MATH 512, SPRING 17
HOMEWORK 3, DUE WED MARCH 22

Recall that $\mathbb{P}$ is homogeneous if for every $p, q \in \mathbb{P}$, there are $p' \leq p, q' \leq q$, such that for every generic G with $p' \in G$, there is a generic H with $q' \in H$ such that $V[G] = V[H]$. Equevalently there is an isomorphism between $\{r \in \mathbb{P} \mid r \leq p'\}$ and $\{r \in \mathbb{P} \mid r \leq q'\}$.

Problem 1. *Show that the Levy collapse $Col(\kappa, \lambda)$ is homogeneous.*

For the next two problems, let U be a normal measure on $\mathcal{P}_\kappa(\lambda)$ and define the supercompact Prikry forcing with respect to U, $\mathbb{P}$ as follows. Conditions are of the form $\langle x_0, ..., x_{n-1}, A\rangle$, where each $x_i \in \mathcal{P}_\kappa(\lambda)$, $A \in U$ and for all $i < n-1$, $x_i \subset x_{i+1}$ and $|x_i| < |\kappa \cap x_{i+1}|$ (this is denoted by $x_i \prec x_{i+1}$). Given $q = \langle x_0^q, ..., x_{k-1}^q, A^q\rangle$ and $p = \langle x_0^p, ..., x_{n-1}^p, A^p\rangle$, we have that $q \leq p$ if:

- $k \geq n$, for each $i < n, x_i^q = x_i^p$,
- for each $n \leq i < k$, $x_i^q \in A^p$,
- $A^q \subset A^p$.

Problem 2. *Show that if G is $\mathbb{P}$-generic, then in $V[G]$, every V-regular cardinal τ with $\kappa \leq \tau \leq \lambda$ has cofinality ω.*

Note: $\mathbb{P}$ preserves κ and cardinals above λ; by the above it follows that $(\kappa^+)^{V[G]} = (\lambda^+)^V$.

Problem 3. *Show that $\mathbb{P}$ is homogeneous.*

Problem 4. *Suppose that j is a λ-supercompact embedding with critical point κ. Let $\kappa \leq \tau < \lambda$ be a regular cardinal. Show that $U := \{X \subset \tau \mid j"\tau \in j(X)\}$ is a normal measure on $\mathcal{P}_\kappa(\tau)$. Define $k : Ult(V, U) \to M$ by $k([f]_U) = jf(j"\tau)$. Show that k is elementary and that $j = k \circ j_U$.*

Problem 5. *Suppose that κ is indestructible supercompact. I.e. after κ-directed closed forcing κ remains supercompact. Show that GCH fails.*

Problem 6. *Suppose that $j : V \to M$ is a μ-supercompact embedding with critical point κ. Suppose that $\mathbb{P}$ is a poset of size at most μ. Show that $M[G]^\mu \cap V[G] \subset M[G]$.*

For the next few problems, suppose that in V, κ is supercompact and $2^\kappa = \kappa^+$. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \kappa\rangle$ be an Easton support iteration, such that for each inaccessible α, $\dot{\mathbb{Q}}_\alpha = \dot{Add}(\alpha, \alpha^{++})$, and it is the trivial poset otherwise. Denote $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle$. In particular, $\mathbb{P} = \mathbb{P}_\kappa * \dot{Add}(\kappa, \kappa^{++})$.

Problem 7. *Let $j : V \to M$ is a λ-supercompact embedding where $\lambda \geq \kappa^{++}$. Show that we can lift j to $j' : V[G] \to M^*$, where G is $\mathbb{P}_\kappa$ -generic. Note* *that here you have to analyze what poset is $j(\mathbb{P}_\kappa)$ and in particular find a generic G^* for it such that $j"G \subset G^*$.*

Problem 8. *Let $j' : V[G] \to M^*$ be the lifted embedding from last problem, where G is $\mathbb{P}_\kappa$ -generic. Now show we can lift j' to $j'' : V[G][H] \to M^{**}$, where H is $Add(\kappa, \kappa^{++})$-generic over $V[G]$.*

Problem 9. *Let $j'' : V[G][H] \to M^*$ be the lifted embedding from last problem. Say $j'' \in V[G][H][K]$. Show that there is a normal measure on κ in $V[G][H]$.*

Note: by similar arguments we can actually show that κ is supercompact in $V[G][H]$.

Problem 10. *Suppose that $V \subset W$ are two models of set theory, such that $(\aleph_{\omega+1})^V = (\aleph_2)^W$. Show that $W \models 2^\omega \geq \aleph_2$. (Use that in V, $\aleph_\omega^\omega \geq \aleph_{\omega+1}$.)*