Next we will define the Mitchell forcing, $\mathbb{M}$. Recall that the Cohen forcing to add $\kappa$ many reals, $Add(\omega, \kappa)$, consists of partial finite functions from $\kappa \times \omega \to \{0, 1\}$. For $\alpha < \kappa$ and $p \in Add(\omega, \kappa)$, $p \upharpoonright \alpha$ will denote $p \upharpoonright \alpha \times \omega$; this is a condition in $Add(\omega, \alpha)$. Also, $Add(\omega_1, 1)$ is the Cohen poset to add one subset of $\omega_1$, i.e. conditions are countable partial functions from $\omega_1 \to \{0, 1\}$. We use the following notation:

- $\mathbb{P} := Add(\omega, \kappa)$,
- for $\alpha < \kappa$, $\mathbb{P}_\alpha := Add(\omega, \alpha) := \mathbb{P} \upharpoonright \alpha$.

Conditions in the Mitchell forcing $\mathbb{M}$ are pairs of the form $(p, q)$, such that

1. $p \in \mathbb{P}$,
2. $\dom(q)$ is a countable subset of $\kappa$.
3. for all $\alpha \in \dom(q)$, $1_{\mathbb{P}_\alpha} \models q(\alpha) \in Add(\omega_1, 1)$.

**Lemma 1.** $\mathbb{M}$ projects to $\mathbb{P}$. 

\textit{Proof.} Let $\pi : \mathbb{M} \to \mathbb{P}$ be $\pi(p, q) = p$. This is a projection. \hfill \Box

**Lemma 2.** $\mathbb{M}$ has the $\kappa$-c.c., and so preserves cardinals greater than or equal to $\kappa$.

\textit{Proof.} Suppose that $\{(p_\eta, q_\eta) \mid \eta < \kappa\}$ are conditions in $\mathbb{M}$. By the $\Delta$-system lemma, applied to the domains of each $p_\eta$ and $q_\eta$, there is an unbounded $I \subset \kappa$, a finite $d \subset \kappa \times \omega$, and countable $b < \kappa$, such that for every $\eta, \delta \in I$, $\dom(p_\eta) \cap \dom(p_\delta) = d$ and $\dom(q_\eta) \cap \dom(q_\delta) = b$.

There are only finitely many possibilities for $p_\eta \upharpoonright d$, so by taking another unbounded $J \subset I$, we may assume that for all $\eta, \delta \in J$, $p_\eta \cup p_\delta$ is a well defined function. Also, for every $\eta \in J$ and $\gamma \in b$, $q_\eta(\gamma)$ is a $\mathbb{P}_\gamma$ name for an element in $Add(\omega_1, 1)$. Since $|P_\gamma| < \kappa$, there are less than $\kappa$ many possibilities for the value of $q_\eta(\gamma)$. So, there are $\eta < \delta$, both in $J$, such that $(p_\eta, q_\eta)$ and $(p_\delta, q_\delta)$ are compatible. \hfill \Box

**Lemma 3.** $\mathbb{M}$ projects to $Add(\omega, \alpha) \ast Add(\omega_1, 1)$

\textit{Proof.} Let $\pi : \mathbb{M} \to Add(\omega, \alpha) \ast Add(\omega_1, 1)$ be $\pi(p, q) = (p \upharpoonright \alpha, q(\alpha))$. We claim that this is a projection. If $(p', q') \leq (p, q)$, then $p' \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} p \upharpoonright \alpha$, and if $\alpha \in \dom(q) \subset \dom(q')$, then $p' \upharpoonright \alpha \models q'(\alpha) \leq q(\alpha)$. (If $\alpha \notin \dom(q)$, then $q'(\alpha) = q(\alpha) = 0 = 1_{Add(\omega_1, 1)}$). So, $\pi$ is order preserving.

For the second requirement of being a projection, suppose that $(r, s) \leq (p, q) = (p \upharpoonright \alpha, q(\alpha))$. Let $p' = p \cup r$. Since $r \leq p \upharpoonright \alpha$, $p'$ is a well defined function, and so $p' \in \mathbb{P}$. Let $q'$ be such that $\dom(q) = \dom(q')$.
Lemma 4. \( \text{Add}(\omega, \alpha) \ast \text{Add}(\omega_1, 1) \) collapses \( \alpha \) to \( \omega_1 \).

Proof. Let \( G \) be \( \text{Add}(\omega, \alpha) \)-generic. In \( V[G] \), let \( \langle r_i \mid i < \alpha \rangle \) be distinct elements of \( 2^\omega \). Let \( H \) be \( \text{Add}(\omega_1, 1) \)-generic over \( V[G] \). In \( V[G][H] \) define \( h : \alpha \to \omega_1 \), by setting \( h(i) \) to be the least limit \( \beta < \omega_1 \) such that there is \( p \in H \) with:

- \( \beta + n \in \text{dom}(p) \) for all \( n \), and
- \( \text{for every } n < \omega, \ p(\beta + n) = r_i(n) \).

Then \( h \) is a one-to-one function from \( \alpha \) to \( \omega_1 \).

□

Corollary 5. \( M \) collapses every uncountable \( \alpha < \kappa \) to \( \omega_1 \).

Definition 6. \( Q := \{ q \mid \text{dom}(q) \subset \kappa, |\text{dom}(q)| < \omega_1, (\forall \alpha \in \text{dom}(q)) 1_{P_\alpha} \models q(\alpha) \in \text{Add}(\omega_1, 1) \} \) (i.e. the second coordinates of conditions in \( M \)). For \( q_1, q_2 \in Q \), we set \( q_2 \leq_{Q} q_1 \) iff:

- \( \text{dom}(q_2) \supset \text{dom}(q_1) \),
- \( \text{for all } \alpha \in \text{dom}(q_1), 1_{P_\alpha} \models q_2(\alpha) \leq q_1(\alpha) \).

Next we consider the product \( P \times Q \). This has the same underlying set as \( M \), but the ordering is different. For example note that \( (p', q') \leq_{P \times Q} (p, q) \) implies that \( (p', q') \leq_{M} (p, q) \), but the converse fails. Actually, \( (p', q') \leq_{P \times Q} (p, q) \) iff \( p' \leq_{P} p \) and \( (1, q') \leq_{M} (1, q) \).

Lemma 7. \( P \times Q \) projects to \( M \).

Proof. Let \( \pi : P \times Q \to M \) be the identity, i.e. \( \pi(p, q) = (p, q) \). We will show that this is a projection. Suppose that \( (p', q') \leq_{P \times Q} (p, q) \). That means that \( p' \leq_{P} p \) and \( q' \leq_{Q} q \). The latter implies that \( \text{dom}(q') \supset \text{dom}(q) \) and for every \( \alpha \in \text{dom}(q) \), \( 1_{P_\alpha} \models q'(\alpha) \leq_{\text{Add}(\omega_1, 1)} q(\alpha) \). Therefore, \( p' \upharpoonright \alpha \models q'(\alpha) \leq_{\text{Add}(\omega_1, 1)} q(\alpha) \), and so \( (p', q') \leq_{M} (p, q) \). So \( \pi \) is order preserving.

To show the second requirement of being a projection, suppose that \( (p', q') \leq_{M} (p, q) = \pi(p, q) \). That means that \( p' \leq_{P} p \); \( \text{dom}(q) \subset \text{dom}(q') \) and for every \( \alpha \in \text{dom}(q) \), \( p' \upharpoonright \alpha \models q'(\alpha) \leq q(\alpha) \). We have to define a condition \( (r, s) \in P \times Q \), such that:

- \( (r, s) \leq_{P \times Q} (p, q) \), and
- \( \pi(r, s) = (r, s) \leq_{M} (p', q') \).

Set \( r = p' \). Let \( s \in Q \) be such that \( \text{dom}(s) = \text{dom}(q') \). For every \( \alpha \in \text{dom}(q) \), we define \( s(\alpha) \) to be a \( P_\alpha \)-name for \( \text{Add}(\omega_1, 1) \), such that \( p' \upharpoonright \alpha \models s(\alpha) = q'(\alpha) \) and if \( t \in P_\alpha \) is incompatible with \( p' \upharpoonright \alpha \), then \( t \not\models_{P_\alpha} s(\alpha) = q(\alpha) \). We can always cook up such a name \(^1\). Then we have:

1. \( 1_{P_\alpha} \models s(\alpha) \leq q(\alpha) \)
2. \( p' \upharpoonright \alpha \models s(\alpha) \leq q'(\alpha) \).

\(^1\)Formally, \( s(\alpha) = \{ (\sigma, t) \mid t \leq p' \upharpoonright \alpha, t \models \sigma \in q'(\alpha) \} \) or \( t \perp p' \upharpoonright \alpha, t \models \sigma \in q(\alpha) \).
The first item guarantees that \((p', s) \leq_{\mathbb{P} \times \mathbb{Q}} (p, q)\), and the second item gives that \((p', s) \leq_{\mathbb{M}} (p', q')\). That concludes the proof that \(\pi\) is a projection.

\[\square\]

**Lemma 8.** \(Q\) is \(\omega_1\)-closed.

**Proof.** Suppose that \(\langle q_n \mid n < \omega \rangle\) is a decreasing sequence of conditions in \(Q\). Define a lower bound \(q\) as follows. Set \(\text{dom}(q) = \bigcup n \text{ dom}(q_n)\). This is still countable. For every \(\alpha \in \text{dom}(q)\), let \(n_{\alpha}\) be such that \(\alpha \in \text{dom}(q_{n_{\alpha}})\). Since the sequence is decreasing, \(\alpha \in \text{dom}(q_{n_{\alpha}})\) for all \(n \geq n_{\alpha}\) and:

- \(1_{\mathbb{P}^\alpha} \vdash "\langle q_n(\alpha) n_{\alpha} \leq n < \omega \rangle\) is a decreasing sequence in Add(\(\omega_1, 1\)"");
- \(1_{\mathbb{P}^\alpha} \vdash "\text{Add}(\omega_1, 1)\) is \(\omega_1\)-closed. It follows that \(1_{\mathbb{P}^\alpha} \vdash "\exists x \in \text{Add}(\omega_1, 1)(\forall n \geq n_{\alpha})(x \leq q_n(\alpha))"\) for all \(n \geq n_{\alpha}^2\). Set \(q(\alpha) = s\).

\[\square\]

**Easton lemma:** Suppose that \(\mathbb{P}\) is \(\tau\)-c.c. and \(\mathbb{Q}\) is \(\tau\)-closed. Then \(1_{\mathbb{P}} \not\vdash \mathbb{Q}\) does not add any new sequences of length less than \(\tau\).

**Lemma 9.** \(M\) preserves \(\omega_1\)

**Proof.** Let \(G_M\) is \(M\)-generic and \(G_P \times G_Q\) be \(\mathbb{P} \times \mathbb{Q}\)-generic, such that \(V[G_M] \subset V[G_P][G_Q]\). Since \(\mathbb{P}\) has the countable chain condition, \(\omega_1\) is still a cardinal in \(V[G_P]\). By Easton’s lemma, every countable sequence from \(V[G_P][G_Q]\) is actually in \(V[G_P]\). So \(\omega_1\) remains a cardinal in \(V[G_P][G_Q]\). (Otherwise there would have been a new countable sequence \(\omega \to \omega_1\).

Since \(V[G_M] \subset V[G_P][G_Q]\), \(\omega_1\) is a cardinal in \(V[G_M]\).

\[\square\]

**Theorem 10.** If \(G_M\) is \(M\)-generic, then \(V[G_M] \models \kappa = 2^{\omega_1} = \omega_2\).

**Proof.** We already showed that \(\omega_1\) and \(\kappa\) remain cardinals, while everything in between is collapsed. It follows that \(V[G_M] \models \kappa = \omega_2\). Also since the forcing projects to \(\text{Add}(\omega, \kappa)\), \(V[G_M] \models \kappa = 2^\omega\).

\[\square\]

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\(2\)This is due to the fact that if \(p \models (\exists x)\phi(x)\), then there is a name \(a\), such that \(p \vdash \phi(a)\). The proof is a good exercise on forcing.