Problem 1.

(1) Suppose \( A \neq \emptyset \) and there is a one-to-one function \( f : A \rightarrow B \). Show that there is a surjective (i.e. onto) function \( g : B \rightarrow A \).

(2) Suppose \( B \) can be well-ordered and there is a surjective function \( g : B \rightarrow A \). Show that there is a one-to-one function \( f : A \rightarrow B \).

Problem 2. In ZF\(^-\) prove the Schröder-Bernstein theorem i.e. that if \( A \preceq B \) and \( B \preceq A \) implies that \( A \approx B \).

Hint: Suppose \( f : A \rightarrow B \) and \( g : B \rightarrow A \) are one-to-one. Set \( A_0 = A \), \( B_0 = B \), \( A_{n+1} = g^{-1}(A_n) \), \( B_{n+1} = f^{-1}(B_n) \), \( A_{\infty} = \bigcap_n A_n \), \( B_{\infty} = \bigcap_n B_n \). Let \( h(x) \) be \( f(x) \) if \( x \in A_\infty \cup \bigcup_n (A_{2n} \setminus A_{2n+1}) \). Otherwise let \( h(x) \) be \( g^{-1}(x) \). Show that \( h \) is well defined and \( h : A \rightarrow B \) is one-to-one and onto.

Problem 3. Show that for infinite cardinals \( \kappa \geq \lambda \),

\[
|\{X \subset \kappa : |X| = \lambda\}| = \kappa^\lambda.
\]

Problem 4. Let \( \lambda \) be an infinite cardinal and \( \kappa \) be any cardinal. Show that

\[
\kappa^{<\lambda} = \sup\{\kappa^\theta : \theta < \lambda, \theta \text{ is a cardinal}\}.
\]

Problem 5. Assume CH (but not GCH). Show that for every natural number \( n > 0 \), \( \omega_n^\omega = \omega_n \).