Lemma 1. (Silver) Let \( \tau < \kappa \) be regular cardinals, such that \( 2^\tau \geq \kappa \). Suppose that \( T \) is a \( \kappa \) tree and \( \mathbb{P} \) is \( \tau^+ \)-closed for some \( \tau < \kappa \). Then forcing with \( \mathbb{P} \) does not add new branches to \( T \).

Proof. Suppose otherwise. Let \( \dot{b} \) be a name for a branch, forced to be such by the empty condition. Working in \( V \), construct \( \langle s_\sigma, p_\sigma \mid \sigma \in 2^{<\tau} \rangle \) by induction on the length of \( \sigma \), such that:

1. Every \( s_\sigma \in T, p_\sigma \in \mathbb{P} \) and \( p_\sigma \Vdash s_\sigma \in \dot{b} \)
2. If \( \sigma_1 \supseteq \sigma_2 \), then \( s_{\sigma_2} \leq_T s_{\sigma_1} \) and \( p_{\sigma_2} \leq p_{\sigma_1} \)
3. For all \( \alpha < \tau \), there is some \( \beta_\alpha < \kappa \), such that for every \( \sigma \in 2^\alpha \)
   \( s_\sigma \in T_{\beta_\alpha} \)
4. For every \( \sigma, s_{\sigma-0} \) and \( s_{\sigma-1} \) are incomparable nodes.

At limit stages we use the closure of \( \mathbb{P} \). More precisely, if \( \alpha \) is limit, \( \sigma \in 2^\alpha \), let \( p_\sigma^* \) be stronger than all \( p_{\sigma\downharpoonright i} \) for \( i < \alpha \). Also let \( \beta_\alpha = \sup_{i<\alpha} \beta_i \). Then let \( p_\sigma \leq p_\alpha^* \) and \( s_\sigma \in T_{\beta_\alpha} \) be such that \( p_\sigma \Vdash s_\sigma \in \dot{b} \). We can find these since \( \dot{b} \) is forced to meet every level.

For the successor stage, suppose that we have constructed \( p_\sigma, s_\sigma \) and \( \beta_\alpha \), where \( \sigma \in 2^\alpha \). Using the splitting lemma, since \( \dot{b} \) is a new branch, we have that there are conditions \( q_{\sigma-0}, q_{\sigma-1} \) stronger than \( p_\sigma \) and nodes \( s_{\sigma-0}, s_{\sigma-1} \), in \( T_{\beta_\alpha+1} \) such that \( q_{\sigma-0} \Vdash s_{\sigma-0} \in \dot{b} \) and \( q_{\sigma-1} \Vdash s_{\sigma-1} \in \dot{b} \).

Now for every \( f \in 2^\tau \), let \( p_f \) be stronger than all \( p_{f\downharpoonright \alpha} \), for \( \alpha < \tau \). Here we use that \( \mathbb{P} \) is \( \tau^+ \)-closed, i.e. sequences of length \( \tau \) have a lower bound. Let \( \beta = \sup_{\alpha<\tau} \beta_\alpha < \kappa \). For every \( f \in 2^\tau \), let \( q_f \leq p_f \) and \( s_f \in T_\beta \) be such that \( q_f \Vdash s_f \in \dot{b} \). Again here we use that \( \dot{b} \) is forced to meet every level (since it is forced to be a branch).

But then by the splitting, we have that whenever \( f \neq g \), \( s_f \neq s_g \). But \( |T_\beta| < \kappa \) and \( 2^\tau \geq \kappa \). Contradiction.

\[ \square \]

Corollary 2. Suppose that \( T \) is an \( \omega_2 \)-tree, \( \mathbb{Q} \) is \( \omega_1 \)-closed, and \( 2^\omega = \omega_2 \). Then \( \mathbb{Q} \) does not add new branches through \( T \).

Let \( G \) be \( \mathbb{M} \)-generic over \( V \). We have to show the tree property in \( V[G] \).
Suppose that \( T \) is a \( \kappa_2 \)-tree in \( V[G] \). Note that since \( \kappa = \kappa_2^{V[G]} \), this means that \( T \) is a \( \kappa \)-tree. We have to show that there is an unbounded branch through \( T \).

Let \( j : V \rightarrow N \) be an elementary embedding with critical point \( \kappa \). Recall that we showed that \( j(\mathbb{M}) \) projects to \( \mathbb{M} \), and so we can lift the embedding to \( j : V[G] \rightarrow N[G^*] \).
Lemma 3. There is a branch $b$ through $T$ in $N[G^*]$ (and so in $V[G^*]$).

Proof. Note that in $N[G^*]$, $j(T)$ is a $j(\kappa)$-tree. Since the sizes of the levels of $T$ are below the critical point, we can also assume that for every level $\alpha < \kappa$, $j(T_\alpha) = T_\alpha = j(T)_\alpha$.

Let $u \in j(T)_\kappa$, i.e. a node on the $\kappa$-th level of $j(T)$. Let $b = \{v \in j(T) \mid v <_{j(T)} u\}$. Since $j(T)$ is a tree, $b$ is a well ordered set. Also, for every $v \in b$, there is some $\alpha < \kappa$, such that $v \in j(T)_\alpha = T_\alpha$. I.e. $b \subset T$. And since the order type of $b$ is $\kappa$, it follows that $b$ is an unbounded branch through $T$.

We want to show that $T$ has a branch in $V[G]$. So far, we have that $T$ has a branch in the bigger model $V[G^*]$. Next we want to use branch preservation lemmas to show that forcing to get from $V[G]$ to $V[G^*]$ could not have added a new branch, i.e. that $b$ must already exists in $V[G]$. The problem is that the forcing to get from $G$ to $G^*$ does not have the nice properties, like closure or Knaster-ness, that are used in the branch preservation lemmas.

To deal with that problem, recall that $M$ is the projection of $\mathbb{P} \times \mathbb{Q}$, where $\mathbb{Q}$ is $\omega_1$-closed in $V$ and $\mathbb{P} = \text{Add}(\omega, \kappa)$. We will show that something similar is true about $j(M)$.

INTERLUDE ON PROJECTIONS:

Suppose that $\mathbb{R}$ and $\mathbb{R}^*$ are any two posets, such that $\mathbb{R}^*$ projects to $\mathbb{R}$. Let $\pi : \mathbb{R}^* \to \mathbb{R}$ be a projection, and suppose that $H$ is $\mathbb{R}$-generic.

Definition 4. In $V[H]$, we set $\mathbb{R}^*/H := \{p \in \mathbb{R}^* \mid \pi(p) \in H\}$.

Lemma 5. If $G$ is $\mathbb{R}^*/H$ generic over $V[H]$, then $G$ is $\mathbb{R}^*$-generic over $V$, and so $V \subset V[H] \subset V[H][G] = V[G]$.

Proof. $G$ is a filter by assumption, so it is enough to show genericity. Suppose that $D \in V$ is a dense subset of $\mathbb{R}^*$. Let $D^* = D \cap \mathbb{R}^*/H$. We claim that $D^*$ is a dense subset of $\mathbb{R}^*/H$. Fix $p \in \mathbb{R}^*/H$. In $V$, let $D_p = \{\pi(q) \mid q \in D, q \leq p\}$.

Claim 6. $D_p$ is dense below $\pi(p)$.

Proof. For any $r \in \mathbb{R}, r \leq \pi(p)$, using that $\pi$ is a projection, let $p' \in \mathbb{R}^*$ be such that $\pi(p') \leq r$. Then let $q \leq p'$ be in $D$. Then $\pi(q) \in D_p$ and $\pi(q) \leq r$.

So, let $r \in D_p \cap H$. Say $r = \pi(q)$ for some $q \in D$, with $q \leq p$. Then $q \in D^*$.

Since $G$ is $\mathbb{R}^*/H$-generic, we have that $D^* \cap G \neq \emptyset$, and so $D \cap G \neq \emptyset$.

Next we give an alternative definition for projections:
Definition 7. \( \mathbb{R}^* \) projects to \( \mathbb{R} \) iff whenever \( G \) is \( \mathbb{R}^* \)-generic, then in \( V[G] \), we can define a \( \mathbb{R} \)-generic filter.

Definition 8. We say that \( \mathbb{R}^* \) is isomorphic to \( \mathbb{R} \) if \( \mathbb{R}^* \) projects to \( \mathbb{R} \) and \( \mathbb{R} \) projects to \( \mathbb{R}^* \).

BACK TO THE MITCHELL THEOREM:

Recall that \( \mathbb{P} \) is \( \text{Add}(\omega, \kappa) \) and \( j : V \to N \) is an elementary embedding with critical point \( \kappa \), and so \( j(\mathbb{P}) = \text{Add}(\omega, j(\kappa)) \). Let \( H \) be \( \mathbb{P} \) generic over \( V \). Define \( \mathbb{P}^* \) to be the set of all conditions \( p \) in \( j(\mathbb{P}) \) such that \( \text{dom}(p) \cap \kappa \times \omega \) is empty. I.e. \( \mathbb{P}^* = \text{Add}(\omega, j(\kappa) \setminus \kappa) \).

Lemma 9. In \( V[H] \), \( \mathbb{P}^* \) is isomorphic to \( j(\mathbb{P})/H = \{ p \in j(\mathbb{P}) \mid p \upharpoonright \kappa \times \omega \in H \} \).

Proof. For the first direction, suppose that \( H^* \) is \( \mathbb{P}^* \)-generic over \( V[H] \). In \( V[H][H^*] \), define \( K := \{ p \in j(\mathbb{P})/H \mid p \upharpoonright j(\kappa) \setminus \kappa \times \omega \in H^* \} \). We want to show that \( K \) is \( j(\mathbb{P})/H \) generic over \( V[H] \). It is a filter because both \( H \) and \( H^* \) are. For genericity, suppose that \( D \in V[H] \) is a dense subset of \( j(\mathbb{P})/H \). Let \( D^* = \{ p \mid j(\kappa) \setminus \kappa \times \omega \mid p \in D \} \). Then \( D \) is a dense subset of \( \mathbb{P}^* \), so there is some \( q \in D \cap H^* \). Let \( p \) witness that \( q \) is in \( D^* \). Then \( p \in D \cap K \).

For the other direction, suppose that \( K \) is \( j(\mathbb{P})/H \) generic over \( V[H] \). In \( V[H][K] \), define \( H^* := K \cap \mathbb{P}^* \). \( H^* \) is a filter because \( K \) is a filter and for any two \( p, q \in \mathbb{P}^* \), \( p \cup q \) is also in \( \mathbb{P}^* \). For genericity, suppose that \( D \in V[H] \) is a dense subset of \( \mathbb{P}^* \). Then the set \( E = \{ p \in j(\mathbb{P})/H \mid p \upharpoonright j(\kappa) \setminus \kappa \times \omega \in D \} \) is a dense subset of \( j(\mathbb{P})/H \). Let \( p \in E \cap K \) and \( q = p \upharpoonright j(\kappa) \setminus \kappa \times \omega \). Then \( q \in D \cap H^* \).

\( \square \)