Practice problems for the 430 final

**Problem 1.** Recall that the soundness theorem says that if $\Gamma \vdash \phi$, then $\Gamma \models \phi$. Prove the soundness theorem, by assuming without proof that every logical axiom is valid (i.e. holds in every model).

**Problem 2.** Suppose that $\Sigma$ is a set of sentences that has arbitrarily large finite models. Show that $\Sigma$ has an infinite model.

**Problem 3.** Let $L = \{<\}$ be a first order language where $<$ is a binary relation symbol.

1. Show that the theory of infinite linear orders is axiomatizable.
2. Show that the theory of infinite linear orders is not finitely axiomatizable.

For the following problem you can make use of the formula $\phi_{\text{code}}(x)$ and that $\mathfrak{A} \models \phi_{\text{code}}[a]$ iff $a$ codes a sequence. You can also use “$\text{lh}(a)$” to refer to the length of the sequence coded by $a$.

**Problem 4.** Write down a $\Sigma_1$ formula $\phi_{\text{exp}}(e,n,k)$, such that $\mathfrak{A} \models \phi_{\text{exp}}[e,n,k]$ iff $e^n = k$. Then write down a $\Pi_1$ formula $\phi'_{\text{exp}}(e,n,k)$ equivalent to $\phi_{\text{exp}}(e,n,k)$.

For the problems below, recall that any model of PA is an end extension of $\mathfrak{A}$, and as a corollary we get that if $\phi$ is $\Sigma_1$ and true in $\mathfrak{A}$, then $\phi$ is true in any model of PA, and so $PA \vdash \phi$. Recall also that we defined a $\Sigma_1$ formula $\phi_{\text{prov}-\theta}(a,b)$ such that, $\mathfrak{A} \models \phi_{\text{prov}-\theta}[a,b]$ iff $T_\theta \vdash \phi_a(b)$. Then setting $e$ to be the Gödel number of $\neg \phi_{\text{prov}-\theta}(v,v)$, we defined

$$\sigma := \neg \phi_{\text{prov}-\theta}(e,e)$$

i.e. $\sigma$ is exactly $\phi_e(e)$. Note that since $\phi_{\text{prov}-\theta}$ is $\Sigma_1$ and $\sigma$ is defined by taking its negation, we have that $\sigma$ is $\Pi_1$.

**Problem 5.**

1. Show that $\mathfrak{A} \models \sigma$ iff $T_\theta \nvdash \sigma$.
2. Prove that $T_\theta \nvdash \sigma$ (and so $\mathfrak{A} \nmodels \sigma$). (Here you will use that $\sigma$ is $\Sigma_1$.)
3. Prove that $\sigma$ is not $\Sigma_1$ (and so not $\Delta_1$).

**Problem 6.** Show that there is no formula $\phi_{\text{true}}(x,y)$ such that

$$\mathfrak{A} \models \phi_{\text{true}}[a,b] \text{ iff } \mathfrak{A} \models \phi_a[b].$$

*Hint: suppose for contradiction that such a formula exists. Define a sentence $\sigma'$ is a similar fashion as $\sigma$ from above. I.e. informally, $\sigma'$ will be the sentence “I am not true”.

Also make sure you know:

- The statements of Soundness, Compactness, Completeness theorems.
- How to show Compactness assuming Completeness; how to prove Soundness.
- The statements and proofs of the First and Second Incompleteness theorems.
• The definition of $\Delta_0$, $\Sigma_1$, $\Pi_1$, $\Delta_1$ formulas.
• How to show that there exists a countable nonstandard model of PA, i.e. a model $\mathcal{B}$ which is not isomorphic to $\mathcal{A}$. Here the proof uses Compactness, to construct a model with an “infinite” element, i.e. an element $b \in \mathcal{B}$ such that for all $n \in \mathbb{N}$, $S_B^n(0_\mathcal{B}) <_\mathcal{B} b$.
• How to prove that every model of PA is an end extension of $\mathcal{A}$.
• As a corollary to the above: that if $\phi$ is $\Sigma_1$ and $\mathcal{A} \models \phi$, then any model of PA $\mathcal{B} \models \phi$. Note that this fact is a key ingredient when showing the First Incompleteness theorem.