MATH 504 HOMEWORK 9

Due Friday, December 7.
This homework is worth two regular homework sets.

**Problem 1.** Let $T$ be a normal Suslin tree, and let $(\mathbb{P}_T, \prec) = (T, \triangleright)$. Show that although $\mathbb{P}_T$ has the countable chain condition, $\mathbb{P}_T \times \mathbb{P}_T$ does not. Hint: for every $x \in T$, pick two immediate successors $p_x, q_x$ of $x$. Look at the set $\{(p_x, q_x) \mid x \in T\} \subset \mathbb{P}_T \times \mathbb{P}_T$.

**Problem 2.** Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are two c.c.c. posets. Show that the following are equivalent:

1. $\mathbb{P} \times \mathbb{Q}$ is c.c.c;
2. $1_\mathbb{P} \forces \mathbb{Q}$ is c.c.c;
3. $1_\mathbb{Q} \forces \mathbb{P}$ is c.c.c;

**Problem 3.** Let $\mathbb{P}$ be a poset such that for every $p \in \mathbb{P}$, there are incompatible $q, r \leq p$. Suppose $G$ is $\mathbb{P}$-generic. Show that $G \times G$ is not $\mathbb{P} \times \mathbb{P}$-generic.

**Problem 4.** Let $\mathbb{P} \in V$ be a poset, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ name for a poset, i.e. $1_\mathbb{P} \forces \dot{\mathbb{Q}}$ is a poset. Suppose that $G$ is $\mathbb{P}$-generic over $V$, and that $H$ is $\dot{\mathbb{Q}}_G$-generic over $V[G]$. Show that $K := G \ast H = \{(p, \dot{q}) \mid p \in G, \dot{q}_G \in H\}$ is $\mathbb{P} \ast \dot{\mathbb{Q}}$-generic over $V$.

**Problem 5.** Suppose that $\mathbb{P} \ast \dot{\mathbb{Q}}$ has the $\kappa$-chain condition. Show that $\mathbb{P}$ has the $\kappa$-chain condition, and $1_\mathbb{P} \forces \" \dot{\mathbb{Q}} \text{ has the } \kappa\text{-chain condition} \"$.

Remark 1. The converse is also true.

**Problem 6.** Suppose that $\mathbb{P}$ is $\kappa$-distributive, and $1_\mathbb{P} \forces \" \dot{\mathbb{Q}} \text{ is } \kappa\text{-distributive} \"$. Show that that $\mathbb{P} \ast \dot{\mathbb{Q}}$ is $\kappa$-distributive.

**Problem 7.** Suppose that $\mathbb{P} = (\mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \mid \beta < \alpha)$ is an iteration of length $\alpha$, and $\mathbb{P}_\beta = \mathbb{P}_\alpha \upharpoonright \beta$ for $\beta < \alpha$. Show that if $G_\alpha$ is $\mathbb{P}_\alpha$-generic and we define $G_\beta := \{p \upharpoonright \beta \mid p \in G_\alpha\}$, then $G_\beta$ is $\mathbb{P}_\beta$-generic and $V[G_\beta] \subset V[G_\alpha]$. (Note that we are not assuming finite support here, just the general definition of an $\alpha$-iteration)

**Problem 8.** Let $\kappa$ be a regular uncountable cardinal and $\mathbb{P}$ be a $\kappa$-closed poset. Show that $\mathbb{P}$ preserves stationary subsets of $\kappa$, i.e. if $S \subset \kappa$ is stationary in the ground model, then $S$ remains stationary in any $\mathbb{P}$-generic extension.

Hint: Given $S$, a name $\dot{C}$, and $p$, such that $p \forces \" \dot{C} \text{ is a club subset of } \kappa \"$, show there is a sequence in the ground model $(p_\alpha, \gamma_\alpha \mid \alpha \leq \kappa)$, such that:

- $(p_\alpha \mid \alpha < \kappa)$ is a decreasing sequence below $p$,
- $(\gamma_\alpha \mid \alpha < \kappa)$ is a club in $\kappa$,
Problem 9. Let \( S \subset \omega_1 \) be a stationary set. Define \( \mathbb{P} := \{ p \in S \mid p \text{ is closed and bounded} \} \), and set \( p \leq q \) if \( p \) end extends \( q \), i.e. for some \( \alpha \), \( p \cap \alpha = q \).

1. Show that \( \mathbb{P} \) is \( \omega \)-distributive, i.e. if \( p \models \dot{f} : \omega \to \text{ON} \), then there is some \( q \leq p \) and a function \( g \) in the ground model, such that \( q \models \dot{f} = \dot{g} \). Note that this implies that \( \mathbb{P} \) adds no countable subsets of \( \omega_1 \), and hence it preserves \( \omega_1 \).

2. What is the best chain condition for \( \mathbb{P} \)? Justify your answer. Use that and the above to show that \( \mathbb{P} \) preserves all cardinals.

3. Suppose that \( T := S \setminus \omega_1 \) is also stationary. Let \( G \) be a \( \mathbb{P} \)-generic filter. Show that in \( V[G] \), \( T \) is nonstationary.

Remark 2. The above is an example of a forcing that destroys a stationary set, without collapsing cardinals. On the other hand you cannot use forcing to destroy a club set. More precisely, if \( V \subset W \) are two models of set theory and \( V \models \text{“}D \text{ is club in } \kappa \text{”} \), then \( W \models \text{“}D \text{ is club in } \kappa \text{”} \). Note that in the above problem it was important that \( S \) was stationary; i.e. you cannot add a new club through a nonstationary set.