathbb{Z}^n} \left| \sum_{|k| \geq N} \hat{\sigma}_\iota(q-k, k) \hat{f}(k) \right|^2 \\ &\lesssim \sum_{q \in \mathbb{Z}^n} \left| \sum_{k \neq q, |k| \geq N} \hat{\sigma}_\iota(q-k, k) \hat{f}(k) \right|^2 + \sum_{|k| \geq N} \left| \hat{\sigma}_\iota(0, k) \hat{f}(k) \right|^2 \\ &\leq A^2 \sum_{q \in \mathbb{Z}^n} \left(\sum_{k \neq q, |k| \geq N} \frac{|\hat{f}(k)|}{|k||q-k|^{n+1}} \right)^2 + A^2 \sum_{|k| \geq N} \frac{|\hat{f}(k)|^2}{|k|^2} \\ &\leq N^{-2} A^2 \left(\sum_{q \in \mathbb{Z}^n} |q|^{-n-1} \right)^2 + N^{-2} A^2 \lesssim N^{-2} A^2. \end{aligned}$$

This proves the lemma. \square

Lemma 5.4. *Let $\{\mathbf{U}_\iota\}_{\iota \in I}$ be as in Lemma 5.3. Let $\mathbf{V}_\kappa = \text{Op}[\tau_\kappa]$, $\kappa \in K$, be another family such that there is a constant $B > 0$ independent of κ such that*

$$(5.8) \quad |\partial_x^\alpha \tau_\kappa(x, \xi)| \leq B, \quad |\xi| \geq 1, \quad |\alpha| \leq n + 1$$

holds for all $\kappa \in K$. Then the family $\{\mathbf{U}_\iota \mathbf{V}_\kappa\}_{\iota \in I, \kappa \in K}$ is uniformly compact.

Proof. Let $N > 0$, $f \in L^2$ with $\text{supp } \hat{f} \subset \{|k| \geq N\}$ be fixed. Let $|q| < N/2$. Using the Cauchy-Schwartz inequality we estimate

$$\begin{aligned} |(\widehat{\mathbf{V}_\kappa f})(q)| &= \left| \sum_{|k| \geq N} \hat{\tau}_\kappa(q - k, k) \hat{f}(k) \right| \leq B \sum_{|k| \geq N} \frac{|\hat{f}(k)|}{|q - k|^{n+1}} \\ &\leq B \sum_{|p| > N/2} \frac{|\hat{f}(q - p)|}{|p|^{n+1}} \leq B \|f\| \left(\sum_{|p| > N/2} |p|^{-2(n+1)} \right)^{1/2} \\ &\lesssim N^{-1} \|f\|. \end{aligned}$$

So, for any fixed $M > 0$ we have

$$\lim_{N \rightarrow \infty} \|\mathbf{P}_M \mathbf{V}_\kappa (\mathbf{I} - \mathbf{P}_N)\| = 0$$

uniformly in $\kappa \in K$. Observe that

$$\|\mathbf{U}_\iota \mathbf{V}_\kappa - \mathbf{U}_\iota \mathbf{V}_\kappa \mathbf{P}_N\| \leq \|\mathbf{U}_\iota (\mathbf{I} - \mathbf{P}_M) \mathbf{V}_\kappa (\mathbf{I} - \mathbf{P}_N)\| + \|\mathbf{U}_\iota \mathbf{P}_M \mathbf{V}_\kappa (\mathbf{I} - \mathbf{P}_N)\|.$$

Thus, using uniform compactness of \mathbf{U}_ι 's and uniform boundedness of \mathbf{V}_κ 's, which follows from Theorem 5.2, we can choose M large enough to make the first summand small uniformly in N, ι, κ . Letting $N \rightarrow \infty$ we make the second summand small too. This finishes the proof. \square

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