THE TREE PROPERTY AT $\aleph_{\omega^2+1}$ AND $\aleph_{\omega^2+2}$

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Abstract. We show that from large cardinals it is consistent to have the tree property simultaneously at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ with $\aleph_{\omega^2}$ strong limit.

1. Introduction

The study of the tree property is motivated by the König infinity lemma [7] which states that every infinite finitely branching tree has an infinite path. It is an instance of compactness for countable objects. The tree property at a cardinal κ states that every tree of height κ with levels of size less than κ has a cofinal branch. In particular, the König infinity lemma is just the tree property for $\aleph_0$. A counterexample to the tree property at κ is called a κ-Aronszajn tree after Aronszajn who constructed an $\aleph_1$-Aronszajn tree [8]. An Aronszajn tree is a canonical example of an incompact object. It is natural to ask: "Is it possible to construct a κ-Aronszajn tree for some $\kappa > \aleph_1$ in ZFC?" This is an important special case of an the area of modern set theory which studies the extent of incompactness in ZFC. In this paper we make a step towards proving that the answer to the above question is no.

Compactness at uncountable cardinals is closely connected to large cardinal axioms. In particular by theorems of Erdös and Tarski [4] and Monk and Scott [11], an inaccessible cardinal has the tree property if and only if it is weakly compact. Progress to towards a negative answer to the above question is equivalent to the *consistency* of the tree property at accessible cardinals. For example, the statement "In ZFC one cannot construct an $\aleph_2$-Aronszajn tree" is equivalent to the existence of a model of ZFC where $\aleph_2$ has the tree property. In light of the connection between the tree property and large cardinals it is not surprising that construction of such a model will require large cardinals.

The first result in this direction is due to Mitchell and Silver [10] who showed that there is a model of ZFC where $\aleph_2$ has the tree property if and only if there is a model of ZFC with a weakly compact cardinal. This theorem is an early example of the use of the techniques of forcing and inner model theory to explore the extent of incompactness in ZFC. For the forward direction, starting in a model V where $\aleph_2$ has the tree property, it

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follows that $\aleph_2^V$ is weakly compact in Gödel's constructable universe L. For the reverse direction, the model of ZFC with the tree property at $\aleph_2$ arises as a forcing extension of a model with a weakly compact cardinal.

In the context of this theorem, we can reformulate our original question as "Is there a model of ZFC where every regular cardinal greater than $\aleph_1$ has the tree property?". Of course, we expect to find such a model as a forcing extension of a model with large cardinals. There are many partial results towards constructing a model where the tree property holds everywhere. We list a few results which take a 'bottom up' approach.

(1) (Abraham [1]) Starting from a supercompact cardinal and a weakly compact above it, there is a forcing extension in which the tree property holds simultaneously at $\aleph_2$ and $\aleph_3$, [1].
(2) (Cummings-Foreman [3]) Starting from ω many supercompact cardinals, there is a forcing extension in which the tree property holds at $\aleph_n$ for every $n \geq 2$, [3].
(3) (Neeman [13]) Starting from ω many supercompact cardinals, there is a forcing extension in which the tree property holds at $\aleph_n$ for every $n \geq 2$ and at $\aleph_{\omega+1}$, where $\aleph_\omega$ is strong limit, [12].
(4) (Unger [19]) Starting from an increasing $\omega + \omega$ sequence of supercompact cardinals, there is a forcing extension in which every regular cardinal in the interval $[\aleph_2, \aleph_{\omega\cdot 2})$ has the tree property. We note that in this model $\aleph_\omega$ is not strong limit.

An important constraint on models where some successor cardinals have the tree property comes from a theorem of Specker [16]. If $\kappa^{<\kappa} = \kappa$, then there is a κ^+-Aronszajn tree. In particular, in Mitchell's model for the tree property at $\aleph_2$ we have the failure of the continuum hypothesis. Further, in a model where the tree property holds at every regular cardinal greater than $\aleph_1$ we must have that the generalized continuum hypothesis fails everywhere and that there are no inacessible cardinals. Recall that a failure of GCH at a singular strong limit cardinal is a failure of the *singular cardinals hypothesis* (SCH). Known models for the failure of SCH involve Prikry type forcing. For some time it was open whether the tree property at κ^+ was consistent with the failure of SCH at κ. Gitik and Sharon [6] developed a new type of Prikry forcing to give a partial solution and Neeman [12] used a version of their forcing to give a positive answer. To continue we would like to have the tree property at κ^{++} as well.

Recently, the first author [15] showed that this situation is consistent for a singular cardinal κ of cofinality ω which is a limit of inaccessibles by improving a result of the first author [17]. In this paper, we show that it is possible to make μ into a small singular cardinal. By necessity the construction is different from the one in [17] and [15].

Theorem 1.1. *If there is an increasing ω-sequence of supercompact cardinals with a weakly compact cardinal above, then there is a forcing extension in which $\aleph_{\omega^2}$ is strong limit and both $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ have the tree property.*

The reader should be advised that this paper uses the ideas of many previous papers in a new more technical setting. We make use of the following:

(1) the argument from the first author's paper [14] on the tree property at $\aleph_{\omega^2+1}$,
(2) Mitchell's poset [10] as presented in [1], and
(3) the proof of Lemma 1.3 in the second author's paper [18].

The paper is outlined as follows. In Section 2, we fix some notation, construct the main forcing and show that the extension has the desired cardinal structure. In Section 3 we prove that $\aleph_{\omega^2+1}$ has the tree property in the extension by repeating an argument from [15] in a new context. In Section 4, we show that $\aleph_{\omega^2+2}$ has the tree property in the extension by proving a new preservation lemma, which resembles Lemma 1.3 of [18]. In Section 5, we give some further applications of the preservation lemma from Section 4. In Section 6, we make some concluding remarks and ask some open questions.

2. The main poset

We start in a model V of GCH. Let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of supercompact cardinals and λ be the least weakly compact cardinal greater than $\sup_{n<\omega} \kappa_n$. For ease of notation we set $\kappa_0 = \kappa$, $\nu = \sup_{n<\omega} \kappa_n$ and $\mu = \nu^+$. We choose a supercompactness measure U on $\mathcal{P}_\kappa(\lambda)$. We define a function $\alpha \mapsto \lambda_\alpha$ where λ_α is the least weakly compact greater than the first ω many $< \lambda_\alpha$-supercompact cardinals above α. We also have functions $\alpha \mapsto \alpha_n$ where α_n is the $n^{th} < \lambda_\alpha$-supercompact cardinal above α. There is a set of inaccessible cardinals $Z \subseteq \kappa$ in the projection of U to a normal measure such that for every element α of Z all of the above functions are defined and moreover α is closed under these functions. We define an iteration of Mitchell-like posets and Levy collapses which form the preparation for our construction.

Definition 2.1. *Let $\rho < \sigma < \tau$ be cardinals. Let $\mathbb{P}(\rho,\tau)$ be $\mathrm{Add}(\rho,\tau)$ and define $\mathbb{M}(\rho,\sigma,\tau)$ to be the collection of pairs (p,f) such that $p \in \mathbb{P}(\rho,\tau)$ and f is a partial function with $\mathrm{dom}(f) \subset \tau \setminus \sigma$ a set of successor ordinals, $|\mathrm{dom}(f)| < \sigma$ and for all $\gamma \in \mathrm{dom}(f)$, $f(\gamma)$ is a $\mathbb{P}(\rho,\tau) \restriction \gamma$-name for an element of $\mathrm{Add}(\sigma,1)$. We set $(p_1,f_1) \le (p_0,f_0)$ if $p_1 \le p_0$ in $\mathbb{P}(\rho,\tau)$, $\mathrm{dom}(f_1) \supseteq \mathrm{dom}(f_0)$ and for all $\gamma \in \mathrm{dom}(f_0)$, $p_1 \restriction \gamma \Vdash f_1(\gamma) \le f_0(\gamma)$.*

Note that $\mathbb{M}(\omega,\omega_1,\tau)$ where τ is weakly compact is Mitchell's original poset as described in [1].

Definition 2.2. *Let $\rho < \sigma < \tau$ be cardinals. Define*

$$\mathbb{Q}(\rho,\sigma,\tau) := \{f \mid (1_{\mathbb{P}}, f) \in \mathbb{M}(\rho,\sigma,\tau)\}.$$

For the order, we say that $f' \le_{\mathbb{Q}} f$ iff $(1_{\mathbb{P}}, f') \le_{\mathbb{M}} (1_{\mathbb{P}}, f)$.

We list some standard claims about these posets. For ease of notation we drop the cardinal parameters ρ, σ and τ.

(1) $\mathbb{M}$ is ρ-closed and τ-cc assuming τ is inaccessible.
(2) $\mathbb{Q}$ is σ-closed and τ-cc assuming τ is inaccessible.
(3) There is a projection map from $\mathbb{P} \times \mathbb{Q}$ to $\mathbb{M}$ given by $(p, f) \mapsto (p, f)$.
(4) The natural restriction map from $\mathbb{M}$ to $\mathbb{M} \upharpoonright \alpha$ is a projection map.
(5) For many $\alpha < \tau$ in the extension by $\mathbb{M} \upharpoonright \alpha$, there are posets $\mathbb{M}', \mathbb{P}'$ and $\mathbb{Q}'$ such that $\mathbb{M}'$ is isomorphic to a dense subset of $\mathbb{M}/\mathbb{M} \upharpoonright \alpha$ and $\mathbb{M}'$ is the projection of $\mathbb{P}' \times \mathbb{Q}'$ as in item (3).

We define an iteration $\mathbb{A}_\kappa$ with reverse Easton support where we do non-trivial forcing at $\alpha \in Z$. For α in Z we force with the full support iteration $\mathbb{L}(\alpha)$ of Levy Collapses to make α_n into α^{+n} and in the extension we force $\mathbb{M}(\alpha) \times \mathrm{Add}(\alpha, \lambda_\alpha^+ \setminus \lambda_\alpha)$ where $\mathbb{M}(\alpha) = \mathbb{M}(\alpha, \alpha^{+\omega+1}, \lambda_\alpha)$.

Let G be $\mathbb{A}_\kappa$-generic and let $H = H_0 * H_1 * H_2$ be generic for $\mathbb{L}(\kappa) * \dot{\mathbb{M}}(\kappa) \times \mathrm{Add}(\kappa, \lambda^+ \setminus \lambda)$. It is not difficult to see that in $V[G * H]$, $\kappa_n = \kappa^{+n}$ for all $n < \omega$ (hence $\nu = \kappa^{+\omega}$), μ is preserved and $\lambda = \mu^+ = \kappa^{+\omega+2}$.

For ease of notation we drop the parameter κ from $\mathbb{L}$ and $\mathbb{M}$. Let $j : V \to M$ be the ultrapower map derived from U. We need a careful lifting of j to the model $V[G * H]$.

Lemma 2.3. *In $V[G * H]$, there are generics $G^* * H^*$ for $j(\mathbb{A}_\kappa * (\mathbb{L} * \mathbb{M} \times \mathrm{Add}(\kappa, \lambda^+ \setminus \lambda))$ such that j extends to $j : V[G * H] \to M[G^* * H^*]$ witnessing that κ is λ-supercompact and for all $\gamma < j(\kappa)$, there is a function $f : \kappa \to \kappa$ such that $j(f)(\kappa) = \gamma$.*

Proof. By the closure of M, $j(\mathbb{A}_\kappa)/G * H$ is λ^+-closed in $V[G * H]$. Moreover since $j(\kappa)$ is inaccessible and $j(\mathbb{A}_\kappa)$ is $j(\kappa)$-cc in M, the poset $j(\mathbb{A}_\kappa)/G * H$ has just $|j(\kappa)| = \lambda^+$ antichains in $M[G * H]$. So in $V[G * H]$ we can find a generic I for $j(\mathbb{A}_\kappa)/G * H$ over $M[G * H]$. We let G^* be the $j(\mathbb{A}_\kappa)$-generic obtained from $G * H * I$. From the work so far we can lift to $j : V[G] \to M[G^*]$.

Next we consider $j(\mathbb{L} * (\dot{\mathbb{M}} \times \dot{\mathrm{Add}}(\kappa, \lambda^+ \setminus \lambda)))$ as computed in $M[G^*]$. First we construct a master condition for $j``H_0$, which is a member of $M[G^*]$ by the closure of $M$. Note that $j``H_0$ is a directed set of cardinality μ in the poset $j(\mathbb{L})$ which is $j(\kappa)$-directed closed. So we can take a lower bound l^* for $j``H_0$. There are λ^+-many antichains, and $j(\mathbb{L})$ is λ^+-closed, so we build a generic H_0^* in $V[G * H]$ for $j(\mathbb{L})$ which contains l^*.

Next, we construct a master condition for $j``H_1$, which again is in $M[G^*]$ by the closure of $M$. We let $p^*$ be the union of the first coordinates of $j``H_1$ and Y be the union of the domains of the second coordinates of $j``H_1$. Since $j(\kappa) > \lambda$, $p^*$ is a potential first coordinate in $j(\mathbb{M})$. Moreover since $H_1$ is a filter, we have that for all $\gamma \in Y$, $p^* \upharpoonright \gamma$ forces that $\{f(\gamma) \mid f$ is a second coordinate of $j``H_1\}$ is a directed set in $j(\mathrm{Add}(\kappa^{+\omega+1}, 1))$. We let $f^*(\gamma)$ be a $j(\mathbb{P}) \upharpoonright \gamma$-name for a condition forced by $p^* \upharpoonright \gamma$ to be a lower bound for this set. As before, $j(\mathbb{M})$ is λ^+-closed and the number of antichains for this poset in $M[G^*][H_0^*]$ is λ^+. So we build a generic H_1^* in $V[G * H]$ for $j(\mathbb{M})$ containing (p^*, f^*). This allows us to lift the embedding further to $j : V[G * H_0 * H_1] \to M[G^* * H_0^* * H_1^*]$.

Next, we find a generic object for $\mathrm{Add}(j(\kappa), j(\lambda^+ \setminus \lambda))$ following an argument from Gitik-Sharon [6]. In preparation let $\langle \eta_\alpha \mid \alpha \in \lambda^+ \setminus \lambda \rangle$ be an enumeration of $j(\kappa)$. Note that j is continuous at λ^+ and hence $\mathrm{Add}(j(\kappa), j(\lambda^+) \setminus j(\lambda)) = \bigcup_{\alpha<\lambda^+} \mathrm{Add}(j(\kappa), j(\alpha) \setminus j(\lambda))$. Moreover if we have an increasing sequence of generics for $\langle H^\alpha \mid \alpha \in (\lambda^+ \setminus \lambda) \cap \mathrm{cof}(\lambda) \rangle$ such that for each α, H^α is generic for $\mathrm{Add}(j(\kappa), j(\alpha) \setminus j(\lambda))$ over $M[G^* * H_0^* * H_1^*]$, then $\bigcup_\alpha H^\alpha$ is generic for $\mathrm{Add}(j(\kappa), j(\lambda^+ \setminus \lambda))$ over the same model.

We construct such H^α by induction. We ensure that each H^α contains the condition $p_\alpha = (\bigcup j``H_2 \restriction \alpha) \cup \{((j(\beta), \kappa), \eta_\beta) \mid \beta < \alpha\}$, where we are thinking of $\mathrm{Add}(j(\kappa), j(\lambda^+ \setminus \lambda))$ as partial functions from $j(\lambda^+ \setminus \lambda) \times j(\kappa)$ to $j(\kappa)$. It is clear that limits do not pose a problem, so it will be enough so show that we can construct H^{α^*} assuming that we have constructed H^α where α^* is the least ordinal of cofinality λ greater than α.

The argument is straight forward. It suffices to build a generic for $\mathrm{Add}(j(\kappa), j(\alpha^*) \setminus j(\alpha))$ over the model $M[G^* * H_0^* * H_1^*][H^\alpha]$ which contains the condition $p_{\alpha^*} \restriction [\alpha, \alpha^*)$. Clearly the poset is λ^+-closed in $V[G * (H_0 \times H_1)]$. Further it has just $|j(\lambda)| = \lambda^+$ many antichains in $M[G^* * H_0^* * H_1^*][H^\alpha]$, so we can build a generic $\bar{H}$ for it and let H^{α^*} be the generic obtained from $H^\alpha \times \bar{H}$.

We can now let $H_2^* = \bigcup_\alpha H^\alpha$ and $H^* = H_0^* * H_1^* * H_2^*$. It is easy to see that we have the desired lifting of j to $j : V[G * H] \to M[G^* * H^*]$. □

Remark 2.4. The reason we forced with $\mathrm{Add}(\lambda^+ \setminus \lambda, \kappa)$ after the Mitchell poset is precisely to get functions from κ to κ to represent ordinals below $j(\kappa)$. This is used in the argument below to get guiding generics for the interleaved collapses in the Prikry forcing.

Working in $V[G * H]$ for $n < \omega$ we define the following:

(1) Let U_n be the supercompactness measure on $\mathcal{P}_\kappa(\kappa^{+n})$ derived from j.
(2) Let $j_n : V[G * H] \to \mathrm{Ult}(V[G * H], U_n) \simeq M_n$
(3) Let k_n be the factor map from M_n to $M[G^* * H^*]$ defined by $j_n(F)(j_n``\kappa^{+n}) \mapsto j(F)(j``\kappa^{+n})$. Then $j = k_n \circ j_n$

The following sequence of claims are standard consequences of the previous lemma, and we only sketch the proofs. For a more detailed presentation in a similar context, see [6].

Claim 2.5. *The critical point of k_n is greater than $j(\kappa)$.*

Proof. We arranged that for every $\gamma < j(\kappa)$, there is $f : \kappa \to \kappa$ with $j(f)(\kappa) = \gamma$. It follows that $j(\kappa) + 1 \subset \mathrm{ran}\, k_n$ □

Claim 2.6. *In $V[G*H]$, there is a generic K for $\mathrm{Coll}(\kappa^{+\omega+3}, < j(\kappa))_{M[G^* * H^*]}$ over $M[G^* * H^*]$.*

Proof. This is by a standard counting argument, using that there are $\kappa^{+\omega+3}$-many antichains to meet and the poset is $\kappa^{+\omega+3}$-closed. □

We set $K_n = \{c \in \mathrm{Coll}(\kappa^{+\omega+3}, < j_n(\kappa))_{M_n} \mid k_n(p) \in K\}$.

Claim 2.7. *K_n is* $\mathrm{Coll}(\kappa^{+\omega+3}, < j_n(\kappa))$*-generic over* M_n.

Proof. K_n is clearly a filter. Now, if $A \in M_n$ is an antichain in $\mathrm{Coll}(\kappa^{+\omega+3}, < j_n(\kappa))$, then by the chain condition, $|A| < j_n(\kappa)$, and so $k_n(A) = k_n``A$ is an antichain in $\mathrm{Coll}(\kappa^{+\omega+3}, < j(\kappa))_{M[G^* * H^*]}$. □

Using the generics K_n and ultrafilters U_n we define a Prikry forcing with interleaved collapses $\mathbb{R}$ as in the first author's [14]. More precisely, for each i, let $X_i = \{x \in \mathcal{P}_\kappa(\kappa^{+i}) \mid \kappa_x \text{ is inaccessible}, o.t.(x) = \kappa_x^{+i}\}$. Conditions are of the from $r = \langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}, A_n, C_n, ...\rangle$, where

- for $i < n$, $x_i \in X_i$, and for $i \geq n$ $A_i \in U_i$, $A_i \subset X_i$.
- for $i < n-1$, $x_i \prec x_{i+1}$ and $c_i \in \mathrm{Coll}(\kappa_{x_i}^{+\omega+3}, < \kappa_{x_{i+1}})$ and $c_{n-1} \in \mathrm{Coll}(\kappa_{x_i}^{+\omega+3}, < \kappa)$,
- if $n > 0$, then $d \in Col(\omega, \kappa_{x_0}^{+\omega})$, otherwise $d \in Col(\omega, \kappa)$,
- for $i \geq n$, $[C_n]_{U_n} \in K_n$

For a condition as above, the stem of r is $\langle d, x_0, c_0, ..., x_{n-1}, c_{n-1}\rangle$ and we denote it by $\mathrm{s}(r)$. Note that two conditions with the same stem are compatible. We denote the weakest common extension (also with the same stem) of two such conditions by $r_1 \wedge r_2$.

Standard arguments show the following:

Proposition 2.8.

(1) *After forcing with $\mathbb{R}$, for each $n \geq 0$, the cofinality of $(\kappa^{+n})^{V[G*H]}$ becomes ω.*
(2) *$\mathbb{R}$ has the $\kappa^{+\omega+1}$ chain condition.*
(3) *$\mathbb{P}$ has the Prikry property: if p is a condition with length at least 1 and ϕ is a formula, then there is a direct extension $p' \leq^* p$ which decides ϕ.*

From this is it is straightforward to show that the extension by $\mathbb{R}$ has the desired cardinal structure.

Claim 2.9. *In the extension of $V[G * H]$ by $\mathbb{R}$, $\kappa = \aleph_{\omega^2}$, $(\kappa^{+\omega+1})^{V[G*H]} = \aleph_{\omega^2+1}$, $\lambda = (\kappa^{+\omega+2})^{V[G*H]} = \aleph_{\omega^2+2}$ and cardinals above λ are preserved.*

Note that $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+3}$. It remains to show that that the tree property holds at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$.

3. The tree property at $\aleph_{\omega^2+1}$

In this section we show that the tree property holds at $\aleph_{\omega^2+1}$ in the extension of $V[G * H]$ by $\mathbb{R}$. We fix an $\mathbb{R}$-name $\dot{T} \in V[G * H]$ forced to be a $\aleph_{\omega^2+1}$-tree and we show that it has a branch. We will work with the name $\dot{T}$ throughout the section and apply arguments from [14] directly to it. We can assume that the underlying set of $\dot{T}$ is $\nu^+ \times \kappa$ and we are forcing the relation $<_T$ on this set. Recall that $\nu = \sup_{n<\omega} \kappa_n$.

Let $\mathbb{S}$ be the quotient forcing $\mathbb{P} \times \mathbb{Q}/H_1$ as defined in $V[G * H]$. Let $H_T = H_0 * (H_1^{\mathbb{P}} \times H_1^{\mathbb{Q}}) \times H_2$ be the generic for $\mathbb{L} * (\mathbb{P} \times \mathbb{Q}) \times \mathrm{Add}(\kappa, \lambda^+ \setminus \lambda)$ obtained by forcing with $\mathbb{S}$. It is not hard to see that $\mathbb{S}$ is $< \kappa^{+\omega+1}$-distributive over $V[G * H]$. It follows that each U_n is still an ultrafilter and each K_n is still generic over the ultrapower of $V[G * H_T]$ by U_n. So in particular $\mathbb{R}$ is a reasonable diagonal Prikry forcing in $V[G * H_T]$. We will show that we have the prerequisites to run the argument from [14] in order to first get a branch in $V[G * H_T]^{\mathbb{R}}$. Then we will use a branch preservation argument to pull the branch back to $V[G * H]^{\mathbb{R}}$.

Lemma 3.1. *In $V[G * H]$ there is an unbounded set $I \subseteq \mu$ and a natural number n^* such that for all $\alpha < \beta$ from I there are a condition r of length n^* and ordinals $\xi, \zeta < \kappa$ such that q forces $(\alpha, \xi) <_T (\beta, \zeta)$.*

This is exactly Lemma 13 of [14]. For the proof we only need an embedding witnessing that κ is $\kappa^{+\omega+1}$ supercompact so that κ is a potential 0^{th} Prikry point in $j(\mathbb{R})$. For this we use the embedding j from Lemma 2.3. Since this is upwards absolute to $V[G * H_T]$, we have the same conclusion in $V[G * H_T]$.

Remark 3.2. We actually have the analog of Remark 14 of [14]. In particular the set of conditions in $\mathbb{R}$ which force the above is dense.

Next we need lifted supercompact embeddings with critical point κ_n for $n < \omega$ that are added by a reasonable forcing.

Lemma 3.3. *For all $n \geq 1$, there is a λ-supercompact embedding j_n^* with $\operatorname{crit}(j_n^*) = \kappa_n$ with domain $V[G * H_T]$ added by the product of κ_{n-1}-closed forcing and $\operatorname{Add}(\kappa, \theta)$ for some θ.*

This is standard. Note that $\mathbb{Q}$ is $\kappa^{+\omega+1}$-directed closed of size λ in $V[G][H_0]$ and hence can be absorbed in the iteration of Levy collapses. This lemma allows us to carry out the arguments from Lemmas 15 and 16 of [14] in $V[G * H_T]$. So we have the following consequence.

Lemma 3.4. *In $V[G * H_T]$ every stem can be extended to a stem h such that there are an unbounded set $J \subseteq \mu$, $\xi < \kappa$, and conditions $r_\alpha \in \mathbb{R}$ for $\alpha \in J$, such that, setting $u_\alpha = \langle \alpha, \xi \rangle$ for $\alpha \in J$, we have:*

1. *each r_α has stem h,*
2. *for all $\alpha < \beta$ from J, $r_\alpha \wedge r_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$.*
3. *each r_α forces that u_α is in the branch.*

We call this condition on h, $\dagger_h$. A version of this property in a context without collapses was first obtained by Neeman in [12]. Note that given $\dagger_h$, and a generic object $\mathcal{R}$ for $\mathbb{R}$ which contains unboundedly many r_α, we can generate a branch b in the extension by taking the upwards closure of all nodes u_α, such that $r_\alpha \in \mathcal{R}$. We let $\dot{b} \in V[G * H_T]$ be an $\mathbb{R}$-name for the branch b generated in this way.

We now want to pull back the existence of the branch from $V[G * H_T]^{\mathbb{R}}$ to $V[G * H]^{\mathbb{R}}$. To do so we follow [15]. Note that forcing with the poset $\mathbb{S}$ takes us from the inner to the outer model above. Unfortunately $\mathbb{S}$ is not particularly nice forcing in the extension by $\mathbb{R}$. In fact it is not even countably closed there. We make up for this by doing our splitting arguments for a fixed stem h and using the fact that $\mathbb{S}$ is κ-closed in $V[G * H]$.

Working in $V[G * H]$ we can view $\dot{b}$ as an $\mathbb{S} \times \mathbb{R}$ name for the branch. We work in the model $V[G * H]$ and give the following definition.

Definition 3.5. *We say that there is an h-splitting at some $\gamma < \mu$ if there are a condition $r \in \mathbb{R}$ with stem h, conditions s, s^0, s^1 in $\mathbb{S}$, and nodes u, u_0, u_1 such that*

1. $(s, r) \Vdash u \in \dot{b} \cap T_\gamma$
2. *$s^0, s^1 \leq s$ and the levels of u_0, u_1 are above γ;*
3. *for $k \in 2$, $(s^k, r) \Vdash u_k \in \dot{b}$;*
4. *r forces that u_0 and u_1 are incompatible in $\dot{T}$.*

The idea is to capture the fact that $\mathbb{S}$ can force different information about the branch relative to a fixed stem h. If $\bar{s}$ forces that $\dagger_h$ holds for some stem h, then we define $\dot{\alpha}_h$ to be an $\mathbb{S}$-name for the supremum over γ for which there is an h-splitting at γ with the witnessing s an element of the $\mathbb{S}$ generic.

Lemma 3.6 (Splitting)**.** *Suppose that $\bar{s}$ forces $\dagger_h$ and $\dot{\alpha}_h = \mu$. There are sequences $\langle s_i \mid i < \nu \rangle$, $\langle r_i \mid i < \nu \rangle$, and $\langle v_i \mid i < \nu \rangle$, such that*

1. *for all $i < \nu$, $s_i \leq \bar{s}$ and the stem of r_i is h,*
2. *for all $i < \nu$, (s_i, r_i) forces that v_i is in $\dot{b}$ and*
3. *for $i < j$, $r_i \wedge r_j$ forces that v_j and v_i are incompatible in $\dot{T}$.*

Proof. Assume the hypotheses. Suppose also that $\bar{s}$ forces $\dagger_h$, as witnessed by $\xi, \dot{J}, \dot{r}_\alpha$. We pass to the generic extension of $V[G * H]$ by $\mathbb{S}$ by a generic $\mathcal{S}$ containing $\bar{s}$. First we need a finer version of h-splitting.

Claim 3.7. *Suppose $\gamma \in J$ and denote $u = \langle \gamma, \xi \rangle$. Then there is $\bar{r} \in \mathbb{R}$ with stem h, conditions s^0, s^1 in $\mathbb{S}$ below $\bar{s}$ and nodes v^0, v^1, such that for $k \in \{0, 1\}$, we have $(s^k, \bar{r}) \Vdash v^k \in \dot{b}$, $\bar{r} \Vdash u <_{\dot{T}} v^k$ and $\bar{r} \Vdash v^0 \perp_{\dot{T}} v^1$.*

Proof. Since $\alpha_h = \mu$, there is h-splitting at a level $\gamma' \geq \gamma$. So, let $s' \in \mathcal{S}$, $s' \leq \bar{s}$, $r \in \mathbb{R}$ with stem h, s^0, s^1 below s', and nodes v, v^0, v^1 of levels higher than γ be such that:

- $(s', r) \Vdash v \in \dot{b} \cap T_{\gamma'}$,
- the levels of v^0, v^1 are above γ',
- for $k \in 2$, $(s^k, r) \Vdash v^k \in \dot{b}$, and
- r forces that v^0 and v^1 are incompatible in $\dot{T}$.

Let $s'' \leq s'$ in $\mathcal{S}$ be such that for some q with stem h, $s'' \Vdash q = \dot{r}_\gamma$. Then since $(s'', q) \Vdash u \in \dot{b}$ and $(s'', r) \Vdash v \in \dot{b}$, it follows that $q \wedge r \Vdash u <_{\dot{T}} v$. Let $\bar{r} = r \wedge q$. Then $\bar{r}, s^0, s^1, v^0, v^1$ are as desired. □

Choose a club $C \subseteq \mu$ such that for all $\beta \in C$ and all $\gamma < \beta$ if there is a splitting at γ as in the conclusion of the above claim, then the witnessing splitting nodes v^0, v^1 can be chosen to have levels below β.

We select increasing sequences γ_i and β_i for $i < \nu$ such that

(1) $\beta_i \in C$,
(2) $\gamma_i \in J$,
(3) $\gamma_i < \beta_i \leq \gamma_{i+1}$.

Denote $u_i = \langle \gamma_i, \xi \rangle$ for $i < \nu$ and let $\bar{s}_i \in \mathcal{S}$ be such that $\bar{s}_i \Vdash \gamma_i \in \dot{J}$ and for some condition q_i, $\bar{s}_i \Vdash q_i = \dot{r}_{\gamma_i}$. Then each $(\bar{s}_i, q_i)$ forces that u_i is in the branch, and for $i < j$, $q_i \wedge q_i \Vdash u_i <_{\dot{T}} u_j$.

Now, for each γ_i, there is a splitting as in the above claim with splitting nodes of levels below β_i. We record the witnesses to this splitting as $\bar{r}_i, s_i^k, v_i^k$. In particular, we have that $(s_i^k, \bar{r}_i) \Vdash v_i^k \in \dot{b}$ and $\bar{r}_i \Vdash u_i <_{\dot{T}} v_i^k$.

Let r be $\bar{r}_i \wedge q_i \wedge q_{i+1}$. Then r forces $u_i <_{\dot{T}} u_{i+1}$ and $u_i <_{\dot{T}} v_i^k$ for $k \in 2$. We take a direct extension r_i of r to decide the statements "$v_i^k <_{\dot{T}} u_{i+1}$" for $k \in 2$. Since T is a tree it must decide one of the statements negatively. If it does so for k, then we set $v_i = v_i^k$ and $s_i = s_i^k$.

By the distributivity of $\mathbb{S}$, the sequence $\langle s_i, r_i, v_i \mid i < \nu \rangle$ is in $V[G * H]$. It is straightforward to see that these sequences satisfy the lemma. □

Lemma 3.8. *If $\bar{s}$ forces $\dagger_h$, then $\bar{s}$ forces $\dot{\alpha}_h < \mu$.*

Proof. Suppose otherwise. Using the splitting lemma we build a tree of conditions similar to Lemma 2.3 in Magidor and Shelah [9]. More precisely, apply the previous lemma to construct conditions $\langle (s_\sigma, r_\sigma) \mid \sigma \in \nu^{<\omega} \rangle$ in $\mathbb{S} \times \mathbb{R}$ and nodes $\langle v_\sigma \mid \sigma \in \nu^{<\omega} \rangle$, such that:

(1) for all σ, r_σ has stem h and if $\sigma' \supset \sigma$, then $(s_{\sigma'}, r_{\sigma'}) \leq (s_\sigma, r_\sigma)$,
(2) for all σ, $s_\sigma \Vdash \text{level}(v_\sigma) \in \dot{J}$, and $(s_\sigma, r_\sigma) \Vdash v_\sigma \in \dot{b}$,
(3) for all σ and $i \neq j$ in ν, $r_{\sigma^\frown i} \wedge r_{\sigma^\frown j}$ forces that $v_{\sigma^\frown i}$ and $v_{\sigma^\frown j}$ are incompatible in $\dot{T}$.

Using the fact that $\mathbb{S}$ is countably closed and the fact that each r_σ has stem h, for each $g \in \nu^\omega$ we can find $s_g \leq s_{g \restriction n}$ and $r_g \leq r_{g \restriction n}$ for all $n < \omega$. Let γ^* be the supremum of the ordinals γ appearing as levels of nodes in the construction. Let $s_g^* \leq s_g$ and $r_g^* \leq r_g$ be such that (s_g^*, r_g^*) decides the value of the branch at level γ^* to be v_g.

Since the number of possible stems is ν, let $g, g' \in \nu^\omega$ be distinct, such that r_g^* and $r_{g'}^*$ have the same stem. Then r_g^* and $r_{g'}^*$ are compatible and by construction $r_g^* \wedge r_{g'}^*$ forces that v_g is incompatible with $v_{g'}$. This is a contradiction since we can take a generic for $\mathbb{R}$ containing $r_g^* \wedge r_{g'}^*$. □

Since there are less than μ many stems, passing to an extension of $V[G*H]$ by $\mathbb{S}$, we get that $\sup_h \alpha_h < \mu$. Let $\bar{s}$ be a condition forcing that $\alpha = \sup_h \dot{\alpha}_h < \mu$. Extending if necessary, suppose also that for some stem $\bar{h}$, $\bar{s}$ forces that $\dagger_{\bar{h}}$ holds, and for some $\bar{r} \in \mathbb{R}$ with stem $\bar{h}$, $(\bar{s}, \bar{r})$ forces that $\dot{b}$ is a cofinal branch. Note that this means $(\bar{s}, \bar{r})$ forces that $\dot{b}$ is generated by $\dagger_{\bar{h}}$.

Let $\gamma > \alpha$ and let $s \leq \bar{s}$ and $r \leq \bar{r}$ be such that for some node u of level γ, $(s, r) \Vdash u \in \dot{b}$. Let $\mathcal{R}$ be $\mathbb{R}$-generic containing r. In $V[G * H * \mathcal{R}]$, we view $\dot{b}$ as its partial interpretation by the generic $\mathcal{R}$. Clearly s forces that u is in $\dot{b}$ over $V[G * H * \mathcal{R}]$.

Now we define $d = \{v >_T u \mid \exists s^* \leq s, s^* \Vdash v \in \dot{b}\}$.

Claim 3.9. *d generates a branch through T.*

Proof. Clearly d meets every level above γ. Next we show that it meets every level exactly once. Suppose that there are distinct v_0 and v_1 in d on level $\bar{\gamma} > \gamma$. Let s_0 and s_1 force v_0 and v_1, respectively, to be in $\dot{b}$ on level $\bar{\gamma}$. Let r_0, r_1 in $\mathcal{R}$ be such that for $k \in 2$, $(s_k, r_k) \Vdash v_k \in \dot{b}$.

Let $r' = r_0 \wedge r_1$, and let h be its stem. Since $r' \in \mathcal{R}$ and h extends $\bar{h}$, it must be that s also forces $\dagger_h$. But then s_0, s_1, r', v_0, v_1 witness an h-splitting at γ, a contradiction since $\gamma > \alpha$. □

Working in $V[G * H]$ it follows that r forces that d is a cofinal branch through $\dot{T}$.

4. The tree property at $\aleph_{\omega^2+2}$

In this section we prove that the tree property holds at $\aleph_{\omega^2+2}$ in the extension of $V[G*H]$ by $\mathbb{R}$. Recall that λ is weakly compact in V and hence in $V[G * H_0]$. Working in $V[G * H_0]$ let $\dot{T}$ be an $\mathbb{M} \times \mathrm{Add}(\kappa, \lambda^+ \setminus \lambda) * \mathbb{R}$-name for a λ-tree. Clearly λ is still weakly compact in $V[G * H_0]$, so we fix an elementary embedding $k : N \to N^*$ with critical point λ where N is a transitive model of size λ with $\dot{T} \in N$.

For ease of notation we set $\mathbb{A} = \mathrm{Add}(\kappa, \lambda^+ \setminus \lambda)$ and $\bar{H} = H_1 \times H_2$. Let $\mathcal{R}$ be $\mathbb{R}$-generic over $V[G * H]$. Since $(\mathbb{M} \times \mathbb{A}) * \mathbb{R}$ has the λ-cc, we can lift k to this extension by forcing over $V[G * H][\mathcal{R}]$ with $k((\mathbb{M} \times \mathbb{A}) * \mathbb{R})/(\bar{H} * \mathcal{R})$. Note that for each n, $U_n \subset k(U_n)$ and $K_n \subset k(K_n)$. So, $\mathbb{R} \subset k(\mathbb{R})$, and moreover $\mathbb{R}$ and $k(\mathbb{R})$ have the same set of stems. Of course there are more measure one sets that can be used in conditions in $k(\mathbb{R})$, since the powerset of κ is increased to $k(\lambda^+)$. But by a characterization of genericity for the Prikry poset, a generic for $k(\mathbb{R})$ induces a generic for $\mathbb{R}$.

The lifted embedding determines a branch through the interpretation of $\dot{T}$. It is enough to show that the forcing to add the embedding cannot add the branch. This will finish the proof of Theorem 1.1. Recall that a poset has the μ-approximation property if it does not add a new set of ordinals x, such that for all $< \mu$-size subsets y in the ground model, $x \cap y$ is in the ground model. Since a new branch through a λ-tree satisfies the hypotheses of the μ-approximation property, it is enough to show the following lemma.

Lemma 4.1. *In $V[G*H][\mathcal{R}]$, $k((\mathbb{M}\times\mathbb{A})*\mathbb{R})/(\bar{H}*\mathcal{R})$ has the μ-approximation property.*

To begin we give some abstract definitions about Prikry forcing. We say that a stem s' extends a stem s, if there are conditions $r' \leq r$ in $\mathbb{R}$ with stems s' and s, respectively.

Definition 4.2. *Suppose that s and s' are stems where s' extends s. If r is a condition with stem s, then we say that points in s' above s are* constrained *by r if there is a condition r' with stem s' such that $r' \leq r$.*

We have the following characterization of when a condition is forced out of the quotient which comes from Cummings and Foreman [3].

Proposition 4.3. *Work in $V[G*H]$. Let $\bar{r} \in \mathbb{R}$, $m \in k(\mathbb{M} \times \mathbb{A})$ and $\dot{r}$ be a $k(\mathbb{M} \times \mathbb{A})/\bar{H}$-name for an element of $k(\mathbb{R})$. We assume that m decides the value of $\mathrm{s}(\dot{r})$. $\bar{r}$ forces $(m, \dot{r}) \notin k(\mathbb{M} \times \mathbb{A} * \mathbb{R})/(\bar{H} * \dot{\mathcal{R}})$ if and only if one of the following holds.*

1. *$m \notin k(\mathbb{M} \times \mathbb{A})/\bar{H}$.*
2. *neither one of $\mathrm{s}(\bar{r})$ or $\mathrm{s}(\dot{r})$ extends the other.*
3. *$\mathrm{s}(\dot{r})$ extends $\mathrm{s}(\bar{r})$ and points in $\mathrm{s}(\dot{r})$ above $\mathrm{s}(\bar{r})$ are not constrained by $\bar{r}$.*
4. *$\mathrm{s}(\bar{r})$ extends $\mathrm{s}(\dot{r})$ and m forces that points in $\mathrm{s}(\bar{r})$ above $\mathrm{s}(\dot{r})$ are not constrained by $\dot{r}$.*

A key point in the proof is that since we are using guiding generics for the collapses, so that Prikry conditions with the same stem are compatible. From this proposition we have the following sufficient condition for forcing conditions into the quotient.

Claim 4.4. *Work in $V[G*H]$. If $\bar{r}$ is in $\mathbb{R}$, $m \in k((\mathbb{M} \times \mathbb{A})/\bar{H}$ and $\dot{r}$ is a $k(\mathbb{M} \times \mathbb{A})/\bar{H}$-name for a condition in $k(\mathbb{R})$ such that*

1. *m decides the value of $\mathrm{s}(\dot{r})$,*
2. *$\mathrm{s}(\bar{r})$ extends $\mathrm{s}(\dot{r})$ and*
3. *m forces that points in $\mathrm{s}(\bar{r})$ above $\mathrm{s}(\dot{r})$ are constrained by $\dot{r}$,*

then there is a direct extension of $\bar{r}$ which forces that $(m, \dot{r}) \in k((\mathbb{M} \times \mathbb{A}) * \mathbb{R})/(\bar{H} * \dot{\mathcal{R}})$.

Proof. Let $\bar{r}_0$ be a direct extension of $\bar{r}$ which decides the statement $(m, \dot{r}) \in k((\mathbb{M} \times \mathbb{A}) * \mathbb{R})/(\bar{H} * \dot{\mathcal{R}})$. It is easy to see that we are not in any of the cases in the previous proposition, so it is not true that $\bar{r}_0 \Vdash (m, \dot{r}) \notin k((\mathbb{M} \times \mathbb{A}) * \mathbb{R})/(\bar{H} * \dot{\mathcal{R}})$. However this means it must force $(m, \dot{r})$ into the quotient. □

We will use the claim above to reproduce the argument of Lemma 1.3 in [18] in the presence of the Prikry forcing. For ease of notation we let $\mathbb{N}$ be the quotient $k((\mathbb{M} \times \mathbb{A}) * \mathbb{R})/(\bar{H} * \mathcal{R})$. We will write conditions in $\mathbb{N}$ in the form $(p, f, \dot{r})$, where $p \in k(\mathbb{P} \times \mathbb{A})$, $f \in k(\mathbb{Q})$ and $\dot{r}$ is a $k(\mathbb{M} \times \mathbb{A})$-name for a condition in $k(\mathbb{R})$. We will say "term ordering" to refer to $\leq_{k(\mathbb{Q})}$.

To simplify the notation of the proof we will prove that there are no new functions τ from μ to 2 added by $\mathbb{N}$ all of whose initial segments are in the ground model, $V[G*H][\mathcal{R}]$. The argument giving μ-approximation is essentially the same. In $V[G*H][\mathcal{R}]$ let $\dot{\tau}$ be a $\mathbb{N}$-name for a function from μ to 2 such that for all $\alpha < \mu$, $\dot{\tau} \restriction \alpha$ is forced to be in $V[G*H][\mathcal{R}]$.

Claim 4.5. *In $V[G*H][\mathcal{R}]$, there is a condition $(p, f, \dot{r}) \in \mathbb{N}$ such that for all $p' \leq p$, x, $\alpha < \mu$, f' which is forced to be below f in the term ordering and all $k(\mathbb{M} \times \mathbb{A})/\bar{H}$-names $\dot{r}'$ for a condition in $k(\mathbb{R})$ if $(p', f', \dot{r}') \in \mathbb{N}$ is below (p, f, r) and forces $\dot{\tau} \restriction \alpha = x$, then $(p, f', \dot{r})$ forces $\dot{\tau} \restriction \alpha = x$.*

Proof. Suppose not. Then working in $V[G * H_0 * H_1]$ there is a condition $(a, \bar{r}) \in \mathbb{A} * \mathbb{R}$ forcing the failure of the claim. The following set is dense below $(a, \bar{r})$ for every name $(p, f, \dot{r})$ for an element in the quotient. We set $(a', \bar{r}')$ in D if and only if there are $p_0, p_1 \in k(\mathbb{P} \times \mathbb{A})$, f^* below f in the term ordering, $k(\mathbb{M} \times \mathbb{A})/\bar{H}$-names r_0, r_1 for elements of $k(\mathbb{R})$, $\alpha < \mu$ and $\mathbb{A} * \mathbb{R}$-names x_0, x_1 for functions from α to 2 which are forced to be distinct, such that $(a', \bar{r}')$ forces that:

- each $(p_i, f^*, \dot{r}_i)$ is in $\mathbb{N}$ below $(p, f, \dot{r})$, and
- each $(p_i, f^*, \dot{r}_i)$ forces that $\dot{\tau} \restriction \alpha = x_i$.

This is immediate from the failure of the claim in $V[G * H][\mathcal{R}]$. If $(p_0, f_0, \dot{r}_0)$ below $(p, f, \dot{r})$ forces a value x_0 for $\tau \restriction \alpha$, but $(p, f_0, \dot{r})$ does not, then we can find $(p_1, f^*, \dot{r}_1)$ below $(p, f_0, \dot{r})$ which forces a value $x_1 \neq x_0$. Furthermore, we can arrange that f^* is below f_0 in the term ordering. We can now select a condition $(a', \bar{r}') \leq (a, \bar{r})$ forcing this situation. In particular, for $i \in 2$ it forces $(p_i, f^*, r_i) \in \mathbb{N}$. Clearly, $(a', \bar{r}') \in D$, and this argument works below any condition stronger than $(a, \bar{r})$. Hence D is dense.

We work in $V[G * H_0 * H_1]$ where the f-parts of conditions in $k(\mathbb{M})/H_1$ are μ-closed under the term ordering. By recursion for $\alpha < \mu$, we construct $p^i_\alpha, x^i_\alpha, x_\alpha, f_\alpha, \dot{r}^i_\alpha, a_\alpha, \bar{r}_\alpha$ and γ_α for $i \in 2$ as follows.

Suppose that we have all of the above for all β below some α. Let $\gamma^* = \sup_{\beta < \alpha} \gamma_\beta$. We choose an $\mathbb{A} * \mathbb{R}$-name for a condition $(p^*, f^*, \dot{r}^*)$ in the quotient which is forced to decide the value of $\dot{\tau} \restriction \gamma^*$ to be x_α and where f^* is forced to be below f_β in the term ordering for $\beta < \alpha$. Using the dense set described above with $(p^*, f^*, \dot{r}^*)$, we can find $(a_\alpha, \bar{r}_\alpha)$ in the dense set and record witnesses $p^i_\alpha, \dot{r}^i_\alpha, x^i_\alpha, \gamma_\alpha$ and f_α. This completes the construction.

We can assume that $(a_\alpha, \bar{r}_\alpha)$ forces that each (p^i_α, f_α) decides the value of $\mathrm{s}(\dot{r}^i_\alpha)$ and that $\mathrm{s}(\bar{r}_\alpha)$ extends this value. By passing to an unbounded subset of μ we can assume that for all $\alpha, \alpha' < \mu$, $\mathrm{s}(\bar{r}_\alpha) = \mathrm{s}(\bar{r}_{\alpha'})$ and $\mathrm{s}(\dot{r}^i_\alpha) = \mathrm{s}(\dot{r}^i_{\alpha'})$ for $i \in 2$. Using the μ-cc of $(k(\mathbb{P} \times \mathbb{A}))^2$, we can find $\alpha < \alpha'$ such that a_α is compatible with $a_{\alpha'}$ and p^i_α is compatible with $p^i_{\alpha'}$ for $i \in 2$. For $i \in 2$ we let p^i be a greatest lower bound for p^i_α and $p^i_{\alpha'}$ and $\dot{r}^i$ be a name for a common extension of $\dot{r}^i_\alpha$ and $\dot{r}^i_{\alpha'}$ with the same stem.

We force with $\mathbb{A}$ below $a_\alpha \cup a_{\alpha'}$ to obtain H_2'. Then applying Claim 4.4, we can find a direct extension of $\bar{r}_\alpha$ and $\bar{r}_{\alpha'}$ which forces that each $(p^i, f_{\alpha'}, \dot{r}^i)$ is in $\mathbb{N}$. We force with a generic $\mathcal{R}'$ containing this extension. Then in $V[G * H_0 * (H_1 * H_2')][\mathcal{R}']$, we have that $(p_i, f_{\alpha'}, \dot{r}^i)$ is in $\mathbb{N}$, and so implies that for $i \in 2$, $x^i_{\alpha'} \restriction \gamma_\alpha = x^i_\alpha$. However, since $\bar{r}_{\alpha'} \in \mathcal{R}'$, we also have that $(p^i, f_{\alpha'}, \dot{r}^i) \in \mathbb{N}$, which decides the value of $\dot{\tau} \restriction \gamma^*$ to be $x_{\alpha'}$, where $\gamma^* = \sup_{\beta < \alpha'} \gamma_\beta$. But then for $i \in 2$, $x^i_{\alpha'} \restriction \gamma^* = x_{\alpha'}$ and this implies that $x^0_\alpha = x^1_\alpha$, a contradiction. □

Towards the proof of Lemma 4.1, we assume for a sake of contradiction that it is forced that $\dot{\tau}$ is not in $V[G * H][\mathcal{R}]$. By our assumption for a contradiction, the set of pairs (n, n') such that n and n' decide different values for some initial segment of $\dot{\tau}$ is dense in $\mathbb{N}^2$. By the previous claim we can get a pair into this dense set by only extending the f-parts of the conditions. Again we fall back to $V[G * H_0 * H_1]$ where the term ordering on the f-parts is μ-closed and $\mathbb{A} * \mathbb{R}$ is μ-cc. Using this we obtain the following splitting assertion.

For a given $(p, f', \dot{r})$ forced to be in $\mathbb{N}$ we can find a maximal antichain A in $\mathbb{A} * \mathbb{R}$ and conditions $(p, f_i, \dot{r}) \leq (p, f', \dot{r})$ for $i \in 2$ such that for all $(a, \bar{r}) \in A$, $(a, \bar{r})$ forces that $(p, f_0, \dot{r})$ and $(p, f_1, \dot{r})$ are in $\mathbb{N}$ and decide different values for $\dot{\tau} \restriction \alpha$ for some α.

Continuing to work in $V[G * H_0 * H_1]$, we repeatedly apply this splitting assertion starting with $(p, f, \dot{r})$ from Claim 4.5 to build a binary tree of conditions $\langle f_s \mid s \in 2^{<\kappa} \rangle$ and maximal antichains A_s in $\mathbb{A} * \mathbb{R}$. We can ensure that every element $(a, \bar{r}) \in A_s$ forces that $(p, f_{s \frown 0}, r)$ and $(p, f_{s \frown 1}, r)$ decide different values for $\tau \restriction \gamma$ for some γ. This gives rise to $\mathbb{A} * \mathbb{R}$-names $\dot{x}_{s \frown i}$ for $i \in 2$ and ordinals $\alpha_s^{a,\bar{r}}$. Let $\alpha^* < \mu$ be the supremum of the ordinals $\alpha_s^{a,\bar{r}}$ used in the construction.

Let $\dot{b}$ be a $k(\mathbb{M} \times \mathbb{A})/\bar{H}$-name for the characteristic function of the first subset of κ added by $k(\mathbb{P})/H_1^{\mathbb{P}}$. By a standard construction on names, there is a lower bound below the sequence $\langle f_{\dot{b} \restriction \eta} \mid \eta < \kappa \rangle$ in the term ordering. In particular, the interpretation b of $\dot{b}$ is in the extension by $\mathbb{P} \times \mathrm{Add}(\kappa, 1)$. Since for each relevant γ, $\langle f_{b \restriction \eta}(\gamma) \mid \eta < \kappa \rangle$ is forced to be decreasing in $V[G * H_0 * H_1^{\mathbb{P}}]$, we can partially interpret each name in $V[G * H_0][\mathbb{P} \times \mathrm{Add}(\kappa, 1)]$ to obtain $k(\mathbb{P})/(\mathbb{P} \times \mathrm{Add}(\kappa, 1))$-names which are forced to be decreasing. For each such sequence we can find a name for a lower bound. The only issue is that $\bigcup_{\eta < \kappa} \mathrm{dom}(f_{b \restriction \eta})$ may not be in $V[G * H_0][H_1^{\mathbb{P}}]$. However it is covered by a set Y in $V[G * H_0][H_1^{\mathbb{P}}]$. We construct a lower bound f^* by taking $\mathrm{dom}(f^*) = Y$ and for each $\gamma \in Y$ we let $f^*(\gamma)$ be a $k(\mathbb{P}) \restriction \gamma$-name such that if $p \in \mathrm{Add}(\kappa, 1)$ forces $\gamma \in \mathrm{dom}(f_{\dot{b} \restriction \eta})$, then p forces $f^*(\gamma)$ to be a lowerbound for $\langle f_{\dot{b} \restriction \eta}(\gamma) \mid \eta < \kappa \rangle$. A very similar construction appears in Mitchell's original paper as Lemma 3.7.

It is straightforward to check that we can force $(p, f^*, \dot{r})$ into the quotient. Passing to the extension $V[G * H][\mathcal{R}]$, we extend $(p, f^*, \dot{r})$ to decide the value of $\dot{\tau} \restriction \alpha^*$ to be x. We can now define the interpretation of $\dot{b}$ in $V[G * H][\mathcal{R}]$, a contradiction. We define $b \restriction \eta + 1$ to be the unique s of length $\eta + 1$ such that for the unique element $(a, \bar{r})$ of $A_{s \restriction \eta}$ in the generic, $(p, f_s, \dot{r})$ forces $x_s = x \restriction \alpha_s^{a,\bar{r}}$.

This concludes the proof of Lemma 4.1 and so we have the tree property at $\aleph_{\omega^2+2}$ in the final model.

5. Other applications of Lemma 4.1

In this section we give an abstract version of the preservation lemma in the previous section, and list some other applications. Let $\kappa < \mu < \lambda$ be cardinals. Let $\mathbb{M}$ be like Mitchell's forcing $\mathbb{M}(\kappa, \mu, \lambda)$ to make $\lambda = \mu^+$ have the tree property with a possible modification that 2^κ can me made as large as we like. In the extension by $\mathbb{M}$, let $\mathbb{R}$ be a forcing of Prikry type which singularizes κ. We define an operator s on $\mathbb{R}$ which returns the stem of a condition. Further we assume that conditions with the same stem are compatible and that there are fewer than μ many stems. So in particular $\mathbb{R}$ has the μ-Knaster property. Let k be an embedding with critical point λ, witnessing that λ is weakly compact in V. As before, we can lift k to to the extension $V[\mathbb{M} * \mathbb{R}]$ by forcing with $k(\mathbb{M} * \mathbb{R})/(\mathbb{M} * \mathbb{R})$.

Lemma 5.1. *In the extension of* $V[\mathbb{M} * \mathbb{R}]$*,* $k(\mathbb{M} * \mathbb{R})/(\mathbb{M} * \mathbb{R})$ *has the* μ*-approximation property.*

The proof is the same, since the proof from Section 4 does not rely on specific properties of the Prikry forcing. And as before, a corollary of this lemma is that $k(\mathbb{M} * \mathbb{R})/(\mathbb{M} * \mathbb{R})$ cannot add a branch through a λ tree T in $V[\mathbb{M} * \mathbb{R}]$.

Next we provide two applications of Lemma 5.1.

Theorem 5.2. *Suppose that* $\kappa < \lambda$ *are cardinals such that* κ *is* λ^+*-hypermeasurable and* λ *is weakly compact. There is a forcing extension in which* κ *is singular strong limit of cofinality* ω*,* $2^\kappa = \kappa^{+++}$ *and the tree property holds at* κ^{++}*.*

This answers a question of Friedman and Halilovic [5]. We expect that this result can be brought down to $\aleph_\omega$, since the preservation lemma would work with a suitable Prikry forcing with interleaved collapses.

Let κ be λ^+-hypermeasurable with λ the least weakly compact cardinal greater than κ. Iterate in a reverse Easton fashion to blow up the powerset of many $\alpha < \kappa$ to λ_α^+ where λ_α is the least weakly compact cardinal greater than α. Let V be the model obtained after the iteration.

Let $\mathbb{M}$ be Mitchell's forcing to make $\lambda = \kappa^{++}$ have the tree property with the small modification that we add λ^+ subets to κ and not just λ. We let $\mathbb{P} = \mathrm{Add}(\kappa, \lambda^+)$ be the first coordinate of $\mathbb{M}$.

Apply a surgery argument to see that κ is still measurable in $V[\mathbb{P}]$. Note that $\mathbb{M}/\mathbb{P}$ is $< \kappa^+$-distributive in $V[\mathbb{P}]$. It follows that κ is measurable in $V[\mathbb{M}]$.

It now suffices to show that in $V[\mathbb{M}]$ if $\mathbb{R}$ is Prikry forcing as defined in $V[\mathbb{M}]$ from some normal measure on κ and k is an elementary embedding witnessing some fragment of the weak compactness of λ in V, then $k(\mathbb{M} * \mathbb{R})/\mathbb{M} * \mathbb{R}$ has the κ^+-approximation property in $V[\mathbb{M} * \mathbb{R}]$. This follows from Lemma 5.1, finishing the proof.

We also give another model for a theorem of Cox and Krueger [2].

Theorem 5.3. *Assume that* $\lambda < \alpha$ *are cardinals such that* λ *is supercompact and* $\mathrm{cf}(\alpha) > \omega$*. There is a forcing extension in which* $\lambda = \aleph_2$*,* $\mathrm{ISP}(\aleph_2)$ *holds and* $2^\omega = \alpha$*.*

Let $\mathbb{M}$ be Mitchell's forcing to make λ into $\aleph_2$ with the change that we add α many subsets of ω instead of just λ. Let j witness that λ is θ-supercompact for some large θ. By standard arguments we can lift the embedding to the extension by $\mathbb{M}$ by forcing with $j(\mathbb{M})$. To establish $\mathrm{ISP}(\omega_2)$ it is enough to show that $j(\mathbb{M})/\mathbb{M}$ has the $\aleph_1$-approximation property. This is immediate from Lemma 5.1 applied with $\mathbb{R}$ as the trivial forcing.

6. Concluding remarks

There are a few natural directions for future research.

First, we ask

Question 1. *Is it consistent that $\aleph_{\omega^2}$ is strong limit and for all $n \geq 1$ $\aleph_{\omega^2+n}$ has the tree property?*

Here there is a natural strategy. Mitchell's poset $\mathbb{M}$ should be replaced with the iteration of Cummings and Foreman [3] or Neeman's revision of it [13].

Second, we ask whether the bottom up approach mentioned in the introduction can be joined with the forcing from our main theorem. In particular we ask:

Question 2. *Is it consistent that every regular cardinal up to $\aleph_{\omega^2+2}$ has the tree property and $\aleph_{\omega^2}$ is strong limit?*

Finally, it is open whether the main result of our paper can be obtained at $\aleph_\omega$. This would require first answering a question of Woodin.

Question 3. *Is it consistent that SCH fails at $\aleph_\omega$ and the tree property holds at $\aleph_{\omega+1}$?*

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