

# COMBINATORICS AT $\aleph_\omega$

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ABSTRACT. We construct a model in which the singular cardinal hypothesis fails at  $\aleph_\omega$ . We use characterizations of genericity to show the existence of a projection between different Prikry type forcings.

## 1. INTRODUCTION

*The Singular Cardinal Problem* is the project to describe a complete set of rules for the behavior of the operation  $\kappa \mapsto 2^\kappa$  for singular cardinals  $\kappa$ . The standard way to blow up the power set of a singular cardinal is to start with a large cardinal  $\kappa$ , add many Cohen subsets to it, and then change its cofinality to  $\omega$  via a Prikry type forcing. In order to bring such results down to smaller cardinal, one has to also interleave collapses. Our construction uses two Prikry forcings with interleaved collapses defined in different models, one over a ground model with (almost) GCH, and the other one over a non GCH ground model. Throughout the course of our proof we give the exact characterization of genericity of both posets, and show that there is a projection between them. The former poset uses *guiding generics* for the collapses constraint functions in the Prikry conditions, but we do not have them for the latter poset.

We will start with a model  $V \models \kappa$  is indestructibly supercompact. Let  $\mathbb{A}$  be the Easton support iteration  $\langle Add(\alpha, \alpha^{+\omega+2}) \mid \alpha \leq \kappa \rangle$ . We will define two Prikry type forcings.  $\bar{\mathbb{P}}$  will be diagonal supercompact Prikry forcing in  $V$  with interleaved collapses using guiding generics.  $\mathbb{P}$  will be diagonal supercompact Prikry forcing in  $V^{\mathbb{A}}$  with interleaved collapses, but no guiding generics. We characterize the genericity for both posets. Then we use that to show that there is a projection from  $\mathbb{P}$  to  $\bar{\mathbb{P}}$ . In the final generic extension  $\kappa = \aleph_\omega$ , and SCH fails at  $\kappa$ . We also show that there is a weak square sequence at  $\kappa$ .

The paper is organized as follows. In section 2 we define the Prikry poset in the ground model  $V$ , where GCH holds. In section 3 we define a Prikry poset in  $V^{\mathbb{A}}$ , where  $\mathbb{A}$  is the forcing to add subsets of  $\kappa$ , while

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Mathematics Subject Classification: 03E04, 03E05, 03E35, 03E55.

Keywords: singular cardinal hypothesis, scales, prikry type forcing, supercompact cardinals.

preserving the supercompactness of  $\kappa$ . In section 4 we show that there is a weak square sequence at  $\aleph_\omega$  in the final model.

## 2. PRIKRY IN THE GROUND MODEL

Suppose that in  $V$ ,  $\kappa$  is indestructibly supercompact and GCH holds above  $\kappa$ . Let  $\bar{U}$  be a normal measure on  $\mathcal{P}_\kappa(\kappa^{+\omega+1})$ . For each  $n$ , let  $\bar{U}_n$  be the projection of  $\bar{U}$  to  $\mathcal{P}_\kappa(\kappa^{+n})$ . For each  $n > 0$ , in  $V$  we build  $K_n$  to be  $\text{Ult}(V, \bar{U}_n)$ -generic for  $(\text{Col}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Col}(\kappa^{+\omega+2}, < j_{\bar{U}_n}(\kappa)))^{\text{Ult}(V, \bar{U}_n)}$  as follows. Using the closure of  $\text{Ult}(V, \bar{U}_n)$  and the poset, we build a decreasing sequence  $\langle q_i^n \mid i < \kappa^{+n+1} \rangle$  of conditions meeting all the maximal antichains in the ultrapower. Define  $K_n = \{q \mid (\exists i)(q_i^n \leq q)\}$ . Define  $K_0$  similarly, but with respect to  $(\text{Col}(\kappa^{+\omega+2}, < j_{\bar{U}_0}(\kappa)))^{\text{Ult}(V, \bar{U}_0)}$ .

We define a ground model version of our Prikry forcing. For use in the definition we let  $\kappa_x =_{\text{def}} x \cap \kappa$  and  $x \prec y$  will denote that  $x \subset y$  and  $|x| < \kappa_y$ .

**Definition 2.1.** *Conditions in  $\bar{\mathbb{P}}$  are of the form  $p = \langle d, \langle p_n \mid n < \omega \rangle \rangle$ , where setting  $l = \text{lh}(p)$ , we have:*

- (1) *For  $0 \leq n < l$ ,  $p_n = \langle x_n, c_n \rangle$  such that:*
  - $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$ , and for  $i < n$ ,  $x_i \prec x_n$ ,
  - $c_0 \in \text{Col}(\kappa_{x_0}^{+\omega+2}, < \kappa_{x_1})$  if  $1 < l$ , and if  $l = 1$ ,  $c_0 \in \text{Col}(\kappa_{x_0}^{+\omega+2}, < \kappa)$ .
  - if  $1 < l$ , for  $0 < n < l - 1$ ,  $c_n \in \text{Col}(\kappa_{x_n}^{+n+2}, \kappa_{x_n}^{+\omega+1}) \times \text{Col}(\kappa_{x_n}^{+\omega+2}, < \kappa_{x_{n+1}})$ , and  $c_{l-1} \in \text{Col}(\kappa_{x_{l-1}}^{+l+1}, \kappa_{x_{l-1}}^{+\omega+1}) \times \text{Col}(\kappa_{x_{l-1}}^{+\omega+2}, < \kappa)$ .
- (2) *For  $n \geq l$ ,  $p_n = \langle A_n, C_n \rangle$  such that:*
  - $A_n \in \bar{U}_n$ , and  $x_{l-1} \prec y$  for all  $y \in A_n$ ,
  - $C_n$  is a function with domain  $A_n$ , for  $y \in A_n$ ,  $C_n(y) \in \text{Col}(\kappa_y^{+n+2}, \kappa_y^{+\omega+1}) \times \text{Col}(\kappa_y^{+\omega+2}, < \kappa)$  if  $n > 0$ , and  $C_n(y) \in \text{Col}(\kappa_y^{+\omega+2}, < \kappa)$  if  $n = 0$ ,
  - $[C_n]_{\bar{U}_n} \in K_n$
- (3) *if  $l > 0$ , then  $d \in \text{Col}(\omega, \kappa_{x_0}^{+\omega})$ , otherwise  $d \in \text{Col}(\omega, \kappa)$ .*

$q = \langle d^q, \langle q_n \mid n < \omega \rangle \rangle \leq p = \langle d^p, \langle p_n \mid n < \omega \rangle \rangle$  if  $\text{lh}(q) \geq \text{lh}(p)$  and:

- $d^q \leq d^p$ , and for all  $n < \text{lh}(p)$ ,  $x_n^p = x_n^q$ ,  $c_n^q \leq c_n^p$ ,
- for  $\text{lh}(p) \leq n < \text{lh}(q)$ ,  $x_n^q \in A_n^p$  and  $c_n^q \leq C_n^p(x_n^q)$ ,
- for  $n \geq \text{lh}(p)$ ,  $A_n^q \subset A_n^p$  and for all  $y \in A_n^q$ ,  $C_n^q(y) \leq C_n^p(y)$

$q \leq^* p$ , i.e.  $q$  is a direct extension of  $p$ , if  $q \leq p$  and  $\text{lh}(q) = \text{lh}(p)$ .

We will use the notation  $p = \langle d^p, x_0^p, c_0^p, \dots, x_{l-1}^p, c_{l-1}^p, A_l^p, C_l^p, \dots \rangle$ , and  $s(p) = \langle d^p, x_0^p, c_0^p, \dots, x_{l-1}^p, c_{l-1}^p \rangle$ . We will also write  $s(p) \prec x$  to denote  $x_{\text{lh}(p)-1}^p \prec x$ .

By adapting the arguments in [6] to our situation, we get:

**Proposition 2.2.**

- (1) (*The Prikry property*) If  $p$  has length at least 1 and  $D$  is a dense open set, then there is a direct extension  $p' \leq^* p$  and  $n < \omega$ , such that every  $n$ -step extension of  $p'$  is in  $D$ .
- (2)  $\bar{\mathbb{P}}$  has the  $\mu = \kappa^{+\omega+1}$  chain condition.
- (3) After forcing with  $\bar{\mathbb{P}}$ ,  $\kappa$  becomes  $\aleph_\omega$  and  $\mu$  becomes  $\aleph_{\omega+1}$ .

If  $\bar{G}$  is  $\bar{\mathbb{P}}$ -generic, then  $\bar{G}$  generates the sequence  $\langle x_n^* \mid n < \omega \rangle$  with each  $x_n^* \in \mathcal{P}_\kappa(\kappa^{+n})$ , such that  $\bigcup_n x_n^* = (\kappa^{+\omega})^V$ . Setting  $\lambda_n := \kappa_{x_n^*}$ , we have that each  $\lambda_n^{+\omega+1}$  is collapsed to  $\lambda_n^{+n+2}$ . As  $n$  increases we preserve more and more cardinals in between  $\lambda_n$  and  $\lambda_n^{+\omega+1}$ , and eventually, over  $\kappa$ ,  $\kappa^{+\omega+1}$  remains a cardinal. In addition, in  $V[\bar{G}]$ , weak square at  $\aleph_\omega$  fails, but  $2^{\aleph_\omega} = \aleph_{\omega+1}$ .

**Proposition 2.3.** (*Characterization of genericity of  $\bar{\mathbb{P}}$* ) The following are the necessary and sufficient conditions for a sequence  $\langle d^*, x_n^*, c_n^* \mid n < \omega \rangle$  to be  $\bar{\mathbb{P}}$ -generic over  $V$ :

- (1) If  $\langle A_n \mid n < \omega \rangle \in V$  with  $A_n \in \bar{U}_n$ , then for all large  $n$ ,  $x_n^* \in A_n$ ;
- (2) For  $n > 0$ ,  $c_n^*$  is  $\text{Col}(\kappa_{x_n^*}^{+n+2}, \kappa_{x_n^*}^{+\omega+1}) \times \text{Col}(\kappa_{x_n^*}^{+\omega+2}, < \kappa_{x_{n+1}^*})$ -generic over  $V$ , and  $c_0^*$  is  $\text{Col}(\kappa_{x_0^*}^{+\omega+2}, < \kappa_{x_1^*})$ -generic over  $V$ ;
- (3)  $d^*$  is  $\text{Col}(\omega, \kappa_{x_0^*}^{+\omega})$ -generic over  $V$ ;
- (4) If  $\langle [C_n]_{\bar{v}_n} \mid n < \omega \rangle \in V$  with  $[C_n]_{\bar{v}_n} \in K_n$ , we have that for all large  $n$ ,  $C_n(x_n^*) \in c_n^*$ .

*Proof.* First we show that the conditions are necessary. Suppose that  $\langle d^*, x_n^*, c_n^* \mid n < \omega \rangle$  is generic for  $\bar{\mathbb{P}}$ . Conditions (1), (2) and (3) follow from standard density arguments. For condition (4), given such  $\langle [C_n]_{\bar{v}_n} \mid n < \omega \rangle$ , the set  $\{p \in \bar{\mathbb{P}} \mid (\forall n \geq \text{lh}(p)) [C_n^p]_{\bar{v}_n} \leq [C_n]_{\bar{v}_n}\}$  is dense, and so (4) follows.

Next we show that the conditions are sufficient. The proof is a generalization of Mathias' arguments [8]. Suppose that (1-4) hold and define  $\bar{G} = \{p \in \bar{\mathbb{P}} \mid d^p \in d^*, \text{ for all } n < \text{lh}(p), x_n^p = x_n^*, c_n^p \in c_n^*, \text{ and for all } n \geq \text{lh}(p), x_n^* \in A_n^p \text{ and } C_n^p(x_n^*) \in c_n^*\}$ . Clearly  $\bar{G}$  is a filter. Suppose that  $D$  is a dense set of  $\bar{\mathbb{P}}$ . We want to find a condition  $p \in D \cap \bar{G}$ .

By the Prikry property, for all stems  $h$ , we can find a condition  $p_h$ , with  $\text{lh}(p_h) = \text{lh}(h)$  and  $s(p_h) \leq h$  in the natural ordering of stems, and some  $n = n_h < \omega$ , such that every  $n_h$ -step extension of  $p_h$  is in  $D$ . By standard diagonalization arguments we can find a condition  $p$  such

that for every stem  $h$ ,  $n < \omega$ , and  $x \in A_n^p$ , we have that if  $s(p^h) \prec x$ , then  $x \in A_n^{p^h}$  and  $C_n^p(x) \leq C_n^{p^h}(x)$ .

Applying (1) and (4), let  $k$  be such that for all  $n \geq k$ ,  $x_n^* \in A_n^p$  and  $C_n(x_n^*) \in c_n^*$ . Let  $h = \langle 1, x_0^*, 1, \dots, x_{k-1}^*, 1 \rangle$ . Using (2) and (3) and that we can choose the stems of  $p_h$  densely often below  $h$ , we can arrange  $p_h$  to have a stem of the form  $h' = \langle d, x_0^*, c_0, \dots, x_{k-1}^*, c_{k-1} \rangle$  where each  $c_i \in c_i^*$  and  $d \in d^*$ . For every  $n \geq k$ ,  $\{x \in A_n^p \mid h' \prec x\} \subset A_n^{p_h}$ , and  $C_n^p(x) \leq C_n^{p_h}(x)$ . It follows that  $p_h$  is in  $\bar{G}$ . Let  $q$  be an  $n_h$ -step extension of  $p_h$  which is in  $\bar{G}$ . Then  $q \in D \cap \bar{G}$ .  $\square$

In the next section we define a Prikry notion of forcing,  $\mathbb{P}$ , in a ground model after adding subsets of  $\kappa$ . We will use the characterization of genericity to show that  $\mathbb{P}$  projects to  $\bar{\mathbb{P}}$ .

### 3. PRIKRY AFTER ADDING THE SUBSETS

Let  $\mathbb{A}$  be the forcing to iterate in Easton fashion adding  $\alpha^{+\omega+2}$  many subsets of  $\alpha$ , for all inaccessible  $\alpha \leq \kappa$ . Let  $E$  be  $\mathbb{A}$ -generic over  $V$ . Then by standard arguments in  $V[E]$ , there is a normal measure  $U$  on  $\mathcal{P}_\kappa(\mu)$ , such that  $\bar{U} \subset U$  and  $j = j_U$  extends  $j_{\bar{U}}$  (see for example [7]). For each  $n$ , let  $U_n$  be the projection of  $U$  to  $\mathcal{P}_\kappa(\kappa^{+n})$  and set  $j_n = j_{U_n}$ ,  $M_n = \text{Ult}(V[E], U_n)$ . Note that  $j_n$  does not extend  $j_{\bar{U}_n}$ . Let  $k_n : M_n \rightarrow \text{Ult}(V[E], U)$  be given by  $k_n([f]_{U_n}) = j(f)(j''\kappa^{+n})$ . As in Gitik-Sharon [6] we can arrange that  $\text{crit}(k_n) > j_n(\kappa)$ . In particular,  $j_n(\kappa) = k_n(j_n(\kappa)) = j(\kappa)$ .

In  $V[E]$ , we define a diagonal supercompact Prikry forcing with respect to  $U$  with interleaved collapses taken from  $V$  to turn  $\kappa$  into  $\aleph_\omega$ . We take collapses from  $V$  since if we took them from  $V[E]$  the resulting forcing would collapse  $2^\kappa = \kappa^{+\omega+2}$ . In fact the forcing corresponding to the ‘tail forcing’  $\mathbb{D}$  described below would be responsible for the collapse. The proof is a relative of the one that adding a Cohen subset of  $\omega_1$  adds a surjection from  $\omega_1$  on to  $2^\omega$ .

**Definition 3.1.** *Conditions in  $\mathbb{P}$  are of the form*

$$p = \langle d, x_0, c_0, \dots, x_{l-1}, c_{l-1}, A_l, C_l, \dots \rangle,$$

where

- (1) for  $0 \leq n < l$ ,  $x_n \in \mathcal{P}_\kappa^V(\kappa^{+n})$ , and for  $i < n$ ,  $x_i \prec x_n$ ;
- (2)  $c_0 \in \text{Col}(\kappa_{x_0}^{+\omega+2}, < \kappa_{x_1})_V$  if  $1 < l$ , and if  $l = 1$ ,  $c_0 \in \text{Col}(\kappa_{x_0}^{+\omega+2}, < \kappa)_V$ ;
- (3) For  $0 < n < l - 1$ ,  $c_n \in \text{Col}(\kappa_{x_n}^{+n+2}, \kappa_{x_n}^{+\omega+1})_V \times \text{Col}(\kappa_{x_n}^{+\omega+2}, < \kappa_{x_{n+1}})_V$ , and  $c_{l-1} \in \text{Col}(\kappa_{x_{l-1}}^{+l+1}, \kappa_{x_{l-1}}^{+\omega+1})_V \times \text{Col}(\kappa_{x_{l-1}}^{+\omega+2}, < \kappa)_V$ ;

- (4) For  $n \geq l$   $A_n \in U_n$ ,  $A_n \subset \mathcal{P}_\kappa^V(\kappa^{+n})$  and  $C_n$  is a function with domain  $A_n$ , for  $y \in A_n$ ,  $C_n(y) \in \text{Col}(\kappa_y^{+n+2}, \kappa_y^{+\omega+1})_V \times \text{Col}(\kappa_y^{+\omega+2}, < \kappa)_V$  if  $n > 0$ , and  $C_n(y) \in \text{Col}(\kappa_y^{+\omega+2}, < \kappa)_V$  if  $n = 0$ ;
- (5) If  $l > 0$ , then  $d \in \text{Col}(\omega, \kappa_{x_0}^{+\omega})_V$ , otherwise  $d \in \text{Col}(\omega, \kappa)_V$ .

The ordering is as for  $\bar{\mathbb{P}}$ .

Note that  $\mathbb{P}$  contains  $\bar{\mathbb{P}}$ . However, due to the absence of the  $K_n$ 's  $\mathbb{P}$  does not satisfy the  $\mu$ -c.c. Next we prove that  $\mathbb{P}$  has the Prikry property. We fix some notation that will be used in the proof. Suppose that  $p \in \mathbb{P}$ , then we let

$$p =_{\text{def}} \langle d^p, x_0^p, c_0^p, \dots, x_{l-1}^p, c_{l-1}^p, A_l^p, C_l^p, \dots \rangle$$

where  $l = \text{lh}(p)$ . Sometimes we will want to refer to the *lower part* of  $p$ . We recall how we defined  $s(p)$ .

$$s(p) =_{\text{def}} \langle d^p, x_0^p, c_0^p, \dots, x_{l-1}^p, c_{l-1}^p \rangle$$

If  $s$  is a lower part, then  $\text{lh}(s)$  refers to  $\text{lh}(p)$  where  $p$  is a condition with  $s(p) = s$ . Similarly if  $s$  is a lower part, then for  $n < \text{lh}(s)$  we refer to  $x_n^s$  and  $c_n^s$  with the obvious interpretation. We make a few remarks and prove a few lemmas before the proof of the Prikry Lemma.

**Remark 3.2.** *There is a natural ordering on stems with the same length and same Prikry points. Suppose that  $s_1, s_2$  are lower parts with  $l = \text{lh}(s_1) = \text{lh}(s_2)$  and for all  $n < l$ ,  $x_n^{s_1} = x_n^{s_2}$ , then we set  $s_1 \leq s_2$  if for all  $n < l$ ,  $c_n^{s_1} \leq c_n^{s_2}$ .*

For each  $n < \omega$ , let  $\mathbb{C}'_n$  be the poset  $(\text{Col}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Col}(\kappa^{+\omega+2}, < j(\kappa)))_{\text{UH}(V, \bar{U})}$ . If  $C_n$  is a typical constraint function in  $\mathbb{P}$ , then the equivalence class of  $C_n$  modulo  $U_n$  is a member of  $\mathbb{C}_n =_{\text{def}} [x \mapsto \text{Col}^V(\kappa_x^{+n+2}, \kappa_x^{+\omega+1}) \times \text{Col}^V(\kappa_x^{+\omega+2}, < \kappa)]_{U_n}$ .

**Proposition 3.3.** *For all  $n < \omega$ ,  $\mathbb{C}'_n \simeq \mathbb{C}_n$*

*Proof.* We observe that  $\mathbb{C}_n \in M_n$  is  $\text{Col}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Col}(\kappa^{+\omega+2}, < j_n(\kappa))$  as computed in  $j_n(V)$  where  $j_n(V)$  is the transitive inner model of  $M_n$  obtained by taking the ultrapower of  $V$  by  $\bar{U}_n$  and using functions from  $V[E]$ . Since  $\text{crit}(k_n) > j_n(\kappa)$  and  $j_n(\kappa)$  is inaccessible in  $M_n$ ,  $k_n \upharpoonright \mathbb{C}_n$  is the identity map and  $k_n(\mathbb{C}_n) = k_n \text{``} \mathbb{C}_n$ . By elementarity and the fact that  $j = k_n \circ j_n$ , we have that  $k_n(\mathbb{C}_n) = \mathbb{C}'_n$ . So we have that  $k_n \upharpoonright \mathbb{C}_n$  is an isomorphism.  $\square$

If we define

$$\mathbb{U}_l =_{\text{def}} \prod_{n \geq l} \mathbb{C}_n \text{ and}$$

$$\mathbb{U}'_l =_{\text{def}} \prod_{n \geq l} \mathbb{C}'_n$$

as full support products in  $V[E]$ , then  $\mathbb{U}_l \simeq \mathbb{U}'_l$ . Note that if  $\langle C_n \mid n \geq l \rangle$  is a typical sequence of constraining functions, then the sequence  $\langle [C_n]_{U_n} \mid n \geq l \rangle$  is an element of  $\mathbb{U}_l$ . In a slight abuse of notation if we have a sequence  $\vec{C} = \langle C_n \mid n \geq l \rangle$ , then we will write  $[\vec{C}]$  in place of  $\langle [C_n]_{U_n} \mid n \geq l \rangle$ . Further if  $p \in \mathbb{P}$ , then we write  $\vec{C}^p$  for the sequence of constraining functions obtained from  $p$ .

**Remark 3.4.** *Note that the ordering on  $\mathbb{U}_l$  has some relation to the ordering on the set  $\{p \upharpoonright [l, \omega) \mid p \in \mathbb{P}, \text{lh}(p) = l\}$  induced by the ordering on  $\mathbb{P}$ . We call this latter poset the poset of upper parts of conditions of length  $l$ . In particular if  $[\vec{C}] \leq [\vec{C}']$  in  $\mathbb{U}'_l$ , then there is a sequence of measure one sets  $\vec{A}$  such that  $(\vec{A}, \vec{C}) \leq (\vec{A}, \vec{C}')$  in the poset of upper parts.*

Each  $\mathbb{C}'_n$  is a member of  $V$  and since  $\mathbb{A}$  is countably closed,  $\mathbb{U}'_l$  is in  $V$ . This allows us to prove the following lemma.

**Lemma 3.5.** *For each  $l < \omega$ ,  $\mathbb{U}'_l$  is  $< \kappa^{l+2}$  distributive in  $V[E]$ .*

*Proof.* We fix  $l < \omega$ . For each  $n \geq l$ ,  $\mathbb{C}'_n$  is  $\kappa^{n+2}$ -closed in  $\text{Ult}(V, \bar{U})$ .  $\text{Ult}(V, \bar{U})$  is closed under  $\kappa^{+\omega+1}$ -sequences in  $V$ . It follows that for each such  $n$ ,  $\mathbb{C}'_n$  is  $\kappa^{n+2}$ -closed in  $V$ . So  $\mathbb{U}_l$  is  $\kappa^{l+2}$ -closed in  $V$ . Since  $\mathbb{A}$  is  $\kappa^+$ -cc,  $\mathbb{U}_l$  is  $< \kappa^{l+2}$ -distributive in  $V[E]$  by Easton's Lemma.  $\square$

It follows that  $\mathbb{U}_l$  is  $< \kappa^{l+2}$ -distributive in  $V[E]$ . We will also need the following.

**Lemma 3.6.** *For every inaccessible  $\alpha < \kappa$  and  $n < \omega$ ,  $\text{Col}(\alpha^{+n+2}, \alpha^{+\omega+1})_V \times \text{Col}(\alpha^{+\omega+2}, < \kappa)_V$  is  $< \alpha^{+n+2}$ -distributive in  $V^{\mathbb{A}}$ .*

The proof is a straightforward application of Easton's Lemma and the fact that  $\mathbb{A}$  is isomorphic to a two step iteration where the first iterate is  $\alpha^+$ -cc and the second iterate is  $\beta$ -closed for some  $\beta > \alpha^{+\omega}$ . We are now ready to prove the Prikry Lemma.

**Lemma 3.7.** *For every  $p \in \mathbb{P}$  with  $\text{lh}(p) \geq 1$  and every dense open set  $D \subseteq \mathbb{P}$ , there are a  $p^* \leq^* p$  and  $n < \omega$  such that every  $n$  step extension of  $p^*$  is in  $D$ .*

Let  $p \in \mathbb{P}$  be a condition of length at least 1 and  $D \subseteq \mathbb{P}$  be a dense open set. We break the argument into three rounds.

**Lemma 3.8.** *There is  $p_0 \leq^* p$  such that  $s(p) = s(p_0)$  and for all  $q \leq p_0$ , if  $q \in D$ , then  $s(q) \frown p_0 \upharpoonright [\text{lh}(q), \omega) \in D$ .*

*Proof.* Let  $s$  be a typical lower part with  $l =_{\text{def}} \text{lh}(s) \geq \text{lh}(p)$ . We define a dense open subset  $D_s$  of  $\mathbb{U}_l$ .

$$D_s =_{\text{def}} \{[\vec{C}] \in \mathbb{U}_l \mid (\exists \vec{A} \ s \frown (\vec{A}, \vec{C}) \in D) \text{ or} \\ (\forall [\vec{C}'] \leq [\vec{C}] \forall \vec{A} \ s \frown (\vec{A}, \vec{C}') \notin D)\}$$

The set  $D_s$  is well-defined, since if  $[\vec{C}] = [\vec{C}']$ , then  $[\vec{C}] \in D_s$  if and only if  $[\vec{C}'] \in D_s$ . Clearly  $D_s$  is dense open in  $\mathbb{U}_l$ . By the distributivity of  $\mathbb{U}_l$  and the fact that there are  $\kappa^{+l-1}$ -many stems of length  $l$ ,

$$D_l =_{\text{def}} \bigcap_{\{s \mid \text{lh}(s)=l\}} D_s$$

is dense open. Let  $\vec{C}$  be a sequence of constraining functions so that for all  $l \geq \text{lh}(p)$ ,  $[\vec{C} \upharpoonright [l, \omega)] \in D_l$  and  $[\vec{C}] \leq [\vec{C}^p]$ . Let  $\vec{A}$  witness that  $[\vec{C}] \leq [\vec{C}^p]$ . For each lower part  $s$ , let  $\vec{A}^s$  be a sequence of measure one sets witnessing the first condition for  $[\vec{C} \upharpoonright [\text{lh}(s), \omega)] \in D_s$  if that condition holds. Let  $\vec{A}'$  be the diagonal intersection of the  $\vec{A}^s$  intersected with  $\vec{A}$ . Let  $\vec{C}'$  be the natural restriction of  $\vec{C}$  to  $\vec{A}'$ . Let  $p_0$  be the condition with  $s(p_0) = s(p)$  and upper part given by  $(\vec{A}', \vec{C}')$ .

We claim that  $p_0$  is as required for the lemma. Suppose that  $q \leq p_0$  with  $q \in D$ . Let  $s = s(q)$  and  $l = \text{lh}(q)$ . It follows from our construction that  $[\vec{C}^{p_0} \upharpoonright [l, \omega)] \in D_s$ . Moreover,  $[\vec{C}^q] \leq [\vec{C}^{p_0} \upharpoonright [l, \omega)]$  and  $q \in D$ . So by the definition of  $D_s$ ,  $\vec{A}^s$  was chosen so that  $s \frown (\vec{A}^s, \vec{C}^{p_0}) \in D$ . We are done since  $s \frown p_0 \upharpoonright [l, \omega) \leq s \frown (\vec{A}^s, \vec{C}^{p_0})$ .  $\square$

Before the next lemma we need another piece of notation. We define

$$s^-(p) =_{\text{def}} \langle d^p, x_0^p, c_0^p, \dots, x_{l-1}^p \rangle,$$

where  $l = \text{lh}(p)$ . Note that  $s^-$  is  $s(p)$  without its topmost collapse condition.

We define a sequence of sets of lower parts  $\langle Y_n \mid n < \omega \rangle$ . Each  $Y_n$  is closed downward in the partial ordering on lower parts. We simultaneously define a decreasing sequence of conditions, which capture membership in the  $Y_n$ .

**Lemma 3.9.** *There is a sequence of sets  $\langle Y_n \mid n < \omega \rangle$  and a decreasing sequence  $\langle p_n \mid n < \omega \rangle$  of direct extensions of  $p_0$  such that*

$$Y_0 =_{\text{def}} \{s \mid s \frown p_0 \upharpoonright [\text{lh}(s), \omega) \in D\} \\ Y_{n+1} =_{\text{def}} \{s \mid \exists A \in U_{\text{lh}(s)} \forall x \in A \ s \frown \langle x, C_{\text{lh}(s)}^{p_n}(x) \rangle \in Y_n\}$$

and for all  $n \geq 1$  and all non-direct extensions  $q \leq p_n$  if  $s(q) \in Y_{n-1}$ , then  $s^-(q) \smallfrown \langle C_{\text{lh}(q)}^{p_n}(x) \rangle \in Y_{n-1}$ .

*Proof.* We work by recursion to define  $p_n$  for  $n \geq 1$ . Let  $s^-$  be a typical restricted lower part whose length  $l$  is at least  $\text{lh}(p_0)$  and let  $x$  be the topmost Prikry point of  $s^-$ . Note that  $x \in \mathcal{P}_\kappa(\kappa^{+(l-1)})$ . We define  $E_{s^-}$  a dense open subset of  $\text{Col}(\kappa_x^{+l+1}, \kappa_x^{+\omega+1})_V \times \text{Col}(\kappa_x^{+\omega+2}, < \kappa)_V$ .

$$E_{s^-} =_{\text{def}} \{c \mid s^- \smallfrown \langle c \rangle \in Y_{n-1} \text{ or } \forall c' \leq c \ s^- \smallfrown \langle c' \rangle \notin Y_{n-1}\}$$

Clearly  $E_{s^-}$  is dense open. By the distributivity of  $\text{Col}(\kappa_x^{+l+1}, \kappa_x^{+\omega+1})_V \times \text{Col}(\kappa_x^{+\omega+2}, < \kappa)_V$  and the fact that there are at most  $\kappa_x^{+(l-1)}$  many  $s^-$  for which  $x$  is the top Prikry point,

$$E_x =_{\text{def}} \bigcap_{\{s^- \mid x \text{ is the top Prikry Point of } s^-\}} E_{s^-}$$

is dense open. For each  $x \in A_{l-1}^{p_{n-1}}$ , we choose  $C_{l-1}^{p_n}(x) \leq C_{l-1}^{p_{n-1}}(x)$  with  $C_{l-1}^{p_n}(x) \in E_x$ . This completely defines  $\vec{C}^{p_n}$ . Let  $p_n =_{\text{def}} s(p) \smallfrown (\vec{A}^{p_{n-1}}, \vec{C}^{p_n})$ . Clearly  $p_n \leq^* p_{n-1}$ .

We claim that  $p_n$  is as required for the lemma. Suppose that  $q \leq p_n$  is a nondirect extension with  $s(q) \in Y_{n-1}$ . For ease of notation, let  $l = \text{lh}(q)$ ,  $s^- = s^-(q)$  and  $x$  be the top Prikry point of  $s^-$ . By the choice of  $p_n$ ,  $C_{l-1}^{p_n}(x) \in E_{s^-}$ . Moreover, since  $s(q) \in Y_{n-1}$ , we chose  $C_{l-1}^{p_n}(x)$  such that  $s^- \smallfrown \langle C_{l-1}^{p_n}(x) \rangle \in Y_{n-1}$  by the definition of  $E_{s^-}$ .  $\square$

With Lemma 3.9, we can complete the proof of the Prikry Lemma.

*Proof of Prikry Lemma.* Let  $p_\omega$  be a direct extension of each of the  $p_n$ . This is possible since the forcing in the upper part is countably closed. It is easy to see that  $p_\omega$  retains the property of each of  $p_n$ . To finish the proof we need to restrict the measure one sets that form the domains of our constraining functions.

For each  $s$  and each  $n < \omega$ , we partition  $A_{\text{lh}(s)}^{p_\omega}$  into two pieces. We consider the collection of  $x$  for which  $s \smallfrown \langle x, C_{\text{lh}(s)}^{p_\omega}(x) \rangle \in Y_n$  and its complement. We call the set which is measure one  $A^n(s)$ . If the above property holds on a measure one set then we say that we are in the *good case*. Let  $A_l^n$  be the diagonal intersection of the  $A^n(s)$  where  $\text{lh}(s) = l$ , and

$$A_l =_{\text{def}} \bigcap_{n < \omega} A_l^n.$$

Let  $p_{\omega+1}$  be the restriction of  $p_\omega$  to the measure one sets  $A_l$  for  $l \geq \text{lh}(p)$ . Let  $q \leq p_{\omega+1}$  with  $q \in D$  and set  $l = \text{lh}(q)$ . We set  $n = l - \text{lh}(p)$ . By Lemma 3.8,  $s(q) \smallfrown p_0 \upharpoonright [\text{lh}(q), \omega) \in D$ .



We prove by induction that for all  $k \leq n$ ,  $s(q) \upharpoonright l - k \in Y_k$ . Suppose that  $k = 0$ , then  $s(q) \upharpoonright l = s(q)$  and  $s(q) \frown p_0 \upharpoonright [\text{lh}(q), \omega) \in D$ , so  $s(q) \in Y_0$ . Next suppose that we have that  $s(q) \upharpoonright l - k \in Y_k$  for some  $k$ . By Lemma 3.9,

$$\langle d^q, c_0^q, x_0^q, \dots, x_{l-k-1}^q, C_{l-k-1}^{p_{\omega+1}}(x_{l-k-1}^q) \rangle \in Y_k$$

By definition of diagonal intersection,  $x_{l-k-1}^q \in A^k(s(q) \upharpoonright l - k - 1)$ . It follows that  $A^k(s(q) \upharpoonright l - k - 1)$  was chosen in the good case. Now by the definition of  $Y_{k+1}$ ,  $s(q) \upharpoonright l - k - 1 \in Y_{k+1}$ . In particular we showed that  $s(q) \upharpoonright l - n \in Y_n$ . Recall that  $l - n = \text{lh}(p)$ . Let  $p^* = s(q) \upharpoonright \text{lh}(p_{\omega+1}) \frown p_{\omega+1} \upharpoonright [\text{lh}(p_{\omega+1}), \omega)$ .

We claim that every  $n$  step extension of  $p^*$  is in  $D$ . Let  $r \leq p^*$  be an  $n$  step extension. We prove by induction that for all  $k \leq n$ ,  $s(r) \upharpoonright \text{lh}(p) + k \in Y_{n-k}$ . If  $k = 0$ , then  $s(r) \upharpoonright \text{lh}(p^*) \leq s(p^*)$  and by the choice of  $p^*$ ,  $s(p^*) \in Y_n$  and hence  $s(r) \upharpoonright \text{lh}(p^*) \in Y_n$ . Suppose that for some  $k < n$  we have  $s(r) \upharpoonright \text{lh}(p) + k \in Y_{n-k}$ . By the definition of  $Y_{n-k}$ ,  $A^{n-k}(s(r) \upharpoonright \text{lh}(p) + k)$  was chosen in the good case. By the definition of diagonal intersection  $x_{\text{lh}(p)+k}^r \in A^{n-k}(s(r) \upharpoonright \text{lh}(p) + k)$ . It follows that  $s(r) \upharpoonright \text{lh}(p) + k + 1 \in Y_{n-k-1}$ , as required. So we have proved that  $s(r) \in Y_0$ , so  $s(r) \frown p_0 \upharpoonright [\text{lh}(r), \omega) \in D$ . By the choice of  $r$ ,  $r \leq s(r) \frown p_0 \upharpoonright [\text{lh}(r), \omega)$ . Therefore  $r \in D$ .  $\square$

Next we list some corollaries of the Prikry lemma.

**Corollary 3.10.** *Let  $p \in \mathbb{P}$  and  $\phi$  be a formula. Then there is  $q \leq^* p$  that decides  $\phi$ .*

*Proof.* Apply the Prikry lemma to the set of all conditions deciding  $\phi$ . Then by further shrinking measure one sets, get  $q \leq^* p$ , so that every  $n$ -step extension decides  $\phi$  the same way for some  $n < \omega$ . Then  $q$  decides  $\phi$ .  $\square$

**Corollary 3.11.** *Forcing with  $\mathbb{P}$  in  $V[E]$  preserves  $\mu$ .*

*Proof.* Suppose otherwise. Then in  $V[E][G]$  the cofinality of  $\mu$  is less than  $\kappa$ . So fix a condition  $p$ ,  $\tau < \kappa$ , and a name  $\dot{f}$ , such that  $p \Vdash$  “ $\dot{f} : \tau \rightarrow \mu$  is unbounded”. For each  $\gamma < \tau$ , let  $D_\gamma$  be the dense open set of conditions deciding values for  $\dot{f}(\gamma)$ . Let  $p' \leq^* p$  be such that  $p'$  satisfies the conclusion of Lemma 3.8 applied to each  $D_\gamma$ . More precisely, for each  $\gamma < \tau$ , if  $q \leq p'$  is such that  $q$  decides  $\dot{f}(\gamma)$ , then  $s(q) \frown p' \upharpoonright [\text{lh}(q), \omega)$  decides  $\dot{f}(\gamma)$ . Here we use that the direct extension relation is  $< \tau^+$ -distributive.

For  $\gamma < \tau$ , let  $\alpha_\gamma < \mu$  be the supremum of all possible values for  $\dot{f}(\gamma)$  forced by conditions below  $p'$ . Then  $\alpha_\gamma < \mu$ . Let  $\alpha = \sup_{\gamma < \tau} \alpha_\gamma < \mu$ . Then  $p' \Vdash (\forall \gamma)(\dot{f}(\gamma) < \alpha)$ . Contradiction.  $\square$

**Corollary 3.12.** *Suppose that  $\dot{X}$  is a  $\mathbb{P}$ -name for a bounded subset of  $\kappa$ , then there is an  $n < \omega$  such that the interpretation of  $\dot{X}$  is in  $V[d^*, c_0^*, \dots, c_{n-1}^*]$  where  $d^*, c_0^*, \dots, c_{n-1}^*$  are generic for an initial segment of the interleaved collapses.*

*Proof.* Suppose that  $\dot{X}$  is a  $\mathbb{P}$ -name for a subset of  $\gamma < \kappa$ . Let  $p \in \mathbb{P}$ . Let  $p'$  be a one step extension of  $p$ , where if  $x$  is the Prikry point chosen then  $\kappa_x > \gamma$ . Set  $n + 1 = \text{lh}(p')$  and so  $x = x_n^{p'}$ . For each  $\alpha < \gamma$ , let  $D_\alpha$  be the dense open set of conditions deciding “ $\alpha \in \dot{X}$ ”. By taking a direct extension, we may assume that  $p'$  satisfies the conclusion of Lemma 3.8 applied to each  $D_\alpha$ . Let  $S_x$  be the set of stems  $s$  of length  $n$ , such that there is a condition  $q \leq p'$  with  $q \upharpoonright n = s$ . For every  $s \in S_x$  and  $\alpha < \gamma$ , let  $D_{s,\alpha} =_{\text{def}} \{c \in \text{Col}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_V \times \text{Col}(\kappa_x^{+\omega+2}, < \kappa)_V \mid (\exists r \leq^* s \hat{\smallfrown} p' \upharpoonright [n, \omega])(r \text{ decides “}\alpha \in \dot{X}\text{”, } c_n^r = c \text{ and } (\forall i > n)([C_i^r]_{U_i} = [C_i^{p'}]_{U_i}))\}$ . Then each  $D_{s,\alpha}$  is dense open because we can apply Lemma 3.10 to conditions of the form  $s \hat{\smallfrown} \langle x, c \rangle \hat{\smallfrown} p' \upharpoonright [n + 1, \omega]$ .

We have that  $\gamma < \kappa_x$  and the size of  $S_x$  is at most  $\text{o.t.}(x) = \kappa_x^{+n}$ , which is less than the distributivity of  $\text{Col}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_V \times \text{Col}(\kappa_x^{+\omega+2}, < \kappa)_V$ . So, let  $c \in \bigcap_{s \in S_x, \alpha < \gamma} D_{s,\alpha}$  with  $c \leq c_n^{p'}$ , and for each  $s, \alpha$ , let  $r_{s,\alpha}$  witness that  $c \in D_{s,\alpha}$ . Let  $p_{s,\alpha} = s \hat{\smallfrown} r_{s,\alpha} \upharpoonright [n, \omega]$ . Finally, set  $p''$  be a direct extension of all  $p_{s,\alpha}$ . Then  $p''$  decides which ordinals are in  $\dot{X}$  up to an extension of the collapse conditions in the stem restricted to length  $n$ . So  $p''$  forces that  $\dot{X} \in V[d^*, \dot{c}_0^* \dots \dot{c}_{n-1}^*]$ .  $\square$

It follows that after forcing with  $\mathbb{A} * \dot{\mathbb{P}}$ ,  $\kappa$  becomes  $\aleph_\omega$  and  $\mu$  becomes  $\aleph_{\omega+1}$ . We have to show that  $\mu^+$  is preserved.

**Proposition 3.13.** *(Characterization of genericity for  $\mathbb{P}$ ) The following are the necessary and sufficient conditions for a sequence  $\langle d^*, x_n^*, c_n^* \mid n < \omega \rangle$  to be  $\mathbb{P}$ -generic over  $V[E]$ :*

- (1) *If  $\langle A_n \mid n < \omega \rangle \in V[E]$  with  $A_n \in U_n$ , then for all large  $n$ ,  $x_n^* \in A_n$ ;*
- (2) *For all  $n > 0$ ,  $c_n^*$  is  $\text{Col}(\kappa_{x_n^*}^{+n+2}, \kappa_{x_n^*}^{+\omega+1})_V \times \text{Col}(\kappa_{x_n^*}^{+\omega+2}, < \kappa_{x_{n+1}^*})_V$ -generic over  $V[E]$ , and  $c_0^*$  is  $\text{Col}(\kappa_{x_0^*}^{+\omega+2}, < \kappa_{x_1^*})_V$ -generic over  $V[E]$ ;*
- (3)  *$d^*$  is  $\text{Col}(\omega, \kappa_{x_0^*}^{+\omega})_V$ -generic over  $V[E]$ ;*

- (4) If  $D$  is a dense subset of  $\prod_{n < \omega} \mathbb{C}_n$  in  $V[E]$ , then there is  $\langle [C_n]_{U_n} \mid n < \omega \rangle \in D$ , such that for all large  $n$ ,  $C_n(x_n^*) \in c_n^*$  (recall that  $\mathbb{C}_n = [x \mapsto \text{Col}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_V \times \text{Col}(\kappa_x^{+\omega+2}, < \kappa)_V]_{U_n}$ .)

*Proof.* First we show that the conditions are necessary. Conditions (1), (2) and (3) follow from standard density arguments. For condition (4), given such a dense set  $D$  in  $\prod \mathbb{C}_n$ , we have that the set  $D^* = \{p \in \mathbb{P} \mid (\exists \langle [C_n]_{U_n} \mid n < \omega \rangle \in D)(\forall n \geq \text{lh}(p))([C_n^p]_{U_n} = [C_n]_{U_n})\}$  is dense in  $\mathbb{P}$ . And so (4) follows.

Next we show that the conditions are sufficient. The proof follows the arguments in the proof of the Prikry property. Suppose that (1), (2), (3), and (4) hold and define  $G = \{p \in \mathbb{P} \mid d^p \in d^*, \text{ for all } n < \text{lh}(p), x_n^p = x_n^*, c_n^p \in c_n^*, \text{ and for all } n \geq \text{lh}(p), x_n^* \in A_n^p \text{ and } C_n^p(x_n^*) \in c_n^*\}$ . First we show that  $G$  is a filter. It is clearly closed upwards. If two conditions  $p, q$  are in  $G$ , by (1), we get that for all large  $n$ ,  $\{x \in A_n^p \cap A_n^q \mid C_n^p(x) \text{ is compatible with } C_n^q(x)\} \in U_n$ , and so we can find a common extension of  $p$  and  $q$  in  $G$ . Now suppose that  $D$  is a dense set of  $\mathbb{P}$ . We want to find a condition  $p \in D \cap G$ .

First we note that the arguments in Lemmas 3.8 and 3.9 used in the proof of the Prikry lemma actually yield the following stronger conclusions:

- For all  $p$  with length at least 1, there is a direct extension  $p' \leq^* p$ , such that for all stems  $h$  with  $\text{lh}(h) \geq \text{lh}(p)$ , if  $q \leq h \hat{\wedge} p' \upharpoonright [\text{lh}(h), \omega]$  is in  $D$ , then  $s(q) \hat{\wedge} p' \upharpoonright [\text{lh}(q), \omega] \in D$ .
- For all  $p$  with length at least 1, there is a sequence of sets  $\langle Y_n \mid n < \omega \rangle$  and a decreasing sequence  $\langle p_n \mid n < \omega \rangle$  of direct extensions of  $p$ , such that the  $Y_n$ 's are defined as in Lemma 3.9, and for each  $n \geq 1$ , for every stem  $h$  and non-direct extensions  $q \leq h \hat{\wedge} p_n \upharpoonright [\text{lh}(h), \omega]$ , if  $s(q) \in Y_{n-1}$ , then  $s^-(q) \hat{\wedge} \langle C_{\text{lh}(q)}^{p_n}(x) \rangle \in Y_{n-1}$ .

Using these properties, we can define  $p_\omega$  and  $p_{\omega+1}$  as in the proof of the Prikry lemma. Now let  $h$  be a stem, and let  $q \leq h \hat{\wedge} p_{\omega+1} \upharpoonright [\text{lh}(h), \omega]$  be in  $D$ . Set  $h' =_{\text{def}} s(q) \upharpoonright \text{lh}(h)$  and  $n_h =_{\text{def}} \text{lh}(q)$ . Then, using similar arguments as in the Prikry lemma, we get that every  $n_h$ -step extension of  $h' \hat{\wedge} p_{\omega+1} \upharpoonright [\text{lh}(h), \omega]$  is in  $D$ .

We can choose such a condition  $p_{\omega+1}$  densely often, and so applying (4) and (1), we can assume that for some  $k$ , we have that for all  $n \geq k$ ,  $x_n^* \in A_n^{p_{\omega+1}}$  and  $C_n^{p_{\omega+1}}(x_n^*) \in c_n^*$ . Let  $h = \langle 1, x_0^*, 1, \dots, x_{k-1}^*, 1 \rangle$ . Using (2) and (3) and that we can choose  $h'$  densely often below  $h$ , we can arrange  $h'$  to be of the form  $h' = \langle d, x_0^*, c_0, \dots, x_{k-1}^*, c_{k-1} \rangle$  where each  $c_i \in c_i^*$  and  $d \in d^*$ . Then  $h' \hat{\wedge} p_{\omega+1} \upharpoonright [k, \omega] \in G$ . Let  $q$  be an  $n_h$ -step extension of  $h' \hat{\wedge} p_{\omega+1} \upharpoonright [k, \omega]$  which is in  $D$ . Then  $q \in D \cap G$ .

□

Set  $\mathbb{D} = \prod_n \mathbb{C}_n / \text{fin}$ . By item (4) of the above proposition in  $V[E]$  there is a projection from  $\mathbb{P}$  to  $\mathbb{D}$ . Let  $H$  be  $\mathbb{D}$  generic over  $V[E]$ .

**Lemma 3.14.** *The quotient forcing  $\mathbb{P}/H$  has the  $\mu$ -chain condition.*

*Proof.* Suppose otherwise. Then in  $V[E][H]$ , there is an antichain  $\langle p_i \mid i < \mu \rangle$  in  $\mathbb{P}/H$ . By passing to an unbounded subset of  $\mu$  if necessary, we may assume all of the conditions have the same length. By shrinking measure one sets, we can arrange that for each  $p_i$ , for all  $k \geq \text{lh}(p_i)$ , for all  $y \in A_k^{p_i}$ , there are  $x_{\text{lh}(p)} \prec x_{\text{lh}(p)+1} \prec \dots \prec x_{k-1} \prec y$ , such that for each  $\text{lh}(p_i) \leq n < k$ ,  $x_n \in A_n^{p_i}$ . The meaning of this is that for every  $y \in A_k^{p_i}$ ,  $p_i$  can be extended to a condition whose stem ends in  $y$ .

Let  $G$  be  $\mathbb{P}/H$ -generic over  $V[E][H]$ . Say,  $G$  generates the sequence  $\langle d^*, x_n^*, c_n^* \mid n < \omega \rangle$ .

**Claim 3.15.** *For every  $p \in \mathbb{P}/H$ , for all large  $n$ ,  $x_n^* \in A_n^p$  and  $C_n^p(x_n^*) \in c_n^*$ .*

*Proof.* For  $p \in \mathbb{P}/H$  the set  $D := \{q \mid \text{for all large } n, [C_n^q]_{U_n} \leq_{\mathbb{C}_n} [C_n^p]_{U_n}\}$  is dense in  $\mathbb{P}/H$ . Let  $q \in G \cap D$ . Then for all large  $n$ ,  $[C_n^q]_{U_n} \leq_{\mathbb{C}_n} [C_n^p]_{U_n}$  and so for all large  $n$ ,  $C_n^q(x_n^*) \leq C_n^p(x_n^*)$ . Then for all large  $n$ ,  $C_n^p(x_n^*) \in c_n^*$

□

For each  $i < \mu$ , let  $k_i < \omega$  be such that for all  $n \geq k_i$ ,  $x_n^* \in A_n^{p_i}$  and  $C_n^{p_i}(x_n^*) \in c_n^*$ . Since  $\mu$  is still regular in  $V[E][G]$ , there is an unbounded subset  $I \subset \mu$  and  $k < \omega$ , such that for all  $i \in I$ ,  $k = k_i$ .

Now, for each  $i < \mu$ , let  $q_i \leq p_i$  in  $\mathbb{P}/H$  be such that  $\text{lh}(q_i) = k+1$  and whenever  $x_k^* \in A_k^{p_i}$ , we have that the stem of  $q_i$  ends in  $\langle x_k^*, C_k^{p_i}(x_k^*) \rangle$ . Note that the set  $I^* := \{i < \mu \mid x_k^* \in A_k^{p_i}\} \in V[E][H]$ , so we can carry out the construction. Here we have  $I^* \supset I \in V[E][G]$ .

Let  $h_i = s(q_i)$ . Then  $\langle h_i \mid i \in I \rangle$  is a sequence of stems of length  $k+1$ , in  $V[E][G]$ . But there are less than  $\mu$  many possible stems. So there are  $i < j$ , forced to be in  $I$ , such that  $h_i = h_j$ . Let  $h = h_i = h_j$ .

**Claim 3.16.**  *$p_i$  and  $p_j$  are compatible in  $\mathbb{P}/H$ .*

*Proof.* Let  $k'$  be such that for all  $n \geq k'$ ,  $[C_n^{p_i}]_{U_n}$  is compatible with  $[C_n^{p_j}]_{U_n}$ . For  $n \geq k'$ , let  $C_n$  be such that  $A_n := \text{dom}(C_n) \subset A_n^{p_i} \cap A_n^{p_j}$  and  $C_n(x) = C_n^{p_i}(x) \cup C_n^{p_j}(x)$ . Also, for every  $n$  with  $k < n < k'$ , let  $c_n = C_n^{p_i}(x_n^*) \cup C_n^{p_j}(x_n^*)$ . Set  $p = h \frown \langle x_n^*, c_n \mid k < n < k' \rangle \frown \langle A_n, C_n \mid k' \leq n < \omega \rangle$  (here we fix a condition deciding the generic sequence up to  $k'$ ). Then  $p \in \mathbb{P}/H$  and  $p$  is stronger than  $p_i$  and  $p_j$ .

□

□

Denote  $\mathbb{C}_n = \mathbb{C}_{n0} \times \mathbb{C}_{n1}$ , where  $\mathbb{C}_{n0} = [x \mapsto \text{Col}^V(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})]_{U_n}$  and  $\mathbb{C}_{n1} = [x \mapsto \text{Col}^V(\kappa_x^{+\omega+2}, < \kappa)]_{U_n}$ , and set  $\mathbb{D}_0 = \prod_n \mathbb{C}_{n0}/\text{fin}$  and  $\mathbb{D}_1 = \prod_n \mathbb{C}_{n1}/\text{fin}$ . Then  $\mathbb{D} = \mathbb{D}_0 \times \mathbb{D}_1$ . Note that by the isomorphism between  $\mathbb{C}_n$  and  $\mathbb{C}'_n$ ,  $\mathbb{D}$  is a poset in  $V$ .

**Proposition 3.17.** *In  $V$ ,*

- (1)  $\mathbb{D}_0$  has the  $\mu^+$  chain condition;
- (2)  $\mathbb{D}_1$  is  $< \mu^+$ -strategically closed;
- (3)  $\mathbb{D}$  is  $(\kappa^{+\omega} + 1)$ -strategically closed.

*Proof.* The proof for (1) is standard. We will just give the proof for (3). The argument for (2) is similar and actually simpler. We just note that (2) holds since  $\mathbb{C}_{n1}$  is isomorphic to  $(\text{Col}(\kappa^{+\omega+2}, < j(\kappa)))^{Ult(V, \bar{U})}$ , which is just  $(\text{Col}(\kappa^{+\omega+2}, < j(\kappa)))^V$ . This is since  $Ult(V, \bar{U})$  is closed under  $\mu$ -sequences of ordinals in  $V$  as an ultrapower by a  $\mu$ -supercompact ultrafilter  $\bar{U}$ , and the last forcing is  $\mu^+$ -closed over  $V$ .

We describe a winning strategy for player II which will produce a decreasing sequence  $\langle q_i \mid i \leq \kappa^{+\omega} \rangle$  of conditions in  $\mathbb{D}$ , such that there are representatives  $a^i \in \prod_n \mathbb{C}_n$  for  $i < \kappa^{+\omega}$ , such that:

- (1) each  $[a^i]/\text{fin} = q_i$ , and
- (2) for all  $i < j$ , if  $j < \kappa^{+n+2}$ , then  $a^j(n) \leq_{\mathbb{C}_n} a^i(n)$ .

Suppose that at some even stage  $j$ , we have the sequences  $\langle q_i, a_i \mid i < \alpha \rangle$  as desired. If  $j = i + 1$ , let  $q_j \leq q_i$  and by making finitely many changes, pick a representative for  $a^j$  for  $q_j$ , such that  $a^j(n) \leq a^i(n)$  for all  $n$ .

Suppose now that  $j$  is limit and  $j < \kappa^{+\omega}$ . Let  $k$  be the least such that  $j < \kappa^{+k+2}$ . Define  $a^j$  by setting  $a^j(n)$  to be a lower bound of  $a^i(n)$  for all  $i < j$  if  $n \geq k$ . Otherwise set  $a^j(n) = 1$ . Player II plays  $q_j = [a^j]$ .

Finally, suppose that  $j = \kappa^{+\omega}$ . Let  $a(n)$  be a lower bound of  $a^i(n)$  for all  $i < \kappa^{+n+1}$ , and set  $q_{\kappa^{+\omega}} = [a]$ . Then for each  $i$ ,  $a(n) \leq a^i(n)$  for all large  $n$ , and so  $q_{\kappa^{+\omega}} \leq q_i$ .

□

It follows that  $\mathbb{D}$  preserves cardinals. So, by the chain condition lemma, in  $V[E][G]$ ,  $\mu^+$  is preserved and becomes  $\aleph_{\omega+2}$ . So we have  $2^{\aleph_\omega} = \aleph_{\omega+2}$  and the singular cardinal hypothesis fails at  $\aleph_\omega$ .

Finally we will show that we can choose a generic object for  $\mathbb{P}$ , that projects to  $\bar{\mathbb{P}}$ . Recall that for each  $n$  we defined a decreasing sequence of conditions  $\langle q_i^n \mid i < \kappa^{+n+1} \rangle$  in  $(\text{Col}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Col}(\kappa^{+\omega+2}, < j_{\bar{U}_n}(\kappa)))_{Ult(V, \bar{U}_n)}$  and used them to build the guiding generic  $K_n$  in  $V$ . Note that each  $q_i^n \in \mathbb{C}_n =_{\text{def}} [x \mapsto \text{Col}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_V \times \text{Col}(\kappa_x^{+\omega+2}, <$

$\kappa)_V]_{U_n}$ . We also showed that

$$k_n''\mathbb{C}_n = k_n(\mathbb{C}_n) = \mathbb{C}'_n =_{def} (\text{Col}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Col}(\kappa^{+\omega+2}, < j(\kappa)))_{\text{Ult}(V, \bar{U})}.$$

$\text{Ult}(V, \bar{U})$  is closed under sequences of length  $\kappa^{+\omega+1}$ , so there is a condition  $q^{*n} \in \mathbb{C}'_n$  which is stronger than all  $k_n(q_i^n)$  for  $i < \kappa^{+n+1}$ . Here we use that, setting  $q_i^n = [f_i^n]_{\bar{U}_n}$ , we have that  $\langle k_n(q_i^n) \mid i < \kappa^{+n+1} \rangle = \langle [y \mapsto f_i^n(y \cap \kappa^{+n})]_{\bar{U}} \mid i < \kappa^{+n+1} \rangle \in V$ . And since  $\mathbb{C}'_n = k_n''\mathbb{C}_n$ , we have that for some  $q^n \in \mathbb{C}_n$ ,  $q^{*n} = k_n(q^n)$ . Then by elementarity,  $q^n$  is stronger than each  $q_i^n$  for  $i < \kappa^{+n+1}$ .

So for each  $n$ , we have a master condition  $q_n \in \mathbb{C}_n$  for the guiding generic  $K_n$  that was used to define  $\bar{\mathbb{P}}$ . Let  $p^m \in \mathbb{P}$  be such that for each  $n \geq \text{lh}(p^m)$ ,  $[C_n^{p^m}]_{U_n} = q^n$ , and let  $G$  be  $\mathbb{P}$ -generic over  $V[E]$  with  $p^m \in G$ . Let  $\langle x_n^* \mid n < \omega \rangle$  and  $\langle d^*, c_n^* \mid n < \omega \rangle$  be the generic sequences added by  $G$ .

**Lemma 3.18.**  $\langle d^*, x_n^*, c_n^* \mid n < \omega \rangle$  is  $\bar{\mathbb{P}}$ -generic over  $V$ .

*Proof.* (1), (2) and (3) of the conditions for genericity of  $\bar{\mathbb{P}}$  in Proposition 2.3 are straightforward. For (4), suppose that  $\langle [C_n]_{\bar{U}_n} \mid n < \omega \rangle \in V$  with each  $[C_n]_{\bar{U}_n} \in K_n$ . The set  $D = \{p \in \mathbb{P} \mid \text{for all large } n, [C_n^p]_{U_n} \leq [C_n]_{U_n}; \text{ or for unboundedly many } n, [C_n^p]_{U_n} \perp [C_n]_{U_n}\}$  is dense in  $\mathbb{P}$ . Let  $q \in D \cap G$ . By our choice of  $p^m$ , it follows that for all large  $n$ ,  $[C_n^q]_{U_n} \leq [C_n]_{U_n}$ . I.e. for all large  $n$ ,  $\{x \in A_n^q \mid C_n^q(x) \leq C_n(x)\} \in U_n$ . Then by genericity of the  $x_n^*$ 's for  $\mathbb{P}$ , we have that for all large  $n$ ,  $C_n^q(x_n^*) \leq C_n(x_n^*)$ . And so, since  $q \in G$ , for all large  $n$ ,  $C_n(x_n^*) \in c_n^*$ .  $\square$

So, we have that  $V[\bar{G}] \subset V[E][G]$ , i.e.  $\mathbb{A} * \dot{\mathbb{P}}$  projects to  $\bar{\mathbb{P}}$ . In particular we have shown that:

**Theorem 3.19.** We can choose a  $\mathbb{P}$ -generic filter  $G$  over  $V[E]$ , such that  $G$  induces a  $\bar{\mathbb{P}}$ -generic filter  $\bar{G}$  over  $V$ .

#### 4. WEAK SQUARE

In this section we show that forcing with  $\mathbb{D}_0$  adds a weak square sequence at  $\kappa^{+\omega}$ . The argument is motivated by a similar result in Assaf Sharon's thesis [11]. For every  $n < \omega$ , fix a  $\mathbb{C}_{n_0}$  name  $\dot{C}_n$  for a club in  $\mu$  of order type  $\kappa^{+n+2}$ . For  $\gamma < \mu$ , we say that  $[c_n]/\text{fin} \in \mathbb{D}_0$  is a  $\gamma$ -condition if for all large  $n$ ,  $c_n$  decides  $\dot{C}_n \cap \gamma$ . Then the set  $D = \{d \in \mathbb{D}_0 \mid d \text{ is a } \gamma\text{-condition}\}$  is dense. Now let  $H_0$  be  $\mathbb{D}_0$ -generic, and let  $\gamma$  be good if there is  $[c_n]/\text{fin} \in H_0$  such that for all large  $n$ ,  $c_n \Vdash_{\mathbb{C}_{n_0}} \gamma \in \lim \dot{C}_n$ . Note that since  $D$  is dense we can always assume that the witness  $[c_n]/\text{fin}$  is a  $\gamma$ -condition. In  $V[H_0]$ , define  $X := \{\gamma < \mu \mid \gamma \text{ is good}\}$ .

**Proposition 4.1.**  *$X$  is  $> \omega$ -club*

*Proof.* First we show closure. Let  $\langle \gamma_\alpha \mid \alpha < \delta \rangle$  be an increasing continuous sequence in  $X$  for some regular, uncountable  $\delta < \mu$ . For each  $\alpha < \delta$  let  $[c_n^\alpha]/fin$  and  $n_\alpha < \omega$  be such that for all  $n \geq n_\alpha$ ,  $c_n^\alpha$  decides  $\dot{C}_n \cap \gamma_\alpha$  and forces that  $\gamma_\alpha$  is a limit point of  $\dot{C}_n$ . Since  $\mathbb{D}_0$  is  $\kappa^{+\omega} + 1$ -strategically closed, there is  $[c_n]/fin \in H_0$  stronger than each  $[c_n^\alpha]/fin$ . By increasing the  $n_\alpha$ 's if necessary, we may assume that for all  $n \geq n_\alpha$ ,  $c_n \leq_{\mathbb{C}_{n_0}} c_n^\alpha$ . Since  $\delta$  has uncountable cofinality, there is some  $n^*$  and unbounded  $I \subset \delta$ , such that for all  $\alpha \in I$ ,  $n^* = n_\alpha$ . Then  $[c_n]/fin$  and  $n^*$  witness that  $\gamma := \sup_{\alpha < \delta} \gamma_\alpha$  is good.

To show that  $X$  is unbounded, fix  $\gamma < \mu$  and a condition  $[c_n]/fin$ . For all  $n$ , let  $\gamma_n^1 > \gamma$  and  $c_n^1 \leq c_n$  be such that  $c_n^1 \Vdash \gamma_n^1 \in \dot{C}_n$ . Let  $\gamma^1 = \sup_n \gamma_n^1$ . Similarly, build increasing  $\langle \gamma^k \mid k < \omega \rangle$  and decreasing  $\langle c_n^k \mid k < \omega \rangle$  in  $\mathbb{C}_{n_0}$ , such that for every  $n, k$   $c_n^k \Vdash (\gamma^{k-1}, \gamma^k) \cap \dot{C}_n \neq \emptyset$ . Let  $c_n^* \in \mathbb{C}_{n_0}$  be stronger than each  $c_n^k$  for  $k < \omega$  and set  $\gamma^* = \sup_k \gamma^k$ . Then each  $c_n^* \Vdash \gamma^* \in \lim \dot{C}_n$ . So, by density, we have that there is a point in  $X$  above  $\gamma$ .  $\square$

Now for every  $\gamma \in X$ , let  $\vec{c}_\gamma$  and  $n_\gamma$  witness that  $\gamma \in X$ . I.e. for all  $n \geq n_\gamma$ ,  $\vec{c}_\gamma(n) \Vdash_{\mathbb{C}_{n_0}} \gamma \in \lim \dot{C}_n$ . We may also assume that each  $\vec{c}_\gamma(n)$  decides  $\dot{C}_n \cap \gamma$ , and let  $Z_{\gamma,n}$  be that value. Define

$$\mathcal{C}_\gamma := \{Y \subset \gamma \mid Y \text{ is a club in } \gamma, (\exists k \geq n_\gamma)(Y \subset \bigcap_{n \geq k} Z_{\gamma,n})\}.$$

Note that for every  $\gamma \in X$ ,  $\mathcal{C}_\gamma$  is nonempty. And for every  $\gamma \in \text{cof}(\omega) \setminus X$ , set  $\mathcal{C}_\gamma = \{Y\}$ , where  $Y$  is any cofinal  $\omega$  sequence in  $\gamma$ .

**Proposition 4.2.**  *$\langle \mathcal{C}_\gamma \mid \gamma \in X \cup \text{cof}(\omega) \rangle$  is a  $\square_{\kappa^{+\omega}}^*$  sequence.*

*Proof.* First note that if  $Y \in \mathcal{C}_\gamma$  for  $\gamma \in X$ , then for some  $n$ ,  $Y \subset Z_{\gamma,n}$ , which has order type less than  $\kappa^{+n+2}$ . So  $|\mathcal{C}_\gamma| \leq |\bigcup_n \mathcal{P}(\kappa^{+n+2})| = \kappa^{+\omega}$ .

Now suppose that  $\gamma \in X$ ,  $Y \in \mathcal{C}_\gamma$  and  $\beta \in \lim(Y)$ . We have to show that  $Y \cap \beta \in \mathcal{C}_\beta$ .

**Claim 4.3.**  *$\beta \in X$ .*

*Proof.* Let  $k \geq n_\gamma$  be such that  $Y \subset \bigcap_{n \geq k} Z_{\gamma,n}$ . For  $n \geq k$ ,  $\vec{c}_\gamma(n)$  decides  $\dot{C}_n \cap \gamma$ , and so it also decides  $\dot{C}_n \cap \beta$ . Since  $Y$  is unbounded in  $\beta$ , it follows that for  $n \geq k$ ,  $Z_{\gamma,n}$  is unbounded in  $\beta$ . I.e.  $\vec{c}_\gamma(n)$  decides that  $\beta$  is a limit point of  $\dot{C}_n$ . So,  $\beta \in X$ .  $\square$

Let  $n^* \geq \max(k, n_\beta)$  be such that for all  $n \geq n^*$ ,  $\vec{c}_\beta(n)$  and  $\vec{c}_\gamma(n)$  are compatible. Then for all  $n \geq n^*$ ,  $Z_{\gamma,n} \cap \beta = Z_{\beta,n}$ . So  $Y \cap \beta \subset \bigcap_{n \geq n^*} Z_{\gamma,n} \cap \beta = \bigcap_{n \geq n^*} Z_{\beta,n}$ . So,  $Y \cap \beta \in \mathcal{C}_\beta$ .  $\square$

**Corollary 4.4.** *In  $V[E][G]$ , there is a weak square sequence at  $\aleph_\omega$ .*

We conclude with some remarks and questions.

**Remark 4.5.** *Although forcing with  $\mathbb{D}_0$ , adds a weak square sequence, in  $V^\mathbb{D}$ , we have  $\text{Refl}(\omega, \mu)$ , and so  $\neg \square_{\kappa^+, \lambda}$  for all  $\lambda < \kappa^+$ .*

**Remark 4.6.** *Suppose that the ground model  $V$  is obtained by doing Laver preparation over  $V_0$ . Then in  $V[E][G]$ , we can define a scale in  $\prod_n \lambda^{+n+1}$ , such that  $S := \{\gamma < \mu \mid \gamma \text{ is a bad point in } V_0\}$  is stationary in  $V[E][G]$ . However, points in  $S$  are not necessarily bad in  $V$ .*

**Remark 4.7.** *Arguing as in [6], in  $V[E][G]$  we can define a very good scale of length  $\mu^+$  in  $\prod_n \lambda^{+n+3}$ . Using this scale standard arguments (see for example [2]) show that  $\text{Refl}(\omega, \mu)$  fails in  $V[E][G]$ .*

**Question 1.** *How much failure of square can we get in the final generic extension?*

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