MATH 320 NOTES, WEEK 8

2.2 The matrix representation of a linear transformation

Recall the definition of a matrix representation of a linear transformation:

Definition 1. *Let V, W be vector spaces over F,* $\dim(V) = n$, $\dim(W) = k$, *$\alpha = \{x_1, ..., x_n\}$ an ordered basis for V, $\beta = \{y_1, ..., y_k\}$ an ordered basis for W, and let $T : V \to W$ be a linear transformation. Define $[T]_\alpha^\beta$ to be the following matrix in $M_{k,n}(F)$: for $1 \leq i \leq n$, the i-th column of $[T]_\alpha^\beta$ is $[T(x_i)]_\beta$.*

The key property of the matrix representation defined as above, is that

$$[T]_\alpha^\beta [x]_\alpha = [T(x)]_\beta.$$

Example 1. Suppose that $A \in M_{n,n}(F)$, and consider the linear transformation $L_A : F^n \to F^n$ defined by $L_A(x) = Ax$. Let e be the standard basis of F^n, then $[L_A]_e = A$.

Example 2. Let $D : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be the derivative operator $D(p) = p'$. Let $e = \{1, x, x^2, x^3\}$ and $\beta = \{1 + x, x, x^2 + x^3, x^3\}$ Then

(1) $[D]_e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(2) $[D]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & -3 & -3 \end{pmatrix}$

We have shown that in the finite dimensional case, linear transformations can be identified as matrices. And as matrices are themselves vector spaces, we can expect that linear transformations form vector spaces, too. It turns out this is indeed the case, regardless of the dimension.

Definition 2. *Let V, W be two vector spaces over a field F. Define $\mathcal{L}(V, W)$ to be the set of all linear transformations $T : V \to W$.*

Next we will prove that $\mathcal{L}(V, W)$ is a vector space over F and then compute its dimension when V and W are finite dimensional.

Lemma 3. *$\mathcal{L}(V, W)$ is a vector space over F with the following operations. For $T, U \in \mathcal{L}(V, W)$ and $c \in F$, set*

(1) *$T + U : V \to W$ is given by $(T + U)(x) = T(x) + U(x)$,*
(2) *$cT : V \to W$, is given by $(cT)(x) = cT(x)$.*

Proof. First, we will show that the two operations are well defined. Let $T, U \in \mathcal{L}(V, W)$ and $c \in F$. We have to show that $T + U$ and cT are also linear.

To that end, let $x, y \in V$ and $d \in F$. Then

$$(T+U)(dx+y) = T(dx+y)+U(dx+y) = dT(x)+T(y)+dU(x)+U(y) =$$
$$= d(T(x) + U(x)) + T(y) + U(y) = d(T + U)(x) + (T + U)(y).$$

Also,

$$(cT)(dx+y) = cT(dx+y) = c(dT(x)+T(y)) = dcT(x)+cT(y) = d(cT)(x)+(cT)(y).$$

Next we check the axioms. Axioms 1,2,5-8 hold because they hold in W. We leave that as an exercise. For axiom 3, recall that the zero transformation T_0 is given by $T_0(x) = \vec{0}$ for all $x \in V$. Then for any other linear transformation T, $T + T_0 = T_0 + T = T$, and so axiom 3 holds. For axiom 4, let T be linear. Then define $-T : V \to W$ by $(-T)(x) = -T(x)$ for all $x \in V$. Then $-T + T = T_0$, and so axiom 4 holds. $\square$

Next we consider the case when V and W are finite dimensional.

Lemma 4. *Suppose that V, W are finite dimensional vector space over F, with $\dim(V) = n, \dim(W) = k$. Let β be a basis for V and γ be a basis for W. Define $\phi : \mathcal{L}(V, W) \to M_{k,n}(F)$ by*

$$\phi(T) = [T]^{\gamma}_{\beta}.$$

Then ϕ is a one-to-one, onto linear transformation.

Proof. First we show linearity. Let $T, U \in \mathcal{L}(V, W)$ and $c \in F$. We want to show that $\phi(cT + U) = c\phi(T) + \phi(U)$. Computing the left and right hand sides, we get:

- LHS= $\phi(cT + U) = [cT + U]^{\gamma}_{\beta}$;
- RHS= $c\phi(T) + \phi(U) = c[T]^{\gamma}_{\beta} + [U]^{\gamma}_{\beta}$

Both of these are k by n matrices. We will show that they have the same columns.

Let $\beta = \{x_1, x_2, ..., x_n\}$ and fix i, $1 \leq i \leq n$. Then, by definition of matrix representation, the ith column of the left hand side is

$$[(cT + U)(x_i)]_{\gamma} = [cT(x_i) + U(x_i)]_{\gamma} = c[T(x_i)]_{\gamma} + [U(x_i)]_{\gamma}.$$

Then first equality is by definition of $cT + U$ and the second equality is by linearity of ϕ_{γ} (that was one of your homework problems).

On the other hand, the ith column of the left hand side, by definition of matrix representations, is exactly

$$c[T(x_i)]_{\gamma} + [U(x_i)]_{\gamma}.$$

It follows that the two are equal, and so ϕ is linear.

Next, we show that ϕ is one-to-one: suppose that $T \in \ker(\phi)$. Then

$$\phi(T) = [T]_\beta^\gamma = O,$$

i.e. it is the zero matrix. So, for every $1 \leq i \leq n$, $[T(x_i)]_\gamma = \vec{0}$. Then for every i, $T(x_i) = \vec{0}$.

Since T sends every vector in the basis β to $\vec{0}$, it follows that for all x, $T(x) = 0$. So, $T = T_0$, i.e. the zero transformations. It follows that $\ker \phi = \{\vec{0}_{\mathcal{L}(V,W)}\}$, and so ϕ is one-to-one.

Finally, we show that ϕ is onto: let $A \in M_{k,n}(F)$. We have to find a linear transformation $T : V \to W$, such that $[T]_\beta^\gamma = A$. To that end, denote the (i, j)-th entry of A by a_{ij}, and denote $\gamma = \{y_1, ..., y_k\}$. let

- $w_1 = a_{11}y_1 + a_{21}y_2 + ...a_{k1}y_k$,
- $w_2 = a_{12}y_1 + a_{22}y_2 + ...a_{k2}y_k$,
- ...
- $w_n = a_{1n}y_1 + a_{2n}y_2 + ...a_{kn}y_k$.

Let $T : V \to W$ be the unique linear transformation such that

$$T(x_1) = w_1, T(x_2) = w_2, ..., T(x_n) = w_n.$$

Then the ith column of $[T]_\beta^\gamma$ is $[T(x_i)]_\gamma = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \cdots \\ a_{in} \end{pmatrix}$

But this is exactly the ith column of the matrix A. It follows that $A = [T]_\beta^\gamma = \phi(T)$. So, ϕ is onto.

□

Corollary 5. *If V, W are finite dimensional vector spaces over F, with $\dim(V) = n, \dim(W) = k$, then $\dim(\mathcal{L}(V, W)) = nk$.*

Proof. Take bases β of V and γ for W, and define ϕ as above. Then since ϕ is a one-to-one, onto linear transformation, by the dimension theorem it follows that $\dim(\mathcal{L}(V, W)) = \dim(M_{k,n}(F)) = nk$.

□

2.3 Composition of linear transformations

Lemma 6. *Suppose that V, W and Z are vector spaces over F and that $T : V \to W$ and $U : W \to Z$ are linear transformations. Then the composition $UT : V \to Z$ defined by*

$$UT(x) = U(T(x))$$

is also a linear transformation.

Proof. Let $x, y \in V$ and $c \in F$. Then $UT(cx + y) = U(T(cx + y)) = U(cT(x) + T(y)) = c(U(T(x))) + U(T(y)) = c(UT)(x) + (UT)(y)$.

□