MATH 512 HOMEWORK 2

Due Wednesday, March 6

**Problem 1.** Suppose that $\kappa$ is measurable and $U$ is a normal measure on $\kappa$. Show that $\{\alpha < \kappa \mid \alpha \text{ is an inaccessible cardinal}\} \in U$. Also show that if $\{\tau < \kappa \mid 2^\tau \leq \tau^+\} \in U$, then $2^\kappa \leq \kappa^+$. 

Recall that $\kappa$ has the tree property if every tree with height $\kappa$ and levels of size less than $\kappa$ has an unbounded branch.

**Problem 2.** Show that if $\kappa$ is measurable, then it has the tree property. In particular, measurable cardinals are weakly compact.

*Hint:* given $j : V \rightarrow M$ and a tree $T$ with height $\kappa$ and levels of size less than $\kappa$, look at a node on the $\kappa$-th level of $j(T)$.

**Problem 3.** Suppose that $\kappa$ is measurable and $U_1, U_2$ are two normal measures on $\kappa$ such that $U_1 \in \mathrm{Ult}(V, U_2)$. I.e. $U_1 < U_2$ in the Mitchell order. Show that $\{\tau < \kappa \mid \tau \text{ is a measurable cardinal}\}$ is stationary.

Recall that $\kappa$ is strongly compact if for every $S$, every $\kappa$-complete filter on $S$ can be extended to a $\kappa$-complete ultrafilter.

**Problem 4.** Show that if $\kappa$ is strongly compact, then it is measurable.

Recall that an algebra $B \subset \mathcal{P}(\kappa)$ is $\kappa$-complete if whenever \( \langle A_\alpha \mid \alpha < \tau \rangle \) are sets in $B$ for some $\tau < \kappa$, then so is $\bigcap_{\alpha < \tau} A_\alpha$.

**Problem 5.** Show that the following are equivalent for a cardinal $\kappa$:

1. $\kappa$ is inaccessible and has the tree property;
2. $\kappa$ is inaccessible and for every $\kappa$-complete algebra $B \subset \mathcal{P}(\kappa)$ of size $\kappa$, every $\kappa$-complete filter on $B$ can be extended to a $\kappa$-complete ultrafilter.

*Each of the above gives a characterization for weak compactness.*

**Remark 1.** A third characterization of weak compactness is the following: $\kappa$ is inaccessible and $\mathcal{L}_{\kappa, \omega}$ satisfies the Weak Compactness Theorem.

Here the language $\mathcal{L}_{\kappa, \omega}$ contains $\kappa$ variables, the usual logical connectives and quantifiers, and infinitary connectives $\bigvee_{\alpha < \tau} \phi_\alpha, \bigwedge_{\alpha < \tau} \phi_\alpha$ for any $\tau < \kappa$ (infinite disjunction and conjunction of size less than $\kappa$). $\mathcal{L}_{\kappa, \omega}$ satisfies the weak Compactness Theorem if for every set of sentences $\Sigma \subset \mathcal{L}_{\kappa, \omega}$ with $|\Sigma| \leq \kappa$, if every $S \subset \Sigma$ with $|S| < \kappa$ has a model, then $\Sigma$ has a model.