SIGMA-PRIKRY FORCING II:
ITERATION SCHEME

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Abstract. In Part I of this series [PRS20], we introduced a class of notions of forcing which we call Σ-Prikry, and showed that many of the known Prikry-type notions of forcing that centers around singular cardinals of countable cofinality are Σ-Prikry. We proved that given a Σ-Prikry poset $P$ and a $P$-name for a non-reflecting stationary set $T$, there exists a corresponding Σ-Prikry poset that projects to $P$ and kills the stationarity of $T$. In this paper, we develop a general scheme for iterating Σ-Prikry posets and, as an application, we blow up the power of a countable limit of Laver-indestructible supercompact cardinals, and then iteratively kill all non-reflecting stationary subsets of its successor. This yields a model in which the singular cardinal hypothesis fails and simultaneous reflection of finite families of stationary sets holds.

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1. Introduction

In the introduction to Part I of this series [PRS20], we described the need for iteration schemes and the challenges involved in devising such schemes, especially at the level of successor of singular cardinals. The main tool available to obtain consistency results at the level of singular cardinals and their successors is the method of forcing with large cardinals and, in particular, Prikry-type forcings. By Prikry-type forcings one usually means to a poset $\mathbb{P} = (\mathbb{P}, \leq)$ having the following property.

Prikry Property. There exists an ordering $\leq^*$ on $\mathbb{P}$ coarser than $\leq$ (typically, of a better closure degree) satisfying that for every sentence $\varphi$ in the forcing language and every $p \in \mathbb{P}$ there exists $q \in \mathbb{P}$ with $q \leq^* p$ deciding $\varphi$.

In this paper, we develop an iteration scheme for Prikry-type posets, specifically, for the class of $\Sigma$-Prikry forcings that we introduced in [PRS20] (see Definition 2.3 below). Of course, viable iteration schemes for Prikry-type posets already exists, namely, the Magidor iteration and the Gitik iteration (see [Git10, §6]). In both these cases the ordering $\leq^*$ witnessing the Prikry Property of the iteration can be roughly described as the finite-support iteration of the $\leq^*$-orderings of its components. As the expectation from the final $\leq^*$ is to have an eventually-high closure degree, the two schemes are typically useful in the context where one carries an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \rho \rangle$ with each $\dot{\mathbb{Q}}_\alpha$ being a $\mathbb{P}_\alpha$-name for either a trivial forcing, or a Prikry-type forcing concentrating on the combinatorics of the inaccessible cardinal $\alpha$. This should be compared with the iteration to control the power function $\alpha \mapsto 2^\alpha$ below some cardinal $\rho$.

In contrast, in this paper, we are interested in carrying out an iteration of length $\kappa^{++}$, where $\kappa$ is a singular cardinal (or, more generally, forced by the first step of the iteration to become one), and all components of the iteration are Prikry-type forcings that concentrate on the combinatorics of $\kappa$ or its successor. For this, we will need to allow a support of arbitrarily large size below $\kappa$. To be able to lift the Prikry property through an infinite-support iteration, members of the $\Sigma$-Prikry class are thus required to possess the following stronger property, which is inspired by the concepts coming from the study of topological Ramsey spaces [Tod10].

Complete Prikry Property. There is a partition of the ordering $\leq$ into countably many relations $\langle \leq_n \mid n < \omega \rangle$ such that, if we denote $\text{cone}_n(q) := \{ r \mid r \leq_n q \}$, then, for every $0$-open $U \subseteq P$ (i.e., $q \in U \implies \text{cone}_0(q) \subseteq U$), every $p \in P$ and every $n < \omega$, there exists $q \leq^* p$ such that $\text{cone}_n(q)$ is either a subset of $U$ or disjoint from $U$.

Another parameter that requires attention when devising an iteration scheme is the chain condition of the components to be used. In view of the goal of solving a problem concerning the combinatorics of $\kappa$ or its successor through an iteration of length $\kappa^{++}$, there is a need to know that all counter-examples to our problem will show up at some intermediate stage of the
iteration, so that we at least have the chance to kill them all. The standard way to secure the latter is to require that the whole iteration $P_{\kappa^+}$ would have the $\kappa^+-$chain condition ($\kappa^+-\text{cc}$). As the $\kappa$-support iteration of $\kappa^+-\text{cc}$ posets need not have the $\kappa^+-\text{cc}$ (see [Ros18] for an explicit counterexample), members of the $\Sigma$-Prikry class are required to satisfy the following strong form of the $\kappa^+-\text{cc}$:

**Linked$_0$ Property.** There exists a map $c : P \to \kappa^+$ satisfying that for all $p, q \in P$, if $c(p) = c(q)$, then $p$ and $q$ are compatible, and, furthermore, $\text{cone}_0(p) \cap \text{cone}_0(q)$ is nonempty.

In particular, our verification of the chain condition of $P_{\kappa^+}$ will not go through the $\Delta$-system lemma; rather, we will take advantage of a basic fact concerning the density of box products of topological spaces.

Now that we have a way to ensure that all counterexamples show up at intermediate stages, we fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^+ \rangle$, and shall want that, for any $\alpha < \kappa^+$, $P_{\alpha+1}$ will amount to forcing over the model $V^{P_\alpha}$ to solve a problem suggested by $z_\alpha$. The standard approach to achieve this is to set $P_{\alpha+1} := P_\alpha * Q_{z_\alpha}$, where $Q_{z_\alpha}$ is a $P_\alpha$-name for a poset that takes care of $z_\alpha$. However, the disadvantage of this approach is that if $P_1$ is a notion of forcing that blows up $2^{\kappa^+}$, then any typical poset $Q_1$ in $V^{P_1}$, which is designed to add a subset of $\kappa^+$ via bounded approximations will fail to have the $\kappa^+-\text{cc}$. To work around this, in our scheme, we set $P_{\alpha+1} := A(P_\alpha, z_\alpha)$, where $A(\cdot, \cdot)$ is a functor that, to each $\Sigma$-Prikry poset $P$ and a problem $z$, produces a $\Sigma$-Prikry poset $A(P, z)$ that projects onto $P$ and solves the problem $z$. A key feature of this functor is that the projection from $A(P, z)$ to $P$ splits, that is, in addition to a projection map $\pi$ from $A(P, z)$ onto $P$, there is a map $\kappa$ that goes in the other direction, and the two maps commute in a very strong sense. The exact details may be found in our definition of forking projection (see Definition 2.13 below).

A special case of the main result of this paper may be roughly stated as follows.

**Main Theorem.** Suppose that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of regular uncountable cardinals, converging to a cardinal $\kappa$. For simplicity, let us say that a notion of forcing $P$ is nice if it has property $D$, $P \subseteq H_{\kappa^+}$ and $P$ does not collapse $\kappa^+$. Now, suppose that:

- $Q$ is a nice $\Sigma$-Prikry notion of forcing;
- $A(\cdot, \cdot)$ is a functor that produces for every nice $\Sigma$-Prikry notion of forcing $P$, and every $z \in H_{\kappa^+}$, a corresponding nice $\Sigma$-Prikry notion of forcing $A(P, z)$. Moreover, $A(\cdot, \cdot)$ admits a forking projection to $P$ with the weak mixing property;
- $2^{2^\kappa} = \kappa^{++}$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++} \rangle$.

Then there exists a sequence $\langle P_\alpha \mid \alpha \leq \kappa^{++} \rangle$ of forcings such that $P_1$ is isomorphic to $Q$, $P_{\alpha+1}$ is isomorphic to $A(P_\alpha, z_\alpha)$, and, for every pair $\alpha \leq \beta$
Moreover, if for each nonzero limit ordinal \( \alpha \leq \kappa^{++} \), \( \mathbb{P}_\alpha \) contains a canonical dense subforcing \( \hat{\mathbb{P}}_\alpha \), then \( \langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++} \rangle \) consists of nice \( \Sigma \)-Prikry forcings.

1.1. Organization of this paper. In Section 2, we recall the definitions of the \( \Sigma \)-Prikry class, forking projections, and introduce property \( D \) and the weak mixing property.

In Section 3, we present our abstract iteration scheme for \( \Sigma \)-Prikry posets, and prove the Main Theorem of this paper (see Lemmas 3.6 and 3.13).

In Section 4, we present the very first application of our scheme. We carry out an iteration of length \( \kappa^{++} \), where the first step of the iteration is the Extender Based Prikry Forcing (EBPF) due to Gitik and Magidor [GM94, §3] for making \( 2^\kappa = \kappa^{++} \), and all the later steps are obtained by invoking the functor \( \mathcal{A}(P, z) \) from [PRS20, §6] for killing a nonreflecting stationary subset \( z \). This functor is due to Sharon [Sha05, §2], and as a corollary, we obtain a correct proof of the main result of [Sha05, §3]:

**Corollary.** If \( \kappa \) is the limit of a countable increasing sequence of supercompact cardinals, then there exists a cofinality-preserving forcing extension in which \( \kappa \) remains a strong limit, every finite collection of stationary subsets of \( \kappa^+ \) reflects simultaneously, and \( 2^\kappa = \kappa^{++} \).

1.2. Notation and conventions. Our forcing convention is that \( p \leq q \) means that \( p \) extends \( q \). We write \( \mathbb{P} \downarrow q \) for \( \{ p \in \mathbb{P} \mid p \leq q \} \). Denote \( E_\theta^\mu := \{ \alpha < \mu \mid \text{cf}(\alpha) = \theta \} \). The sets \( E_{\leq \theta}^\mu \) and \( E_{> \theta}^\mu \) are defined in a similar fashion. For a stationary subset \( S \) of a regular uncountable cardinal \( \mu \), we write \( \text{Tr}(S) := \{ \delta \in E_{> \omega}^\mu \mid S \cap \delta \text{ is stationary in } \delta \} \). \( H_\nu \) denotes the collection of all sets of hereditary cardinality less than \( \nu \). For every set of ordinals \( x \), we denote \( cl(x) := \{ \sup(x \cap \gamma) \mid \gamma \in \text{Ord}, x \cap \gamma \neq \emptyset \} \), \( \text{acc}(x) := \{ \gamma \in x \mid \sup(x \cap \gamma) = \gamma > 0 \} \) and \( nacc(x) := x \setminus \text{acc}(x) \).

2. The \( \Sigma \)-Prikry class and forking projections

In this section, we recall some definitions and facts from [PRS20, §2] and [PRS20, §4], and then continue developing the theory of forking projections. The reader is not assumed to be familiar with [PRS20].

2.1. The \( \Sigma \)-Prikry class and Property \( D \).

**Definition 2.1.** We say that \( (\mathbb{P}, \ell) \) is a graded poset iff \( \mathbb{P} = (P, \leq) \) is a poset, \( \ell : P \to \omega \) is a surjection, and, for all \( p \in P \):

- For every \( q \leq p \), \( \ell(q) \geq \ell(p) \);
- There exists \( q \leq p \) with \( \ell(q) = \ell(p) + 1 \).

**Convention 2.2.** For a graded poset as above, we denote \( P_n := \{ p \in P \mid \ell(p) = n \} \), \( P^p_n := \{ q \in P \mid q \leq p, \ell(q) = \ell(p) + n \} \), and sometime write \( q \leq^n p \) (and say the \( q \) is an \( n \)-step extension of \( p \)) rather than writing \( q \in P^p_n \).
**Definition 2.3.** Suppose that $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}$, and that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$. Suppose that $\mu$ is a cardinal such that $1 \Vdash_{\mathbb{P}} \mu = \kappa^+$. For functions $\ell : P \to \omega$ and $c : P \to \mu$, we say that $(\mathbb{P}, \ell, c)$ is a notion of forcing with a greatest element, if exists, is unique. By convention, a greatest element, if exists, is unique.

**Fact 2.6 ([PRS20, Lemma 2.8]).** Let $p \in P$.

1. For every $n < \omega$, $W_n(p) := \{ w(p, q) \mid q \in P_n \}$, and $W_{\geq n}(p) := \{ w(p, q) \mid \exists m \in \omega \setminus n[q \in P_m]\}$. The object $W(p) := \bigcup_{n < \omega} W_n(p)$ is called the p-tree.

**Fact 2.7 ([PRS20, Lemma 2.10]).**

1. $\mathbb{P}$ does not add bounded subsets of $\kappa$.

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1. By convention, a greatest element, if exists, is unique.
2. Note that $w(p, q)$ is the weakest $n$-step extension of $p$ above $q$. 
(2) For every regular cardinal $\nu \geq \kappa$, if there exists $p \in P$ for which $p \Vdash \text{cf}(\nu) < \kappa$, then there exists $p' \leq p$ with $|W(p')| \geq \nu$.\textsuperscript{3}

**Definition 2.8.** We say that $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a good enumeration of a set $A$ iff $A$ is a cardinal, $\vec{r}$ is injective, and $\{r_\xi \mid \xi < \chi \} = A$.

**Definition 2.9** (Diagonalizability). Given $p \in P$, $n < \omega$, and a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ of $W_n(p)$, we say that $\vec{q} = \langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable (with respect to $\vec{r}$) iff the two hold:

(a) $q_\xi \leq^0 r_\xi$ for every $\xi < \chi$;

(b) there is $p' \leq^0 p$ such that for every $q' \in W_n(p')$, $q' \leq^0 q_\xi$, where $\xi$ is the unique index to satisfy $r_\xi = w(p,q')$.

**Definition 2.10** (Diagonalizability game). Given $p \in P$, $n < \omega$, a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ of $W_n(p)$, and a dense subset $D$ of $\mathbb{P}_{\ell_\nu(p)+n}$, $\mathbb{D}_p(p,\vec{r},D)$ is a game of length $\chi$ between two players $I$ and $II$, defined as follows:

- At stage $\xi < \chi$, $I$ plays a condition $p_\xi \leq^0 p$ compatible with $r_\xi$, and then $II$ plays $q_\xi \in D$ such that $q_\xi \leq p_\xi$ and $q_\xi \leq^0 r_\xi$;

- $I$ wins the game iff the resulting sequence $\vec{q} = \langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable.

In the special case that $D$ is all of $\mathbb{P}_{\ell_\nu(p)+n}$, we omit it, writing $\mathbb{D}_p(p,\vec{r})$.

The following lemma will be useful later.

**Lemma 2.11.** Given $p \in P$, $n < \omega$, a good enumeration $\vec{r}$ of $W_n(p)$, and a dense subset $D$ of $\mathbb{P}_{\ell_\nu(p)+n}$, $I$ has a winning strategy for $\mathbb{D}_p(p,\vec{r},D)$ iff it has a winning strategy for $\mathbb{D}_p(p,\vec{r})$.

**Proof.** Only the forward implication requires an argument. Write $\vec{r}$ as $\langle r_\xi \mid \xi < \chi \rangle$: we shall describe a winning strategy for $I$ in the game $\mathbb{D}_p(p,\vec{r})$ by producing sequences of the form $\langle (p_\eta, q_\eta, q'_\eta) \mid \eta < \xi \rangle$, where $\langle (p_\eta, q_\eta) \mid \eta < \xi \rangle$ is an initial play (consisting of $\xi$ rounds) in the game $\mathbb{D}_p(p,\vec{r})$, and $\langle (p_\eta, q'_\eta) \mid \eta < \xi \rangle$ is an initial play in the game $\mathbb{D}_p(p,\vec{r},D)$.

Assuming that $I$ has a winning strategy for $\mathbb{D}_p(p,\vec{r},D)$, here is a description of our winning strategy for $I$ in the game $\mathbb{D}_p(p,\vec{r})$:

- For $\xi = 0$, we play a condition $p_0$ according to the winning strategy of $I$ in the game $\mathbb{D}_p(p,\vec{r},D)$. Then, $II$ plays $q_0 \leq p_0$ such that $q_0 \leq^0 r_0$. Since $D$ is dense in $\mathbb{P}_{\ell_\nu(p)+n}$, we then pick $q'_0 \in D$ with $q'_0 \leq^0 q_0$.

- Suppose that $\xi < \chi$ is nonzero and that $\langle (p_\eta, q_\eta, q'_\eta) \mid \eta < \xi \rangle$ has already been defined. Let $p_\xi$ be given by the winning strategy of $I$ for the game $\mathbb{D}_p(p,\vec{r},D)$ with respect to the initial play $\langle (p_\eta, q'_\eta) \mid \eta < \xi \rangle$. Then, $II$ plays $q_\xi \leq p_\xi$ such that $q_\xi \leq^0 r_\xi$. Finally, pick $q'_\xi \in D$ such that $q'_\xi \leq^0 q_\xi$.

At the end of the above process, since $\langle (p_\xi, q'_\xi) \mid \xi < \chi \rangle$ is a play in the game $\mathbb{D}_p(p,\vec{r},D)$ using the winning strategy of $I$, we may fix $p' \leq^0 p$\textsuperscript{3}For future reference, we point out that this fact relies only on Clauses (1), (2), (4) and (7) of Definition 2.3. Furthermore, we do not need to know that $I$ decides a value for $\kappa^+$.
witnessing that \( \langle q'_\xi \mid \xi < \chi \rangle \) is diagonalizable. So, for every \( q' \in W_n(p') \),
if \( \xi \) is the unique index to satisfy \( r_\xi = w(p, q') \), then \( q' \leq^0 q'_\xi \leq^0 q_\xi \). In
particular, \( p' \) witnesses that \( \langle q_\xi \mid \xi < \chi \rangle \) is diagonalizable, as desired. \( \square \)

**Definition 2.12 (Property \( D \)).** We say that \((\mathbb{P}, \ell_\mathbb{P})\) has property \( D \) iff for
any \( p \in \mathbb{P}, \ n < \omega \) and any good enumeration \( \vec{r} = \langle r_\xi \mid \xi < \chi \rangle \) of \( W_n(p) \), \( \mathbb{I} \)
has a winning strategy for the game \( \mathcal{D}(p, \vec{r}) \).

2.2. Forking projections. In this and the next subsection, we continue the work started in [PRS20, §4] concerning forking projections. This will
play a key role in Section 3, where we deal with iterating \( \Sigma \)-Prikry posets.

**Definition 2.13 ([PRS20, Definition 4.1]).** Suppose that \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) is a \( \Sigma \-
Prikry triple, \( \mathbb{A} = (A, \subseteq) \) is a notion of forcing, and \( \ell_\mathbb{A} \) and \( c_\mathbb{A} \) are functions
with \( \text{dom}(\ell_\mathbb{A}) = \text{dom}(c_\mathbb{A}) = A \).

A pair of functions \((\mathfrak{h}, \pi)\) is said to be a forking projection from \((\mathbb{A}, \ell_\mathbb{A})\)
to \((\mathbb{P}, \ell_\mathbb{P})\) iff all of the following hold:

1. \( \pi \) is a projection from \( \mathbb{A} \) onto \( \mathbb{P} \), and \( \ell_\mathbb{A} = \ell_\mathbb{P} \circ \pi; \)
2. for all \( a \in A \), \( \mathfrak{h}(a) \) is an order-preserving function from \( \langle \mathbb{P} \downarrow \pi(a), \leq \rangle \) to \((A \downarrow a, \leq)\); 
3. for all \( p \in \mathbb{P}, \ \{ a \in A \mid \pi(a) = p \} \) admits a greatest element, which
   we denote by \( \lceil p \rceil^A \);  
4. for all \( n, m < \omega \) and \( b \leq^{n+m} a \), \( m(a, b) \) exists and satisfies:
   \[ m(a, b) = \mathfrak{h}(a)(m(\pi(a), \pi(b))); \]
5. for all \( a \in A \) and \( q \leq \pi(a), \ \pi(\mathfrak{h}(a)(q)) = q; \)
6. for all \( a \in A \) and \( q \leq \pi(a), \ a = \lceil \pi(a) \rceil^A \) iff \( \mathfrak{h}(a)(q) = \lceil q \rceil^A; \)
7. for all \( a \in A, a' \leq^0 a \) and \( r \leq^0 \pi(a'), \ \mathfrak{h}(a)(r) \leq \mathfrak{h}(a)(r). \)

The pair \((\mathfrak{h}, \pi)\) is said to be a forking projection from \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) to \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) iff, in addition to all of the above, the following holds:

8. for all \( a, a' \in A, \) if \( c_\mathbb{A}(a) = c_\mathbb{A}(a') \), then \( c_\mathbb{P}(\pi(a)) = c_\mathbb{P}(\pi(a')) \) and, for
   all \( r \in P_0(\pi(a)) \cap P_0(\pi(a')), \ \mathfrak{h}(a)(r) = \mathfrak{h}(a')(r). \)

**Remark 2.14.** Intuitively speaking, \( \mathfrak{h}(a) \) is an operator that, for each condition \( p \in \mathbb{P} \downarrow \pi(a) \), provides the \( \subseteq \)-greatest condition \( b \leq a \) with \( \pi(b) = p \).

**Example 2.15.** Suppose that \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) is a \( \Sigma \)-Prikry triple. Let \( \mu \) denote the cardinal such that \( \mathbb{I} \models \mu = \kappa^+ \). We define the following objects:

- \( \mathbb{A} = (A, \subseteq) \), where \( A := P \times \mu \) and \( (p, \alpha) \leq (p, \beta) \) iff \( p \leq q \) and \( \alpha \supseteq \beta; \)
- \( \ell_\mathbb{A} : A \to \omega \) via \( \ell_\mathbb{A}(p, \alpha) := \ell_\mathbb{P}(p); \)
- \( c_\mathbb{A} : A \to \mu \times \mu \) via \( c_\mathbb{A}(p, \alpha) := (c_\mathbb{P}(p), \alpha); \)
- \( \pi : A \to P \) via \( \pi(p, \alpha) := p; \)
- for \( a = (p, \alpha) \in A \), define \( \mathfrak{h}(a) : \mathbb{P} \downarrow p \to A \) via \( \mathfrak{h}(a)(q) := (q, \alpha). \)

Then \((\mathfrak{h}, \pi)\) is a forking projection from \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) to \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\).

\[ \text{For future reference we point out that } [a]^A = (\pi(a), 0) \text{ for all } a \in A. \]
Definition 2.16. Given two posets $\mathbb{P} = (P, \leq)$ and $\mathbb{A} := (A, \leq)$, and a projection $\pi$ from $\mathbb{A}$ to $\mathbb{P}$, we denote by $\mathbb{A}^\pi$ the poset $(A, \leq^\pi)$, where $a \leq^\pi b$ iff $a \leq b$ and $\pi(a) = \pi(b)$.

For a subposet $\dot{\mathbb{A}} := (A, \leq)$ of $\mathbb{A}$, we likewise denote $\dot{\mathbb{A}}^\pi := (\dot{A}, \leq^\pi)$.

Lemma 2.17. Suppose that $(\dot{\mathbb{A}}, \pi)$ is a forking projection from $(\dot{\mathbb{A}}, \ell_{\dot{\mathbb{A}}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$. For every $a \in A$, $\dot{\mathbb{A}}(a)^{(\pi(a))} = (a, a)$. 

Proof. By Definition 2.13(4), using $(m, n, b) := (0, 0, a)$, we infer that $\dot{\mathbb{A}}(a)(\pi(a)) = \dot{\mathbb{A}}(a)(w(\pi(a), \pi(a))) = w(a, a) = a$. □

Lemma 2.18 (Canonical form). Suppose that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ and $(\dot{\mathbb{A}}, \ell_{\dot{\mathbb{A}}}, c_{\dot{\mathbb{A}}})$ are both $\Sigma$-Prikry notions of forcing. Denote $\mathbb{P} = (P, \leq)$ and $\dot{\mathbb{A}} = (A, \leq)$.

If $(\dot{\mathbb{A}}, \ell_{\dot{\mathbb{A}}}, c_{\dot{\mathbb{A}}})$ admits a forking projection to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ as witnessed by a pair $(\dot{\mathbb{A}}, \pi)$, then we may assume that all of the following hold true:

1. each element of $A$ is a pair $(x, y)$ with $\pi(x, y) = x$;
2. for all $a \in A$, $[\pi(a)]^\dot{A} = (\pi(a), \emptyset)$;
3. for all $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{\dot{\mathbb{A}}}(\dot{[p]}^\dot{A}) = c_{\dot{\mathbb{A}}}(\dot{[q]}^\dot{A})$.

Proof. By applying a bijection, we may assume that $A = |A|$ with $1_{\dot{\mathbb{A}}} = \emptyset$. To clarify what we are about to do, we agree to say that “$a$ is a lift” iff $a = [\pi(a)]^\dot{A}$. Now, define $f : A \to P \times A$ via:

$$f(a) := \begin{cases} (\pi(a), \emptyset), & \text{if } a \text{ is a lift;} \\
(\pi(a), a), & \text{otherwise.} \end{cases}$$

Claim 2.18.1. $f$ is injective.

Proof. Suppose $a, a' \in A$ with $f(a) = f(a')$.

$\blacktriangleright$ If $a$ is not a lift and $a'$ is not a lift, then from $f(a) = f(a')$ we immediately get that $a = a'$.

$\blacktriangleright$ If $a$ is a lift and $a'$ is a lift, then from $f(a) = f(a')$, we infer that $\pi(a) = \pi(a')$, so that $a = [\pi(a)]^\dot{A} = [\pi(a')]^\dot{A} = a'$.

$\blacktriangleright$ If $a$ is not a lift, but $a'$ is a lift, then from $f(a) = f(a')$, we infer that $a = \emptyset = 1_{\dot{\mathbb{A}}}$, contradicting the fact that $1_{\dot{\mathbb{A}}} = [1_{\mathbb{P}}]^\dot{A} = [\pi(1_{\dot{\mathbb{A}}})]^\dot{A}$ is a lift. So this case is void. □

Let $B := \operatorname{Im}(f)$ and $\alpha_B := \{(f(a), f(b)) \mid a \leq b\}$, so that $\mathbb{B} := (B, \leq_B)$ is isomorphic to $\dot{\mathbb{A}}$. Define $\ell_{\mathbb{B}} := \ell_{\dot{\mathbb{A}}} \circ f^{-1}$ and $\pi_{\mathbb{B}} := \pi \circ f^{-1}$. Also, define $\dot{\mathbb{B}}$ via $\dot{\mathbb{B}}(b)(p) := f(\dot{\mathbb{A}}(f^{-1}(b))(p))$. It is clear that $b \in B$ is a lift iff $f^{-1}(a)$ is a lift if $b = (\pi_{\mathbb{B}}(b), \emptyset)$.

Next, define $c_{\mathbb{B}} : B \to \mu \times 2$ by letting for all $b \in B$:

$$c_{\mathbb{B}}(b) := \begin{cases} (c_{\mathbb{P}}(\pi_{\mathbb{B}}(b)), 0), & \text{if } b \text{ is a lift;} \\
(c_{\dot{\mathbb{A}}}(f^{-1}(b)), 1), & \text{otherwise.} \end{cases}$$

Claim 2.18.2. Suppose $b_0, b_1 \in B$ with $c_{\mathbb{B}}(b_0) = c_{\mathbb{B}}(b_1)$. Then $c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_0)) = c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_1))$ and, for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\dot{\mathbb{B}}(b_0)(r) = \dot{\mathbb{B}}(b_1)(r)$. 
Proof. We focus on verifying that for all $r \in P_0^{\pi_R(b_0)} \cap P_0^{\pi_R(b_1)}$, $\hat{\eta}_B(b_0)(r) = \hat{\eta}_B(b_1)(r)$. For each $i < 2$, denote $a_i := f^{-1}(b_i)$ and $p_i := \pi_R(b_i)$, so that $\pi(a_i) = p_i$. Suppose $r \in P_0^{\pi_R} \cap P_0^{\pi_R}$.

$\mathbf{\triangledown}$ If $b_0$ is a lift, then so are $b_1, a_0, a_1$. Therefore, for each $i < 2$, Definition 2.13(6) implies that $\hat{\eta}_B(b_i)(r) = f(\hat{\eta}(a_i)(r)) = f([r]^A) = [r]^B$. In effect, $\hat{\eta}_B(b_0)(r) = \hat{\eta}_B(b_1)(r)$, as desired.

$\mathbf{\triangledown}$ Otherwise, $c_A(a_0) = c_A(a_1)$. As $r \in P_0^{\pi(a_0)} \cap P_0^{\pi(a_1)}$, $\hat{\eta}_B(b_0)(p) = f(\hat{\eta}(a_0)(p)) = f(\hat{\eta}(a_1)(p)) = \hat{\eta}_B(b_1)(p)$.

This completes the proof. □

Setup 2. Throughout the rest of this section, suppose that:

- $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $1_\mathbb{P}$;
- $\mathbb{A} = (A, \preceq)$ is a notion of forcing with a greatest element $1_\mathbb{A}$;
- $\Sigma = (\kappa_n \mid n < \omega)$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$, and $\mu$ is a cardinal such that $1_\mathbb{P} \force \check{\mu} = \check{\kappa}^+$;
- $\ell_\mathbb{P}$ and $c_\mathbb{P}$ are functions witnessing that $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ is a $\Sigma$-Prikry;
- $\ell_\mathbb{A}$ and $c_\mathbb{A}$ are functions with $\text{dom}(\ell_\mathbb{A}) = \text{dom}(c_\mathbb{A}) = A$;
- $(\hat{\eta}, \pi)$ is a forking projection from $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$.

The next two facts will help verifying Clauses (1) and (3) of Definition 2.3 for the different stages of the iteration of Section 3.

**Fact 2.19** ([PRS20, Lemma 4.3]). Suppose that $(\hat{\eta}, \pi)$ is a forking projection from $(\mathbb{A}, \ell_\mathbb{A})$ to $(\mathbb{P}, \ell_\mathbb{P})$, or, just a pair of maps satisfying Clauses (1), (2) and (4) of Definition 2.13. For each $a \in A$, the following holds:

1. $\hat{\eta}(a) \restriction W(\pi(a))$ forms a bijection from $W(\pi(a))$ to $W(a)$;
2. For all $n < \omega$ and $r \in P_0^{\pi(a)}$, $\hat{\eta}(a)(r) \in A_0^n$.

In particular, $(\mathbb{A}, \ell_\mathbb{A})$ is a graded poset.

**Fact 2.20** ([PRS20, Lemma 4.7]). Suppose that $(\hat{\eta}, \pi)$ is a forking projection from $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$, or, just a pair of maps satisfying Clauses (1), (2), (4), (7) and (8) of Definition 2.13. For all $a, a' \in A$, if $c_\mathbb{A}(a) = c_\mathbb{A}(a')$, then $A_0^n \cap A_0^{a'}$ is non-empty. In particular, if $|\text{Im}(c_\mathbb{A})| \leq \mu$, then $(\mathbb{A}, \ell_\mathbb{A})$ is $\mu^{+,-}$-linked.

**Lemma 2.21.** Suppose that $(\mathbb{A}, \ell_\mathbb{A})$ has property $\mathcal{D}$. Then it has the CPP.

**Proof.** Let $U \subseteq A$ be a 0-open set, $a \in A$ and $n < \omega$. Let $\tilde{r} = \langle r_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$. Let $\langle (a_\xi, b_\xi) \mid \xi < \chi \rangle$ list the rounds of the game $\varnothing_A(a, \tilde{r})$ in which, in round $\xi$, $I$ plays according to their winning strategy and $II$ plays $b_\xi \leq^n a_\xi$ such that

1. $b_\xi \leq^n r_\xi$, and
2. if $A_0^n \cap U \neq \emptyset$, then $b_\xi \in U$.

Let $a' \leq^n a$ be a condition witnessing the diagonalizability of $\langle b_\xi \mid \xi < \chi \rangle$. Set $p := \pi(a)$ and $p' := \pi(a')$. By Fact 2.19, $W(a) = \hat{\eta}(a)^*W(p)$, hence, for
each \( q \leq^n p \), we may let \( \xi(q) < \chi \) be such that \( \check{\eta}(a)(w(p, q)) = r_{\xi(q)} \). Set \( \bar{U} := \{ q \in P^b_n \mid b_{\xi(q)} \in U \} \). As \( q' \leq^0 q \leq^n p \) implies \( \xi(q) = \xi(q') \), the set \( \bar{U} \) is 0-open. Now, since \( (\mathbb{P}, \mathbb{E}, c) \) is \( \Sigma \)-Prikry (see Setup 2), applying CPP to \( \bar{U}, p' \), and \( n \), we find \( \bar{p} \leq^0 p' \) such that either \( P^b_n \subseteq \bar{U} \) or \( P^b_n \cap \bar{U} = \emptyset \).

Set \( \bar{a} := \check{\eta}(a')(\bar{p}) \). Since \( \bar{p} \leq^0 p' \leq^0 p \), Clauses (1) and (2) of Definition 2.13 yield \( \bar{a} \leq^0 a' \leq^0 a \).

**Claim 2.21.1.** Let \( b \in A^a_n \). Then:

1. \( b \leq^0 b_{\xi(\pi(b))} \);
2. If \( b \in U \), then \( P^b_n \subseteq \bar{U} \).

**Proof.** Denote \( q := \pi(b) \).

(1) Since \( w(a', b) \in W_n(a') \) and \( a' \) is a witness to diagonalizability of \( \langle b_\xi \mid \xi < \chi \rangle \), \( b \leq^0 w(a', b) \leq^0 b_\xi \), where \( \xi \) is the unique index to satisfy \( r_\xi = w(a, b) \).

By Clause (4) of Definition 2.13,

\[
\bar{r}_\xi = w(a, b) = \check{\eta}(a)(w(p, q)) = r_{\xi(q)},
\]

so that \( \xi = \xi(\pi(b)) \).

(2) Assuming that \( b \in U \), we altogether infer that \( b_{\xi(q)} \in U \). By the definition of \( \bar{U} \), then, \( q \in \bar{U} \cap P^\bar{p}_n \). So, by the choice of \( \bar{p} \), furthermore \( P^\bar{p}_n \subseteq \bar{U} \).

It thus follows that if \( A^a_n \cap U \neq \emptyset \), then for every \( b \in A^a_n \), \( \pi(b) \in P^b_n \subseteq \bar{U} \), so that \( b_{\xi(\pi(b))} \in U \). By the preceding claim, \( b \leq^0 b_{\xi(\pi(b))} \), so, since \( U \) is 0-open, \( b \in U \). Thus we have shown that if \( A^a_n \cap U \neq \emptyset \), then \( A^a_n \subseteq U \).\( \square \)

**Proposition 2.22.** Let \( a \in A \), \( n < \omega \) and \( \bar{s} = \langle s_\xi \mid \xi < \chi \rangle \) be a good enumeration of \( W_n(a) \).

Suppose that \( \langle b_\xi \mid \xi < \chi \rangle \) is a sequence of conditions in \( \mathbb{A} \downarrow a \) such that:

(a) \( \langle \pi(b_\xi) \mid \xi < \chi \rangle \) is diagonalizable with respect to \( \langle \pi(s_\xi) \mid \xi < \chi \rangle \), as witnessed by \( p' \leq^0 \pi(a) \);\(^5\)

(\( \beta \)) \( b \) is a condition in \( \mathbb{A} \) with \( \pi(b) = p' \) such that, for all \( q' \in W_n(p') \),

\[
\check{\eta}(b)(q') \leq^0 b_\xi,
\]

where \( \xi \) is the unique index such that \( \pi(s_\xi) = w(\pi(a), q') \).

Then \( b \) witnesses that \( \langle b_\xi \mid \xi < \chi \rangle \) is diagonalizable with respect to \( \bar{s} \).

**Proof.** We go over the two clauses of Definition 2.9:

(a) Let \( \xi < \chi \). By Clause (a) above, \( \pi(b_\xi) \leq^0 \pi(s_\xi) \). Together with Definition 2.3(6), it follows that

\[
w(\pi(a), \pi(b_\xi)) \leq^0 w(\pi(a), \pi(s_\xi)) = \pi(s_\xi).
\]

Finally, Clauses (1), (4) and (5) of Definition 2.13 yield

\[
b_\xi \leq^0 w(a, b_\xi) = \check{\eta}(a)(w(\pi(a), \pi(b_\xi))) \leq^0 \check{\eta}(a)(\pi(s_\xi)) = s_\xi.
\]

\(^5\)By Fact 2.19, \( \langle \pi(s_\xi) \mid \xi < \chi \rangle \) is a good enumeration of \( W_n(\pi(a)) \), hence the Clause (a) is well-posed.
(b) Let \( b' \in W_n(b) \), and we shall show that \( b' \leq 0 b_\xi \), where \( \xi \) is the unique index to satisfy \( s_\xi = w(a,b') \). Set \( q' := \pi(b') \). As \( \pi(b) = p' \), we infer from Definition 2.13(4) that \( b' = \hat{\pi}(b)(q') \) and \( q' \in W_n(p') \). Thus, by Clause (β) above \( b' = \hat{\pi}(b)(q') \leq 0 b_\xi \), where \( \xi \) is the unique index such that \( \pi(s_\xi) = w(\pi(a), q') \). Again by Definition 2.13(4),

\[
s_\xi = \hat{\pi}(a)(\pi(s_\xi)) = \hat{\pi}(a)(w(\pi(a), q')) = w(a,b'),
\]
as desired. \( \Box \)

2.3. Types and the weak mixing property. In this subsection, we will provide a sufficient condition for \((\hat{\mu}, \ell_\mu)\) to inherit property \( D \) from \((\mathbb{P}, \ell_\mathbb{P})\).

While reading the next two definitions, the reader may want to have a simple example in mind. Such an example is given by Lemma 2.27 below.

Definition 2.23 (Types). A type over \((\hat{\pi}, \pi)\) is a map \( tp : A \to \langle \mathbb{P}, \mathbb{P}_\mathbb{P} \rangle \) having the following properties:

1. for each \( a \in A \), either \( \text{dom}(tp(a)) = \alpha + 1 \) for some \( \alpha < \mu \), in which case we define \( mtp(a) := tp(a)(\alpha) \), or \( tp(a) \) is empty, in which case we define \( mtp(a) := 0 \);
2. for all \( a, b \in A \) with \( b \leq a \), \( \text{dom}(tp(a)) \leq \text{dom}(tp(b)) \) and for each \( i \in \text{dom}(tp(a)) \), \( tp(b)(i) \leq tp(a)(i) \);
3. for all \( a \in A \) and \( q \leq \pi(a) \), \( \text{dom}(tp(\hat{\pi}(a)(q))) = \text{dom}(tp(a)) \);
4. for all \( a \in A \), \( tp(a) = \emptyset \) iff \( a = [\pi(a)]^\hat{\mu} \);
5. for all \( a \in A \) and \( \alpha \in \mu \setminus \text{dom}(tp(a)) \), there exists a stretch of \( a \) to \( \alpha \), denoted \( a^\alpha \), and satisfying the following:
   a. \( a^\alpha \leq \pi a \);
   b. \( \text{dom}(tp(a^\alpha)) = \alpha + 1 \);
   c. \( tp(a^\alpha)(i) \leq mtp(a) \) whenever \( \text{dom}(tp(a)) \leq i \leq \alpha \);
6. for all \( a, b \in A \) with \( \text{dom}(tp(a)) = \text{dom}(tp(b)) \), for every \( \alpha \in \mu \setminus \text{dom}(tp(a)) \), if \( b \leq a \), then \( b^\alpha \leq a^\alpha \);
7. For each \( n < \omega \), the poset \( \hat{A}_n \) is dense in \( A_n \), where \( \hat{A}_n := (A_n, \subseteq) \) and \( \hat{A}_n := \{ a \in A_n \mid \pi(a) \in P_n \land mtp(a) = 0 \} \).

Remark 2.24. Note that Clauses (2) and (3) imply that for all \( m, n < \omega \), \( a \in A_m \) and \( q \leq \pi(a) \), if \( q \in P_n \) then \( \hat{\pi}(a)(q) \in A_n \).

The next definition is a weakening of [PRS20, Definition 4.11].

Definition 2.25 (Weak Mixing property). The forking projection \((\hat{\mu}, \pi)\) is said to have the weak mixing property if it admits a type \( tp \) satisfying that for every \( n < \omega \), there exists an ordering \( \square^n \subseteq \leq^0 \) such that for all \( a \in A, \hat{\tau} \), and \( p' \leq^0 \pi(a) \), and for every function \( g : W_n(\pi(a)) \to \hat{\mu} \downarrow a \), if there exists an ordinal \( \iota \) such that all of the following hold:

1. \( \hat{\tau} = (r_\xi \mid \xi < \chi) \) is a good enumeration of \( W_n(\pi(a)) \);
2. \( \langle \pi(g(r_\xi)) \mid \xi < \chi \rangle \) is diagonalizable with respect to \( \hat{\tau} \), as witnessed by \( p'^\iota \).

\( ^6 \)In particular, \( \ell_\mu(g(r_\xi)) = \ell_\mu(a) + n \) for every \( \xi < \chi \).
(3) for every $\xi < \chi$: 
- if $\xi < \iota$, then $\text{dom}(tp(g(\xi))) = 0$; 
- if $\xi = \iota$, then $\text{dom}(tp(g(\xi))) \geq 1$; 
- if $\xi > \iota$, then $(\sup_{\eta<\xi} \text{dom}(tp(g(r_\eta)))) + 1 < \text{dom}(tp(g(\xi)))$; 

(4) for all $\xi \in (\iota, \chi)$ and $i \in [\text{dom}(tp(a))]} \cup \{\sup_{\eta<\xi} \text{dom}(tp(g(r_\eta)))\}$, 
$$tp(g(r_\xi))(i) \leq mtp(a),$$ 

(5) $\sup_{\xi<\chi} mtp(g(r_\xi)) < \omega$, 
then there exists $b \subseteq \mathbb{N}$ with $\pi(b) = p'$ such that, for all $q' \in W_n(p')$, 
$$\text{\textit{\&}(b)(q') \subseteq g(w(\pi(a)), q')).$$

Remark 2.26. Note that there may be more than one witness for a forking projection to admit the weak mixing property. For instance, if $(\textit{\&}, \pi)$ has the weak mixing property as witnessed by some type $tp$ and a sequence of “fusion” orderings $\langle \subseteq^n \mid n < \omega \rangle$, then the weak mixing property is also witnessed by $tp$ and the constant sequence $\langle \subseteq^0 \mid n < \omega \rangle$. The more we know about these witnesses, the better form of mixing we obtain. The role of the witnesses will only become apparent when carrying out a transfinite iteration (see Lemma 3.10 and Claim 4.23.2 below). For the main example of these fusion orderings, see Definition 4.12.

Lemma 2.27. The forking projection $(\textit{\&}, \pi)$ from Example 2.15 has the weak mixing property.

Proof. We attach a type $tp : A \rightarrow \langle\nu\omega$ as follows. For every $a = (p, \alpha) \in A$, with $\alpha > 0$, let $tp(a)$ be the constant $(\alpha + 1)$-sequence whose sole value is 0. Otherwise, let $tp(a) := \emptyset$. We shall verify that $tp$ and the constant sequence $\langle \subseteq^0 \mid n < \omega \rangle$ witness the weak mixing property. To this end, suppose that we are given $n < \omega$, $a \in A$, $\vec{r} = (r_\xi \mid \xi < \chi)$, $p' \subseteq^0 \pi(a)$, a function $g : W_n(\pi(a)) \rightarrow A \downarrow a$ and an ordinal $\iota$ satisfying Clauses (1)–(4) of Definition 2.25. For each $\xi < \chi$, write $(q_\xi, \alpha_\xi) := g(r_\xi)$. Note that by Clause (4) of Definition 2.23 and Footnote 4, $\alpha_\xi = 0$ for all $\xi < \iota$.

Set $b := (p', \alpha')$, for $\alpha' := \sup_{\xi<\chi} \alpha_\xi$. Note that, by regularity of $\mu$, $\alpha' < \mu$. Now, since $p'$ witnesses that $\langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable, for every $q' \in W_n(p')$, if we let $\xi$ denote the unique index to satisfy $r_\xi = w(\pi(a), q')$, then $q' \subseteq q_\xi$. As $\alpha' \geq \alpha_\xi$, it altogether follows that $(q', \alpha') = \textit{\&}(b)(q') \subseteq^0 g(w(\pi(a), q')) = (q_\xi, \alpha_\xi)$. □

Lemma 2.28. Suppose that $(\textit{\&}, \pi)$ has the weak mixing property and that $(\mathbb{P}, \ell_\mathbb{P})$ has property $\mathcal{D}$. Then $(\mathcal{A}, \ell_{\mathcal{A}})$ has property $\mathcal{D}$, as well.

Proof. Let $a \in A$, $n < \omega$ and $\vec{s} = \langle s_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$, with $\chi = |W_n(a)|$. By Lemma 2.11 and Definition 2.23(7), it is enough to show that $\mathbf{I}$ has a winning strategy in $\mathcal{D}_\mathcal{A}(a, \vec{s}, D)$, where $D :=$  

\textsuperscript{7}The role of the $\iota$ would be to keep track of the support when we apply the weak mixing lemma in the iteration (see, e.g. Lemma 3.10 and Claim 3.11.6).
For each $\xi < \chi$, let $r_\xi := \pi(s_\xi)$. By Fact 2.19, $s_\xi = \hat{\pi}(a)(r_\xi)$, and $\vec{r} := (r_\xi \mid \xi < \chi)$ forms a good enumeration of $W_n(\pi(a))$.

Fix any type $\mathfrak{tp}$ witnessing the weak mixing property of $(\hat{\pi}, \pi)$. We may assume that $\mathbb{C}^n$ is nothing but $\leq^0$. We shall describe a winning strategy for $I$ in the game $\hat{\mathcal{D}}_A(\vec{s}, D)$ by producing sequences of the form $(\langle p_\eta, a_\eta, b_\eta, q_\eta \rangle \mid \eta < \xi)$, where $(\langle a_\eta, b_\eta \rangle \mid \eta < \xi)$ is an initial play (consisting of $\xi$ rounds) in the game $\hat{\mathcal{D}}_A(\vec{s}, D)$, and $(\langle p_\eta, q_\eta \rangle \mid \eta < \xi)$ is an initial play in the game $\hat{\mathcal{D}}_I(\pi(a), \vec{r})$.

For $\xi = 0$, we first play a condition $p_0$ according to the winning strategy for $I$ in the game $\hat{\mathcal{D}}_I(\pi(a), \vec{r})$. In particular, $p_0 \leq^0 \pi(a)$. As $p_0$ is compatible with $r_0$, fix a condition $r' \leq p_0, r_0$, and note that it follows from Definition 2.13(2) that $\hat{\pi}(a)(r') \leq \hat{\pi}(a)(p_0), \hat{\pi}(a)(r_0)$. Now set $\alpha_0 := \text{dom}(\mathfrak{tp}(a)) + 1$ and $a_0 := \hat{\pi}(a)(p_0)^{\alpha_0}$. By Definition 2.23(5), $a_0 \leq^0 a$, $\pi(a_0) = p_0$, and it can also be shown that $a_0$ is compatible with $s_0$. Next, let $\mathbf{I}$ play $b_0 \in D$ such that $b_0 \leq a_0$ and $b_0 \leq^0 s_0$. Finally, let $q_0 := \pi(b_0)$.

For $\xi < \chi$ is nonzero and that $(\langle p_\eta, a_\eta, b_\eta, q_\eta \rangle \mid \eta < \xi)$ has already been defined. Let $p_\xi$ be given by the winning strategy for $I$ in the game $\hat{\mathcal{D}}_I(\pi(a), \vec{r})$ with respect to the initial play $\langle p_\eta, q_\eta \rangle \mid \eta < \xi)$. As in the previous case, we may fix a condition $r'$ such that $\hat{\pi}(a)(r') \leq \hat{\pi}(a)(p_\xi), s_\xi$.

Set $\vec{\alpha}_\xi := (\sup_{\eta < \xi} \text{dom}(\mathfrak{tp}(b_\eta))) + 1$. Then, by Clauses (5) and (6) of Definition 2.23, we may let $a_\xi := \hat{\pi}(a)(p_\xi)^{\vec{\alpha}_\xi}$, and infer that:

- $a_\xi \leq^\pi \hat{\pi}(a)(p_\xi)$;
- $\text{dom}(\mathfrak{tp}(a_\xi)) = \vec{\alpha}_\xi + 1$;
- $\mathfrak{tp}(a_\xi)(i) \leq \text{mtp}(\hat{\pi}(a)(p_\xi))$, whenever $\text{dom}(\mathfrak{tp}(\hat{\pi}(a)(p_\xi))) \leq i \leq \vec{\alpha}_\xi$;
- $\hat{\pi}(a)(r')^{\vec{\alpha}_\xi} \leq \hat{\pi}(a)(p_\xi)^{\vec{\alpha}_\xi} = a_\xi$ and $\hat{\pi}(a)(r')^{\vec{\alpha}_\xi} \leq \hat{\pi}(a)(r') \leq s_\xi$.

In particular, $a_\xi \leq^0 a$, $\pi(a_\xi) = p_\xi$ and $a_\xi$ is compatible with $s_\xi$. Next, let $\mathbf{II}$ play $b_\xi \in D$ such that $b_\xi \leq a_\xi$ and $a_\xi \leq^0 s_\xi$. Finally, let $q_\xi := \pi(b_\xi)$.

At the end of the game, we have produced a sequence $(\langle p_\xi, a_\xi, b_\xi, q_\xi \rangle \mid \xi < \chi)$, such that $\langle p_\xi, q_\xi \rangle \mid \xi < \chi)$ is the outcome of a game $\hat{\mathcal{D}}_I(\pi(a), \vec{r})$ in which $\mathbf{I}$ played according to a winning strategy, we may fix $p' \leq^0 \pi(a)$ witnessing that $\langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable.

It follows that if we define a function $g : W_n(\pi(a)) \to D$ via $g(r_\xi) := b_\xi$, then all the requirements of Definition 2.25 are fulfilled with respect to $\iota = 0$. For instance, to see that Clause (4) of Definition 2.25 holds, notice that by Clauses (2) and (5) of Definition 2.23, for all $\xi < \chi$ and $i \in \text{dom}(\mathfrak{tp}(a_\xi), \text{dom}(\mathfrak{tp}(a_\xi)))$,

$\mathfrak{tp}(b_\xi)(i) \leq \mathfrak{tp}(a_\xi)(i) \leq \text{mtp}(\hat{\pi}(a)(p_\xi)) \leq \text{mtp}(a)$.

In effect, we may pick $b \leq^0 a$ with $\pi(b) = p'$ such that for all $q' \in W_n(p')$, $\hat{\pi}(b)(q') \leq^0 g(w(\pi(a), q'))$.

---

8 For details see the upcoming argument in the case $\xi > 0$.

9 Note that by our construction, $\text{dom}(\mathfrak{tp}(a_\xi)) > 0$ for all $\xi < \chi$. 

---
By definition, for each \( q' \in W_n(p') \), \( g(w(\pi(a), q')) = b_\xi \), where \( \xi \) is the unique index such that \( \pi(s_\xi) = w(\pi(a), q') \). Therefore, invoking Proposition 2.22 we infer that \( b \) diagonalizes \( \langle b_\xi \mid \xi < \chi \rangle \), as desired.

\textbf{Corollary 2.29.} If \( (\mathcal{P}, \ell_\mathcal{P}) \) has property \( \mathcal{D} \), and \((\mathfrak{h}, \pi)\) has the weak mixing property, then \((\mathfrak{h}, \ell_\mathfrak{h})\) has the CPP.

\textit{Proof.} By Lemmas 2.21 and 2.28.

\textbf{Lemma 2.30.} Suppose that \((\mathfrak{h}, \pi)\) is as in Setup 2 or, just a pair of maps satisfying Clauses (1), (2), (5) and (7) of Definition 2.13.

Let \( n < \omega \). If \((\mathfrak{h}, \pi)\) admits a type, and \( \mathcal{A}_n \) is defined according to the last clause of Definition 2.13, if \( \mathcal{A}_n \) is \( \kappa_n \)-directed-closed, then so is \( \mathcal{A}_n \).

\textit{Proof.} The proof is very similar to that of [PRS20, Lemma 4.6], bearing Remark 2.24 in mind.

3. Iteration Scheme

In this section, we present our iteration scheme for \( \Sigma \)-Prikry posets. Throughout the section, assume that \( \Sigma = \langle \kappa_n \mid n < \omega \rangle \) is a non-decreasing sequence of regular uncountable cardinals. Denote \( \kappa := \sup_{n<\omega} \kappa_n \). Also, assume that \( \mu \) is some cardinal satisfying \( \mu^{< \mu} = \mu \), so that \( |H_\mu| = \mu \).

The following convention will be applied hereafter:

\textbf{Convention 3.1.} For all ordinals \( \gamma \leq \alpha \leq \mu^+ \):

1. \( \emptyset_\alpha := \alpha \times \{ \emptyset \} \) denotes the \( \alpha \)-sequence with constant value \( \emptyset \);
2. For a \( \gamma \)-sequence \( p \) and an \( \alpha \)-sequence \( q \), \( p \ast q \) denotes the unique \( \alpha \)-sequence satisfying that for all \( \beta < \alpha \):

\[
(p \ast q)(\beta) = \begin{cases} q(\beta), & \text{if } \gamma \leq \beta < \alpha; \\ p(\beta), & \text{otherwise}. \end{cases}
\]
3. Let \( \mathbb{P}_\alpha := (P_\alpha, \leq_\alpha) \) and \( \mathbb{P}_\gamma := (P_\gamma, \leq_\gamma) \) be forcing posets such that \( P_\alpha \subseteq \alpha H_\mu^{+} \) and \( P_\gamma \subseteq \gamma H_\mu^{+} \). Also, assume \( p \mapsto p \upharpoonright \gamma \) defines a projection between \( \mathbb{P}_\alpha \) and \( \mathbb{P}_\gamma \). We denote by \( i^\alpha_\gamma : V^{\mathbb{P}_\alpha} \to V^{\mathbb{P}_\gamma} \) the map defined by recursion over the rank of each \( \mathbb{P}_\gamma \)-name \( \sigma \) as follows:

\[ i^\alpha_\gamma(\sigma) := \{(i^\alpha_\gamma(\tau), p \ast \emptyset_\alpha) \mid (\tau, p) \in \sigma \}. \]

Our iteration scheme requires three building blocks:

\textbf{Building Block I.} We are given a \( \Sigma \)-Prikry triple \((Q, \ell, c)\) such that \( Q = (Q, \leq_Q) \) is a subset of \( H_\mu^{+} \), \( \mathbb{I}_Q \vDash_Q \mu = \kappa^+ \) and \( \mathbb{I}_Q \vDash_Q "\kappa \text{ is singular}".\footnote{At the behest of the referee, we stress that the last hypothesis plays a rather isolated role; see Footnote 22 on page 34.}

Additionally, we assume that \((Q, \ell)\) has property \( \mathcal{D} \). To streamline the matter, we also require that \( \mathbb{I}_Q \) be equal to \( \emptyset \).

\textbf{Building Block II.} For every \( \Sigma \)-Prikry triple \((\mathcal{P}, \ell_\mathcal{P}, c_\mathcal{P})\) having property \( \mathcal{D} \) such that \( \mathcal{P} = (P, \leq) \) is a subset of \( H_\mu^{+} \), \( \mathbb{I}_\mathcal{P} \vDash_\mathcal{P} \mu = \kappa^+ \) and \( \mathbb{I}_\mathcal{P} \vDash_\mathcal{P} \).
“κ is singular”, every \( r^* \in P \), and every \( \mathbb{P} \)-name \( z \in H_{\mu^+} \), we are given a corresponding \( \Sigma \)-Prikry triple \((A, \ell_A, c_A)\) having property \( \mathcal{D} \) such that:

(a) \((A, \ell_A, c_A)\) admits a forking projection \((\check{\pi}, \pi)\) to \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) that has the weak mixing property;
(b) for each \( n < \omega \), \( \check{\mathcal{A}}_n^n \) is \( \kappa_n \)-directed-closed;\(^{11}\)
(c) \( \mathbb{I}_A \models \check{\kappa}_n = \kappa^n; \)
(d) \( A = (A, \leq) \) is a subset of \( H_{\mu^+} \).

By Lemma 2.18, we may streamline the matter, and also require that:
(e) each element of \( A \) is a pair \((x, y)\) with \( \pi(x, y) = x; \)
(f) for every \( a \in A \), \([\pi(a)]^A = (\pi(a), \emptyset); \)
(g) for every \( p, q \in P \), if \( c_P(p) = c_P(q) \), then \( c_A([p]^A) = c_A([q]^A) \).

**Building Block III.** We are given a function \( \psi : \mu^+ \to H_{\mu^+} \).

**Goal 3.2.** Our goal is to define a system \( \{(\mathbb{P}_\alpha, \ell_{\alpha}, c_\alpha, (\check{n}_{\alpha, \gamma} \mid \gamma \leq \alpha)) \mid \alpha \leq \mu^+\} \) in such a way that for all \( \gamma \leq \alpha \leq \mu^+; \)

(i) \( \mathbb{P}_\alpha \) is a poset \((P_\alpha, \leq_{\alpha})\), \( P_\alpha \subseteq \mathcal{A} \mathcal{P}_{\mu^+} \), and, for all \( p \in P_\alpha \), \( |B_p| < \mu \), where \( B_p := \{ \beta + 1 : \beta \in dom(p) \land p(\beta) \neq \emptyset\} \);
(ii) The map \( \pi_{\alpha, \gamma} : P_\alpha \to P_\gamma \) defined by \( \pi_{\alpha, \gamma}(p) := p \upharpoonright \gamma \) forms a projection from \( \mathbb{P}_\alpha \) to \( \mathbb{P}_\gamma \), and \( \ell_\alpha = \ell_\gamma \circ \pi_{\alpha, \gamma}; \)
(iii) \( \mathbb{P}_0 \) is a trivial forcing, \( \mathbb{P}_1 \) is isomorphic to \( \mathbb{Q} \) given by Building Block I, and \( \mathbb{P}_{\alpha+1} \) is isomorphic to \( A \) given by Building Block II when invoked with \( (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha) \) and a pair \((r^*, z)\) which is decoded from \( \psi(\alpha); \)
(iv) If \( \alpha > 0 \), then \( (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha) \) is a \( \Sigma \)-Prikry triple having property \( \mathcal{D} \) whose greatest element is \( \emptyset_\alpha, \ell_\alpha = \ell_1 \circ \pi_{\alpha, 1}, \) and \( \emptyset_\alpha \models \mathbb{P}_\alpha \models \check{\kappa}_\alpha = \kappa^\alpha; \)
(v) If \( 0 < \gamma < \alpha \leq \mu^+; \) then \( (\check{n}_{\alpha, \gamma}, \pi_{\alpha, \gamma}) \) is a forking projection from \((\mathbb{P}_\alpha, \ell_\alpha)\) to \((\mathbb{P}_\gamma, \ell_\gamma)\); in case \( \alpha < \mu^+; \) \((\check{n}_{\alpha, \gamma}, \pi_{\alpha, \gamma}) \) is furthermore a forking projection from \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)\) to \((\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)\), and in case \( \alpha = \gamma + 1, (\check{n}_{\alpha, \gamma}, \pi_{\alpha, \gamma}) \) has the weak mixing property;
(vi) If \( 0 < \gamma \leq \beta \leq \alpha \), then, for all \( p \in P_\alpha \) and \( r \leq \gamma \), \( \check{n}_{\beta, \gamma} \upharpoonright (\check{n}_{\beta, \gamma} \upharpoonright \beta)(r) = (\check{n}_{\alpha, \gamma}(p)(r)) \upharpoonright \beta. \)

**Remark 3.3.** Note the asymmetry between the cases \( \alpha < \mu^+ \) and \( \alpha = \mu^+; \)

(1) By Clause (i), we will have that \( \mathbb{P}_\alpha \subseteq H_{\mu^+} \) for all \( \alpha < \mu^+, \) but \( \mathbb{P}_{\mu^+} \not\subseteq H_{\mu^+}. \) Still, \( \mathbb{P}_{\mu^+} \) will nevertheless be isomorphic to a subset of \( H_{\mu^+}, \) as we may identify \( P_{\mu^+} \) with \( \{ p \upharpoonright (\sup(B_p) + 1) \mid p \in P_{\mu^+}\}. \)

(2) Clause (v) puts a weaker assertion for \( \alpha = \mu^+. \) In order to avoid trivialities, let us assume that \( \mu^+\)-many stages in our iteration \( \mathbb{P}_{\mu^+} \) are non-trivial. To see the restriction in Clause (v) is necessary note that, by the pigeonhole principle, there must exist two conditions \( p, q \in P_{\mu^+} \) and an ordinal \( \gamma < \mu^+ \) for which \( c_{\mu^+}(p) = c_{\mu^+}(q), B_p \subseteq \gamma, \) but \( B_q \not\subseteq \gamma. \) Now, towards a contradiction, assume there is a map \( \check{n} \) such that \( (\check{n}, \pi_{\alpha, \gamma}) \) forms a forking projection from \((\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})\)

\(^{11}\)\( A_\alpha \) denotes the poset of Definition 2.23(7) regarded with respect to the type witnessing Clause (a) of Building Block II.
to \((\mathbb{P}, \ell, c)\). By Definition 2.13(8), then, \(c_\gamma(p \upharpoonright \gamma) = c_\gamma(q \upharpoonright \gamma)\), so that by Definition 2.3(3), we should be able to pick \(r \in (P_\gamma)_{0} \cap (P_\gamma)_{0}^{[\gamma]}\), and then by Definition 2.13(8), \(\cap (\cap(q)(r))\). Finally, as \(B_p \subseteq \gamma\), \(p = [p \upharpoonright \gamma]^{P_{\mu}^+}\), so that, by Definition 2.13(6), \(\cap (\cap(q)(r)) = [r]^{P_{\mu}^+}\). But then \(\cap (\cap(q)(r)) = [r]^{P_{\mu}^+}\), so that, by Definition 2.13(6), \(q = [q \upharpoonright \gamma]^{P_{\mu}^+}\), contradicting the fact that \(B_q \not\subseteq \gamma\).

3.1. Defining the iteration. For every \(\alpha < \mu^+\), fix an injection \(\phi_\alpha : \alpha \to \mu\). As \(|H_\mu| = \mu\), by the Engeling-Karlowicz theorem, we may also fix a sequence \((e^i : i < \mu)\) of functions from \(\mu^+\) to \(H_\mu\) such that for every function \(e : C \to H_\mu\) with \(C \in [\mu^+]^{<\mu}\), there is \(i < \mu\) such that \(e \subseteq e^i\).

The upcoming definition is by recursion on \(\alpha \leq \mu^+\), and we continue as long as we are successful. We shall later verify that the described process is indeed successful.

\(\square\) Let \(F_0 := (\{\emptyset\}, \leq_0)\) be the trivial forcing. Let \(\ell_0\) and \(c_0\) be the constant function \(\{\emptyset, e_0\}\), and let \(\cap_0 = \emptyset\) be the constant function \(\{(\emptyset, \emptyset)\}\), so that \(\cap_0(\emptyset)\) is the identity map.

\(\square\) Let \(F_1 := (P_1, \leq_1)\), where \(P_1 := 1^Q\) and \(p \leq p'\) iff \(p(0) \leq Q p'(0)\).

Let \(\ell_1\) and \(c_1\) by stipulating \(\ell_1(p) := \ell(p(0))\) and \(c_1(p) = c(p(0))\). For all \(p \in P_1\), let \(\cap_1(p) : \{\emptyset\} \to \{p\}\) be the constant function, and let \(\cap_1(\emptyset)\) be the identity map.

\(\square\) Suppose \(\alpha < \mu^+\) and that \(\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, \langle \cap_\beta \rangle \gamma \leq \beta) \rangle \beta \leq \alpha\) has already been defined. We now define the triple \((\mathbb{P}_{\alpha+1}, \ell_{\alpha+1}, c_{\alpha+1})\) and the sequence of maps \(\langle \cap_{\alpha+1} \rangle \gamma \leq \alpha + 1\).

\(\square\) If \(\psi(\alpha)\) happens to be a triple \((\beta, r, \sigma)\), where \(\beta < \alpha\), \(r \in P_\beta\) and \(\sigma\) is a \(\mathbb{P}_\beta\)-name, then we appeal to Building Block II with \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)\), \(r^* := r * \emptyset_\alpha\) and \(z := \sigma^{c_\beta}_\alpha(\sigma)\) to get a corresponding \(\Sigma\)-Prikry poset \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\).

\(\square\) Otherwise, we obtain \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) by appealing to Building Block II with \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)\), \(r^* := \emptyset_\alpha\) and \(z := \emptyset\).

In both cases, we also obtain a forcing projection \((\cap, \pi)\) from \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) to \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)\). Furthermore, each condition in \(\mathbb{A} = (A, \leq)\) is a pair \((x, y)\) with \(\pi(x, y) = x\), and, for every \(p \in \mathbb{P}_\alpha\), \([p]^{\mathbb{A}} = (p, 0)\). Now, define \(P_{\alpha+1} := (P_{\alpha+1}, \leq_{\alpha+1})\) by letting \(P_{\alpha+1} := \{x^\beta \mid (x, y) \in A\}\), and then let \(p \leq_{\alpha+1} p'\) iff \((p \upharpoonright \alpha, p(\alpha)) \leq \langle p' \upharpoonright \alpha, p'(\alpha)\rangle\). Put \(\ell_{\alpha+1} := \ell_1 \circ \pi_{\alpha+1, 1}\) and define \(c_{\alpha+1} : P_{\alpha+1} \to H_\mu\) via \(c_{\alpha+1}(p) := c_\alpha(p \upharpoonright \alpha, p(\alpha))\).

Next, let \(p \in P_{\alpha+1}\), \(\gamma \leq \alpha + 1\) and \(r \leq \gamma\) be arbitrary; we need to define \(\cap_{\alpha+1, \gamma}(p)(r)\). For \(\gamma = \alpha + 1\), let \(\cap_{\alpha+1, \gamma}(p)(r) := r\), and for \(\gamma \leq \alpha\), let \(\cap_{\alpha+1, \gamma}(p)(r) := x^\gamma \cap (p \upharpoonright \alpha, p(\alpha))(\cap_{\alpha+1, \gamma}(p \upharpoonright \alpha)(r)) = (x, y)\).

\(\square\) Suppose \(\alpha \in \text{acc}(\mu^+ + 1)\), and that \(\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, \langle \cap_\beta \rangle \gamma \leq \beta) \rangle \beta < \alpha\) has already been defined. Define \(\mathbb{P}_\alpha := (P_{\alpha \leq \alpha})\) by letting \(P_{\alpha} \text{ be all } \alpha\)-sequences } p \text{ such that } |B_p| < \mu \text{ and } \forall \beta < \alpha(p \upharpoonright \beta \in \mathbb{P}_\beta)\). Let \(p \leq \alpha q\) iff

\[ \text{This is a consequence of the fact that } p = (p \upharpoonright \gamma) \ast \emptyset_{\mu}^+ = [p \upharpoonright \gamma]^{P_{\mu}^+}. \text{ See the discussion at the beginning of Lemma 3.6.} \]
∀β < α (p ∣ β ≤ β q ∣ β). Let ℓα := ℓ1 ∗ πα,1. Next, we define cα : Pα → Hµ, as follows.

- If α < µ⁺, then, for every p ∈ Pα, let
  \[ cα(p) := \{(φα(γ), cγ(p ∣ γ)) \mid γ ∈ B_p\}. \]

- If α = µ⁺, then, given p ∈ Pα, first let C := cl(Bp), then define a function e : C → Hµ by stipulating:
  \[ e(γ) := (φα[C ∩ γ], cγ(p ∣ γ)), \]
  and then let cα(p) := i for the least i < µ such that e ⊆ e^i.

Finally, let p ∈ Pα, γ ≤ α and r ≤ γ p ∣ γ be arbitrary; we need to define ˆnα,γ(p)(r). For γ = α, let ˆnα,α(p)(r) := r, and for γ < α, let ˆnα,γ(p)(r) := ∪\{ ˆnβ,γ(p ∣ β)(r) \mid γ ≤ β < α\}.

**Convention 3.4.** Even though (P0, ℓ0) is not a graded poset, in order to smooth up inductive claims that come later, we define 0 to be ≤0, and likewise, for every p ∈ P0, we interpret (P0)0 as \{ q ∈ P0 ∣ q ≤0 p \}.

**3.2. Verification.** We now verify that for all α ≤ µ⁺, (Pα, ℓα, cα, ˆn) fulfills requirements (i)–(vi) of Goal 3.2. By the recursive definition given so far, it is obvious that Clauses (i) and (iii) hold, so we focus on the rest. We commence with an expanded version of Clause (vi).

**Lemma 3.5.** For all γ ≤ α ≤ µ⁺, p ∈ Pα and r ∈ Pγ with r ≤ γ p ∣ γ, if we let q := ˆnα,γ(p)(r), then:

1. q ∣ β = ˆnβ,γ(p ∣ β)(r) for all β ∈ [γ, α];
2. Bq = Bp ∪ Br;
3. q ∣ γ = r;
4. If γ = 0, then q = p;
5. p = (p ∣ γ) ∗ 0α iff q = r ∗ 0α;
6. for all p’ ≤0 p, if r ≤0 p’ ∣ γ, then ˆnα,γ(p’)(r) ≤α ˆnα,γ(p)(r).

**Proof.** Clause (3) follows from Clause (1) and the fact that ˆnα,γ(p ∣ γ) is the identity function. Clause (5) follows from Clauses (2) and (3).

We now prove Clauses (1), (2), (4) and (6) by induction on α ≤ µ⁺:

- The case α = 0 is trivial, since, in this case, all the conditions under consideration (and their corresponding B-sets) are empty, and all the maps under consideration are the identity.
- The case α = 1 follows from the fact that, by definition, ˆn1,0(p)(r) = p and ˆn1,1(p)(r) = r.
- Suppose α ≥ 2 is a successor ordinal, say α = α’ + 1, and that the claim holds for α’. Fix arbitrary γ ≤ α, p ∈ Pα and r ∈ Pγ with r ≤ γ p ∣ γ. Denote q := ˆnα,γ(p)(r). Recall that Pα = Pα’ + 1 was defined by feeding (Pα’, ℓα’, cα’) into Building Block II, thus obtaining a Σ-Prikry triple (Å, ℓÅ, cÅ) along with a forking projection (ˆn, π), such that each condition in the poset Å = (Å, ≤) is a pair
where $p$ so that

$$\text{where the rightmost equality follows from Lemma 2.17. Altogether,}$$

$$\text{So, since}$$

$$\text{Denote}$$

$$\text{Clause (7) of Definition 2.13, indeed}$$

$$\text{(In particular,}$$

$$\text{by Definition 2.13(5), is equal to}$$

$$\text{In addition, the case}$$

$$\text{the fact that}$$

$$\text{It thus follows from Clause (f) of Building Block II together with}$$

$$\text{the case}$$

$$\text{If}$$

$$\text{If}$$

$$\text{To avoid trivialities, assume}$$

$$\text{To avoid trivialities, assume}$$

$$\text{that}$$

$$\text{as desired.}$$

$$\text{If}$$

$$\text{(6) To avoid trivialities, assume that}$$

$$\text{Fix}$$

$$\text{Now, by the induction hypothesis,}$$

$$\text{Denote}$$

$$\text{By Clause (7) of Definition 2.13, indeed}$$
Suppose \( \alpha \in \text{acc}(\mu^+ + 1) \) is an ordinal such that, for all \( \alpha' < \alpha, \beta \in [\gamma, \alpha], p \in P_{\alpha'}, \) and \( r \in P_\gamma \) with \( r \leq_\gamma p \restriction \gamma, \)
\[ \langle \dot{\eta}_{\alpha', \gamma}(p \restriction \beta)(r) \rangle = (\dot{\eta}_{\alpha', \gamma}(p \restriction \alpha')(r)) \restriction \beta. \]

Fix arbitrary \( \gamma \leq \alpha, p \in P_\alpha \) and \( r \in P_\gamma \) with \( r \leq_\gamma p \restriction \gamma. \) Denote \( q := \langle \dot{\eta}_{\alpha, \gamma}(p)(r) \rangle. \) By our definition of \( \dot{\eta}_{\alpha, \gamma} \) at the limit stage, we have:
\[ q = \bigcup \{ \dot{\eta}_{\beta, \gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha \}. \]

By the induction hypothesis, \( \langle \dot{\eta}_{\beta, \gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha \rangle \) is a \( \subseteq \)-increasing sequence, and \( B_{\dot{\eta}_{\beta, \gamma}(p \restriction \beta)(r)} = B_p \restriction B_r \) whenever \( \gamma \leq \beta < \alpha. \) It thus follows that \( q \) is a legitimate condition, and Clauses (1), (2), (4) and (6) are satisfied.

Our next task is to verify Clauses (ii) and (v) of Goal 3.2:

**Lemma 3.6.** Suppose that \( \alpha \leq \mu^+ \) is such that for all nonzero \( \gamma < \alpha, (\mathbb{P}_\gamma, c_\gamma, \ell_\gamma) \) is \( \Sigma \)-Prikry. Then:

- for all nonzero \( \gamma \leq \alpha, (\dot{\eta}_{\alpha, \gamma}, \pi_{\alpha, \gamma}) \) is a forking projection from \( (\mathbb{P}_\alpha, \ell_\alpha) \) to \( (\mathbb{P}_\gamma, \ell_\gamma), \) where \( \pi_{\alpha, \gamma} \) is defined as in Goal 3.2(ii);
- if \( \alpha < \mu^+, \) then \( (\dot{\eta}_{\alpha, \gamma}, \pi_{\alpha, \gamma}) \) is furthermore a forking projection from \( (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha) \) to \( (\mathbb{P}_\gamma, \ell_\gamma, c_\gamma) \)
- if \( 0 < \alpha < \mu, \) then \( (\dot{\eta}_{\alpha+1, \gamma}, \pi_{\alpha+1, \gamma}) \) has the weak mixing property.

**Proof.** Let us go over the clauses of Definition 2.13.

Clause (5) is covered by Lemma 3.5(3), and Clause (7) is covered by Lemma 3.5(6). Clause (3) is obvious, since for all nonzero \( \gamma < \alpha \) and \( p \in P_\gamma, \) a straightforward verification makes it clear that \( p \ast 0_\alpha \) is the greatest element of \( \{ q \in P_\alpha \mid \pi_{\alpha, \gamma}(q) = p \}. \) In effect, Clause (6) follows from Lemma 3.5(5).

Thus, we are left with verifying Clauses (1),(2),(4) and (8). The next claim takes care of the first three.

**Claim 3.6.1.** For all nonzero \( \gamma \leq \alpha \) and \( p \in P_\alpha:

1. \( \pi_{\alpha, \gamma} \) forms a projection from \( \mathbb{P}_\alpha \) to \( \mathbb{P}_\gamma, \) and \( \ell_\alpha = \ell_\gamma \circ \pi_{\alpha, \gamma}; \)
2. \( \dot{\eta}_{\alpha, \gamma}(p) \) is an order-preserving function from \( \mathbb{P}_\gamma \downarrow (p \restriction \gamma), \leq_\gamma \) to \( (\mathbb{P}_\alpha \downarrow p, \leq_\alpha); \)
3. for all \( n, m < \omega \) and \( q \leq_{\alpha}^n m \) \( p, m(p, q) \) exists and, furthermore, \( m(p, q) = \dot{\eta}_{\alpha, \gamma}(p)(m(p \restriction \gamma, q \restriction \gamma)). \)

**Proof.** We commence by proving (2) and (3) by induction on \( \alpha \leq \mu^+: \)

- The case \( \alpha = 1 \) is trivial, since, in this case, \( \gamma = \alpha. \)
- Suppose \( \alpha = \alpha' + 1 \) is a successor ordinal and that the claim holds for \( \alpha'. \) Let \( \gamma \leq \alpha \) and \( p \in P_\alpha \) be arbitrary. To avoid trivialities, assume \( \gamma < \alpha. \) By the induction hypothesis, \( \dot{\eta}_{\alpha', \gamma}(p \restriction \alpha')(r) \) is an order-preserving function from \( \mathbb{P}_\gamma \downarrow (p \restriction \gamma) \) to \( \mathbb{P}_{\alpha'} \downarrow (p \restriction \alpha'). \)

Recall that \( \mathbb{P}_\alpha = \mathbb{P}_{\alpha' \downarrow 1} \) was defined by feeding \( (\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'}) \) into Building Block II, thus obtaining a \( \Sigma \)-Prikry triple \( (\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A}) \) along
with the pair \((\alpha, \pi)\). Now, as \(\mathcal{h}(p \upharpoonright \alpha', p(\alpha'))\) and \(\mathcal{h}_{\alpha', \gamma}(p \upharpoonright \alpha')\) are both order-preserving, the very definition of \(\mathcal{h}_{\alpha, \gamma}(p \upharpoonright \gamma)\) and \(\leq_{\alpha' + 1}\) implies that \(\mathcal{h}_{\alpha, \gamma}(p \upharpoonright \gamma)\) is order-preserving. In addition, as \((x, y)\) is a condition in \(\mathcal{A}\) if \(x^{-}(y) \in P_\alpha\) and as \(\mathcal{h}(p \upharpoonright \alpha', p(\alpha'))\) is an order-preserving function from \(\mathcal{P}_{\alpha'} \downarrow (p \upharpoonright \alpha')\) to \(\mathcal{A} \downarrow (p \upharpoonright \alpha', p(\alpha'))\), we infer that, for all \(r \leq_{\alpha} \gamma\), \(\mathcal{h}_{\alpha, \gamma}(p \upharpoonright \gamma)(r)\) is in \(\mathcal{P}_{\alpha} \downarrow p\).

Let \(q \leq_{\alpha} \gamma \leq_{\alpha'} \alpha\) for some \(n, m < \omega\). Let
\[
(x, y) := m((p \upharpoonright \alpha', p(\alpha'))), (q \upharpoonright \alpha', q(\alpha'))).
\]
Trivially, \((m, p, q)\) exists and is equal to \(x^{-}(y)\). We need to show that \(m(p, q) = \mathcal{h}_{\alpha, \gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma))\). By Definition \ref{def:induction}(4),
\[
(x, y) = \mathcal{h}(p \upharpoonright \alpha', p(\alpha'))(m(p \upharpoonright \alpha', q \upharpoonright \alpha')).
\]
By the induction hypothesis,
\[
m(p \upharpoonright \alpha', q \upharpoonright \alpha') = \mathcal{h}_{\alpha', \gamma}(p \upharpoonright \alpha')(m(p \upharpoonright \gamma, q \upharpoonright \gamma)),
\]
and so it follows that
\[
(x, y) = \mathcal{h}(p \upharpoonright \alpha', p(\alpha'))(\mathcal{h}_{\alpha', \gamma}(p \upharpoonright \alpha')(m(p \upharpoonright \gamma, q \upharpoonright \gamma))).
\]
Thus, by the definition of \(\mathcal{h}_{\alpha, \gamma}\) and the above equation, we have that \(\mathcal{h}_{\alpha, \gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma))\) is indeed equal to \(x^{-}(y)\).

\begin{itemize}
\item Suppose \(\alpha \in \text{acc}(\mu^+ + 1)\) is an ordinal for which the claim holds below \(\alpha\). Let \(\gamma \leq \alpha\) and \(p \in P_\alpha\) be arbitrary. To avoid trivialities, assume \(\gamma < \alpha\). By Lemma \ref{lem:order preservation}(1), for every \(r \in \mathcal{P}_\gamma \downarrow (p \upharpoonright \gamma)\):
\[
\mathcal{h}_{\alpha, \gamma}(p)(r) = \bigcup_{\gamma \leq \gamma' < \alpha} \mathcal{h}_{\alpha', \gamma}(p \upharpoonright \alpha')(r).
\]
As for all \(q, q' \in P_\alpha\), \(q \leq_{\alpha} q'\) iff \(\forall \alpha' < \alpha(q \upharpoonright \alpha' \leq_{\alpha'} q' \upharpoonright \alpha')\), the induction hypothesis implies that \(\mathcal{h}_{\alpha, \gamma}(p)\) is an order-preserving function from \(\mathcal{P}_\gamma \downarrow (p \upharpoonright \gamma)\) to \(\mathcal{P}_\alpha \downarrow p\).

Finally, let \(q \leq_{\alpha} p\); we shall show that \((m, p, q)\) exists and is, in fact, equal to \(\mathcal{h}_{\alpha, \gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma))\). By Lemma \ref{lem:order preservation}(1) and the induction hypothesis,
\[
\mathcal{h}_{\alpha, \gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma)) = \bigcup_{\gamma \leq \gamma' < \alpha} m(p \upharpoonright \alpha', q \upharpoonright \alpha'),
\]
call it \(r\). We shall show that \(r\) plays the role of \((m, p, q)\).

By the definition of \(\leq_{\alpha}\), it is clear that \(q \leq_{\alpha} r \leq_{\alpha} p\), so it remains to show that it is the greatest condition in \((P_\alpha^m)_n\) to satisfy this. Fix an arbitrary \(s \in (P_\alpha^m)_n\) with \(q \leq_{\alpha} s\). For each \(\alpha' < \alpha\), \(q \upharpoonright \alpha' \leq_{\alpha'} s \upharpoonright \alpha' \leq_{\alpha'} p \upharpoonright \alpha'\), so that \(s \upharpoonright \alpha' \leq_{\alpha'} m(p \upharpoonright \alpha', q \upharpoonright \alpha')\), and thus \(s \leq_{\alpha} r\). Altogether this shows that \(r = m(p, q)\).

This completes the proof of Clauses (2) and (3) above.
We are left to prove (1). The case $q = \alpha$ is trivial, so assume $\gamma < \alpha$. Clearly, $\pi_{\alpha,\gamma}$ is order-preserving and also $\pi_{\alpha,\gamma}(\emptyset_\alpha) = \emptyset_\gamma$. Let $p \in P_\alpha$ and $q \in P_\gamma$ be such that $q \leq_\gamma \pi_{\alpha,\gamma}(p)$. By Lemma 3.5(3), $\pi_{\alpha,\gamma}(q^*) = q$ and by Clause (2) of this claim, $q^* \leq_\alpha p$. Altogether, $\pi_{\alpha,\gamma}$ is indeed a projection. For the second part, recall that, for all $\beta \leq \mu^+$, $\ell_\beta := \ell_1 \circ \pi_{\alpha,\gamma}$, hence $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1} = \ell_1 \circ (\pi_{\gamma,1} \circ \pi_{\alpha,\gamma}) = (\ell_1 \circ \pi_{\gamma,1}) \circ \pi_{\alpha,\gamma} = \ell_\gamma \circ \pi_{\alpha,\gamma}$. \hfill \Box

We are left with verifying Clause (8) of Definition 2.13 to show that $(\pi_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is a forking projection from $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$.

**Claim 3.6.2.** Suppose $\alpha \neq \mu^+$. For all $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$ and all nonzero $\gamma \leq \alpha$:

- $c_\gamma(p \restriction \gamma) = c_\gamma(p' \restriction \gamma)$, and
- $\pi_{\alpha,\gamma}(p)(r) = \pi_{\alpha,\gamma}(p')(r)$ for every $r \in (P_\gamma)_0^{\gamma} \cap (P_\gamma)_0^{\gamma}$.

**Proof.** By induction on $\alpha < \mu^+$:

- The case $\alpha = 1$ is trivial, since, in this case, $\gamma = \alpha$.
- Suppose $\alpha = \alpha' + 1$ is a successor ordinal and that the claim holds for $\alpha'$. Fix an arbitrary pair $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$.

  Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a $\Sigma$-Prikry triple $(\mathcal{A}, \ell_{\mathcal{A}}, c_{\mathcal{A}})$ along the pair $(\pi, \pi)$. By the definition of $c_{\alpha'+1}$, we have

  $$c_{\alpha'}(p \restriction \alpha', p(\alpha')) = c_\alpha(p) = c_\alpha(p') = c_\alpha(p' \restriction \alpha', p'(\alpha')).$$

  So, as $(\pi, \pi)$ is a forking projection from $(\mathcal{A}, \ell_{\mathcal{A}}, c_{\mathcal{A}})$ to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$, we have $c_{\alpha'}(p \restriction \alpha') = c_{\alpha'}(p' \restriction \alpha')$, and, for all $r \in (P_{\alpha'})_0^{\alpha'} \cap (P_{\alpha'})_0^{\alpha'}$,

  $$\pi_{\alpha',\gamma}(p)(r) = \pi_{\alpha',\gamma}(p')(r) = \pi(\alpha', p(\alpha'))(r).$$

  Now, as $c_{\alpha'}(p \restriction \alpha') = c_{\alpha'}(p' \restriction \alpha')$, the induction hypothesis implies that $c_{\alpha'}(p \restriction \gamma) = c_{\alpha'}(p' \restriction \gamma)$ for all nonzero $\gamma \leq \alpha'$. In addition, the case $\gamma = \alpha$ is trivial.

  Finally, fix a nonzero $\gamma \leq \alpha$ and $r \in (P_\gamma)_0^{\gamma} \cap (P_\gamma)_0^{\gamma}$, and let us prove that $\pi_{\alpha,\gamma}(p)(r) = \pi_{\alpha,\gamma}(p')(r)$. To avoid trivialities, assume $\gamma < \alpha$. It follows from the definition of $\pi_{\alpha,\gamma}$ that $\pi_{\alpha,\gamma}(p)(r) = x^\gamma(y)$ and $\pi_{\alpha,\gamma}(p')(r) = x'^\gamma(y')$, where:

  - $(x, y) := \pi_{\alpha',\gamma}(p(\alpha'))(\pi_{\alpha,\gamma}(p \restriction \alpha')(r))$, and
  - $(x', y') := \pi_{\alpha',\gamma}(p'(\alpha'))(\pi_{\alpha,\gamma}(p' \restriction \alpha')(r)).$

  But we have already pointed out that the induction hypothesis implies that $\pi_{\alpha',\gamma}(p(\alpha') \restriction r') = \pi_{\alpha',\gamma}(p'(\alpha') \restriction r')$, call it, $r'$. So, we just need to prove that $\pi(p \restriction \alpha', p(\alpha'))(r') = \pi(p' \restriction \alpha', p'(\alpha'))(r')$. But we also have $c_\alpha(p \restriction \alpha, p(\alpha')) = c_\alpha(p') = c_{\mathcal{A}}(p' \restriction \alpha, p'(\alpha'))$ and by our choice of $r$ and Clause (2) of Claim 3.6.1, $r' \in (P_{\alpha'})_0^{\alpha'} \cap (P_{\alpha'})_0^{\alpha'}$.

  So, as $(\pi, \pi)$ is a forking projection from $(\mathcal{A}, \ell_{\mathcal{A}}, c_{\mathcal{A}})$ to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$, Clause (8) of Definition 2.13 implies that

  $$\pi_{\alpha',\gamma}(p(\alpha'))(r') = \pi_{\alpha',\gamma}(p'(\alpha'))(r'),$$

  as desired.
Suppose \( \alpha \in \text{acc}(\mu^+) \) is an ordinal for which the claim holds below \( \alpha \). For any condition \( q \in \bigcup_{\alpha' < \alpha} P_{\alpha'} \), define a function \( f_q : B_q \rightarrow H_\mu \) via \( f_q(\alpha') := c_{\alpha'}(q \upharpoonright \alpha') \). Now, fix an arbitrary pair \( p, p' \in P_\alpha \) with \( c_\alpha(p) = c_\alpha(p') \). By the definition of \( c_\alpha \) this means that

\[
\{(\phi_\alpha(\gamma), c_\gamma(p \upharpoonright \gamma)) \mid \gamma \in B_p\} = \{(\phi_\alpha(\gamma), c_\gamma(p' \upharpoonright \gamma)) \mid \gamma \in B_{p'}\}.
\]

As \( \phi_\alpha \) is injective, \( f_p = f_{p'} \). Next, let \( \gamma \leq \alpha \) be nonzero; we need to show that \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \). The case \( \gamma = \alpha \) is trivial, so assume \( \gamma < \alpha \).

Now, if \( \text{dom}(f_p) \setminus \gamma \) is nonempty, then for \( \alpha' := \min(\text{dom}(f_p) \setminus \gamma) \), we have \( c_{\alpha'}(p \upharpoonright \alpha') = f_p(\alpha') = f_{p'}(\alpha') = c_{\alpha'}(p' \upharpoonright \alpha') \), and then the induction hypothesis entails \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \). In particular, if \( \text{dom}(f_p) \) is unbounded in \( \alpha \), then \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \) for all \( \gamma \leq \alpha \).

Next, suppose that \( \text{dom}(f_p) \) is bounded in \( \alpha \) and let \( \delta < \alpha \) be the least ordinal to satisfy \( \text{dom}(f_p) \subseteq \delta \). We already know that \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \) for all \( \gamma < \delta \), and now prove by induction that \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \) for all \( \gamma \in [\delta, \alpha) \). For a successor ordinal \( \gamma \), this follows from Clauses (f) and (g) of Building Block II, and for a limit ordinal \( \gamma \), this follows from the fact that the injectivity of \( \phi_\gamma \) and the equality \( f_p\upharpoonright\gamma = f_p = f_{p'} = f_{p'\upharpoonright\gamma} \) implies that \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \).

Finally, fix a nonzero \( \gamma \leq \alpha \) and \( r \in (P_\gamma)^{p\upharpoonright\gamma}_0 \cap (P_\gamma)^{p'\upharpoonright\gamma}_0 \), and let us prove that \( \bar{\eta}_{\alpha, \gamma}(p)(r) = \bar{\eta}_{\alpha, \gamma}(p')(r) \). To avoid trivialities, assume \( \gamma < \alpha \). We already know that, for all \( \alpha' \in [\gamma, \alpha) \), \( c_{\alpha'}(p \upharpoonright \alpha') = c_{\alpha'}(p' \upharpoonright \alpha') \), and so the induction hypothesis implies that \( \bar{\eta}_{\alpha', \gamma}(p \upharpoonright \alpha')(r) = \bar{\eta}_{\alpha', \gamma}(p' \upharpoonright \alpha')(r) \), and then by Lemma 3.5(1):

\[
\bar{\eta}_{\alpha, \gamma}(p)(r) = \bigcup_{\gamma \leq \alpha' < \alpha} \bar{\eta}_{\alpha', \gamma}(p \upharpoonright \alpha')(r) = \bigcup_{\gamma \leq \alpha' < \alpha} \bar{\eta}_{\alpha', \gamma}(p' \upharpoonright \alpha')(r) = \bar{\eta}_{\alpha, \gamma}(p')(r),
\]

as desired. \( \square \)

**Claim 3.6.3.** If \( 0 < \alpha < \mu^+ \) then \( (\bar{\eta}_{\alpha+1, \alpha}, \pi_{\alpha+1, \alpha}) \) has the weak mixing property.

**Proof.** Once again, recall that \( P_{\alpha+1} \) was defined by feeding \( (P_\alpha, \ell_\alpha, c_\alpha) \) into Building Block II, thus obtaining a \( \Sigma \)-Prikry triple \( (A, \ell_\kappa, c_\kappa) \), along with a pair \( (\bar{\eta}, \pi) \) having the weak mixing property. Let \( \text{tp} \) be a type over \( (\bar{\eta}, \pi) \) and \( (\Sigma^n \mid n < \omega) \) be a sequence of orderings witnessing this. For each \( p \in P_{\alpha+1} \), set \( \text{tp}_{\alpha+1}(p) := \text{tp}(p \mid \alpha, p(\alpha)) \). Also, for each \( n < \omega \), derive

\[
\Sigma^n_{\alpha+1} := \{ (p, q) \in P_{\alpha+1} \times P_{\alpha+1} \mid (p \mid \alpha, p(\alpha)) \Sigma^n (q \upharpoonright \alpha, q(\alpha)) \}.
\]

The canonical isomorphism from \( A \) to \( P_{\alpha+1} \) (i.e., \( (x, y) \mapsto x^-(y) \)) makes it clear that \( \text{tp}_{\alpha+1} \) and \( (\Sigma^n_{\alpha+1} \mid n < \omega) \) witness together that \( (\bar{\eta}_{\alpha+1, \alpha}, \pi_{\alpha+1, \alpha}) \) has the weak mixing property. \( \square \)
This completes the proof of Lemma 3.6.

\[\square\]

**Definition 3.7.** For each nonzero \( \alpha < \mu^+ \), we let \( \text{tp}_{\alpha+1} \) and \( \langle \Xi_{\alpha+1}^n \mid n < \omega \rangle \) be the witnesses to the weak mixing property of \((\mathbb{P}_{\alpha+1}, \pi_{\alpha+1, \alpha})\), as defined in the proof of Subclaim 3.6.3.

Recalling Definition 2.3(2), for all nonzero \( \alpha \leq \mu^+ \) and \( n < \omega \), we need to identify a candidate for a dense subposet \( \mathbb{P}_n = (\hat{\mathbb{P}}_n, \leq_{\alpha}) \) of \( \mathbb{P}_n \). We do this next.

**Definition 3.8.** Let \( n < \omega \). Set \( \hat{\mathbb{P}}_n := 1(Q_n) \).

\[1\]

Then, for each \( \alpha \in [2, \mu^+] \), define \( \hat{\mathbb{P}}_n \) by recursion:

\[
\hat{\mathbb{P}}_n := \begin{cases} 
\{ p \in \mathbb{P}_n \mid \pi_{\alpha, \beta}(p) \in \hat{\mathbb{P}}_{\beta n} \& \text{mtp}_{\beta+1}(p) = 0 \}, & \text{if } \alpha = \beta + 1; \\
\{ p \in \mathbb{P}_n \mid \pi_{\alpha, 1}(p) \in \hat{\mathbb{P}}_n \& \forall \gamma \in B_p \text{ mtp}(\pi_{\alpha, \gamma}(p)) = 0 \}, & \text{otherwise.}
\end{cases}
\]

**Lemma 3.9.** Let \( n < \omega \) and \( 1 \leq \beta < \alpha \leq \mu^+ \). Then:

1. \( \pi_{\alpha, \beta}^* \hat{\mathbb{P}}_n \subseteq \hat{\mathbb{P}}_{\beta n} \).
2. For every \( p \in \hat{\mathbb{P}}_{\beta n} \), \( p \upharpoonright_{\alpha} \in \hat{\mathbb{P}}_n \).

\[\square\]

Proof. By straight-forward induction, relying on Clause (4) of Definition 2.23.

We are now left with addressing Clause (iv) of Goal 3.2. Prior to that we will provide a sufficient condition securing that for each \( \alpha \in \text{acc}(\mu^+ + 1) \), the pair \( (\mathbb{P}_{\alpha}, \ell_{\alpha}) \) has property \( \mathcal{D} \). For this, we establish a version of the Weak Mixing Property (see Definition 2.25) for limit stages.

**Lemma 3.10.** Let \( \alpha \in \text{acc}(\mu^+ + 1) \). For all \( \alpha \in \mathbb{P}_n \), \( n < \omega \), \( \vec{r} \), and \( p' \leq_0 \pi_{\alpha, 1}(a) \), if we are given a function \( g : W_n(\pi_{\alpha, 1}(a)) \rightarrow \mathbb{P}_{\alpha} \downarrow a \) for which the following hold:

1. \( (B_{g(\xi) \mid \xi < \chi}) \) is \( \subseteq \)-weakly increasing. Put \( B := \bigcup_{\xi < \chi} B_{g(\xi)} \), and for each \( \gamma \in B \), let \( \iota_{\gamma} := \min\{ \xi < \chi \mid \gamma \in B_{g(\xi)} \} \);
2. \( \vec{r} = (r_{\xi} \mid \xi < \chi) \) is a good enumeration of \( W_n(\pi_{\alpha, 1}(a)) \);
3. \( (\pi_{\alpha, 1}(g(r_{\xi})) \mid \xi < \chi) \) is diagonalizable with respect to \( \vec{r} \) as witnessed by \( p' \);
4. for all \( \gamma \in B \), \( \xi \in (\iota_{\gamma}, \chi) \), and \( i \in [\text{dom}(\pi_{\alpha, \gamma}(a)) \mid \sup_{\eta < \xi} \text{dom}(\pi_{\alpha, \gamma}(g(r_{\eta}))))], \)

\[
\text{tp}_{\gamma}(\pi_{\alpha, \gamma}(g(r_{\xi}))(i) \leq \text{mtp}(\pi_{\alpha, \gamma}(a)),
\]

\[\text{Here, } Q_n \text{ is obtained from Clause (2) of Definition 2.3 with respect to the triple } (Q, \ell, c) \text{ given by Building Block I.}\]
(5) for all $\gamma \in B$, $\sup_{\xi \leq \chi} \mathsf{mtp}_\gamma(\pi_{\alpha,\gamma}(g(r_\xi))) < \omega$, then there exists $b \in P_\alpha$ such that:

(a) $\pi_{\alpha,1}(b) = p'$;
(b) for all $\gamma \in B_\alpha$, $\pi_{\alpha,\gamma}(b) \subseteq \sup_\gamma \pi_{\alpha,\gamma}(a)$;
(c) for all $q' \in W_n(p')$, $\eta_{\alpha,1}(b)(q') \leq g(w(\pi_{\alpha,1}(a), q'))$.

**Proof.** Let $\chi < \mu$ Goal 3.2(i) and by (II) above, we may let $\text{dom}(\tau) < \theta$. Let $\langle \text{Clauses (a) and (b) of the lemma hold.} \rangle$

Suppose there is a sequence $\langle \langle \tau \rangle \rangle$ be the increasing enumeration of $B = \bigcup_{\xi < \chi} B_\xi$. From Goal 3.2(i) and $\chi < \mu$, we infer that $\theta < \mu$. For each $\tau < \theta$:

- as $\gamma_\tau$ is a successor ordinal, we let $\beta_\tau$ denote its predecessor;
- for every $\xi < \chi$, let $r_\xi^\tau := \eta_{\beta_\tau,1}(a \upharpoonright \beta_\tau)(r_\xi)$. By Fact 2.19, $r_\tau^\tau := \langle r_\xi^\tau \mid \tau < \theta \rangle$ is a good enumeration of $W_n(a \upharpoonright \beta_\tau)$;
- derive a map $g_\tau : W_n(a \upharpoonright \beta_\tau) \rightarrow P_{\gamma_\tau} \downarrow (a \upharpoonright \gamma_\tau)$ via

$$g_\tau(r_\xi^\tau) := g(r_\xi) \upharpoonright \gamma_\tau.$$

**Claim 3.10.1.** Suppose there is a sequence $\langle \langle b_\tau, p_\tau \rangle \rangle$ be the increasing enumeration of $\mathbb{B} = \bigcup_{\xi < \chi} B_\xi$. From Goal 3.2(i) and $\chi < \mu$, we infer that $\theta < \mu$. For each $\tau < \theta$:

(I) $b_0 \upharpoonright 1 = p_0 \upharpoonright 1 = p'$;
(II) $b_\tau \upharpoonright \tau' = b_{\tau'}$ for all $\tau' < \tau$;
(III) $b_\tau$ witnesses the conclusion of Definition 2.25 with respect to the tuple $(a \upharpoonright \gamma_\tau, r_\tau^\tau, p_\tau, g_\tau, \tau_\gamma)$. In particular, $p_\tau^\tau \leq_{\beta_\tau} (a \upharpoonright \beta_\tau)$ diagonalizes $(g_\tau(r_\xi^\tau) \upharpoonright \beta_\tau \mid \xi < \chi)$.

Then there is $b \in P_\alpha$ as in the conclusion of the Lemma.

**Proof.** By (II) above, we may let $b^* := \bigcup_{\tau < \theta} b_\tau$, so that $b^* \in P_\delta$ for $\delta := \text{dom}(b^*)$. For each $\tau < \theta$, Clauses (II) and (III) yield

$$b^* \upharpoonright \gamma_\tau = b_\tau \subseteq_{\tau_\gamma} a \upharpoonright \gamma_\tau,$$

and hence $b^* \upharpoonright \gamma_\tau \leq_{\delta} (a \upharpoonright \delta)$. So we may let $b := \eta_{\alpha,\delta}(a)(b^*)$, and infer from (I) that $b \upharpoonright 1 = p'$. Also, we have that $b \upharpoonright \gamma \subseteq_{\gamma_\tau} a \upharpoonright \gamma$, for each $\gamma \in B_\alpha$. This shows that Clauses (a) and (b) of the lemma hold.

We are now left with verifying Clause (c). Let $q' \in W_n(p')$; we want to show that $\eta_{\alpha,1}(b)(q') \leq_{\gamma_\tau} g(w(a \upharpoonright 1, q'))$. Note that by Lemma 3.5(2), $B_\alpha \subseteq B_\beta = B_\delta$, so that $b = b^* \upharpoonright \alpha$. Hence, $\langle \gamma_\tau \mid \tau < \theta \rangle$ is cofinal in $B_\beta$, and so it suffices to prove that, for each $\tau < \theta$,

$$\eta_{\alpha,1}(b)(q') \upharpoonright \tau \leq_{\gamma_\tau} g(w(a \upharpoonright 1, q')) \upharpoonright \gamma_\tau.$$

For each $\tau < \theta$, combining Clause (II) with Lemma 3.5(1) we have

$$\eta_{\alpha,1}(b)(q') \upharpoonright \tau = \eta_{\gamma_\tau,1}(b \upharpoonright \gamma_\tau)(q') = \eta_{\gamma_\tau,1}(b_\tau)(q'),$$

hence it suffices to check that

\[ \tag{\star} \eta_{\gamma_\tau,1}(b_\tau)(q') \leq_{\gamma_\tau} g(w(a \upharpoonright 1, q')) \upharpoonright \gamma_\tau. \]

By (\star) from Page 16 it is not hard to check that

\[ \tag{\star\star} \eta_{\gamma_\tau,1}(b_\tau)(q') = \eta_{\gamma_\tau,\beta_\tau}(b_\tau)(\eta_{\beta_\tau,1}(b_\tau \upharpoonright \beta_\tau)(q')). \]
Since \( b_r \upharpoonright 1 = p' \) and \( q' \in W_n(p') \), Lemma 3.6 yields \( r := \check{\eta}_{\beta_r,1}(b_r \upharpoonright \beta_r)(q') \) is in \( W_n(b_r \upharpoonright \beta_r) \). Combining equation (**) with (III), we infer that
\[
\check{\eta}_{\gamma_r,1}(b_r)(q') = \check{\eta}_{\gamma_r,\beta_r}(b_r)(r) \leq_{\gamma_r} g_r(w(a \upharpoonright \beta_r, r)) = g(w(a \upharpoonright 1, q')) \upharpoonright \gamma_r,
\]
where the rightmost equality follows from the definition of \( g \) and the fact that \( r \upharpoonright 1 = q' \). This verifies equation (*) and yields the claim. \( \square \)

Let us now argue by induction that such \( \langle (b_r, p^r) \mid \tau < \theta \rangle \) exists.

**Claim 3.10.2.** There is a pair \( (b_0, p^0) \) for which Clauses (I)–(III) hold.

**Proof.** Clause (II) is trivial at this stage. Setting \( p^0 := \check{\eta}_{\beta_0,1}(a \upharpoonright \beta_0)(p') \) takes care of the second part of Clause (I), and we shall come back to the first part towards the end. Now, let us examine the tuple \( (a \upharpoonright \gamma_0, r^0, p^0, g_0, \tau_{\gamma_0}) \) against the clauses of Definition 2.25 with respect to the forking projection \( (\check{\eta}_{\gamma_0,\beta_0}, \tau_{\gamma_0,0}) \): Clause (1) is obvious and Clauses (3), (4) and (5) follow by combining the corresponding clauses in the lemma with the definition of \( g_0 \).

Regarding Clause (2), we claim that \( p^0 \) diagonalizes \( \langle g_0(r^0_{\xi}) \upharpoonright \beta_0 \mid \xi < \chi \rangle \). To that effect we will check (a) and (β) of Proposition 2.22, when this is regarded with respect to the forking projection \( (\check{\eta}_{\beta_0,1}, \pi_{\beta_0,1}) \), and the parameters \( a \upharpoonright \beta_0, r^0, \langle g_0(r^0_{\xi}) \upharpoonright \beta_0 \mid \xi < \chi \rangle, p^r \) and \( p^0 \), respectively.

(a) Note that \( g_0(r^0_{\xi}) \upharpoonright 1 = g(r_{\xi}) \upharpoonright 1 \), for each \( \xi < \chi \). Therefore, Clause (1) implies that \( p^r \) diagonalizes \( \langle g_0(r^0_{\xi}) \upharpoonright 1 \mid \xi < \chi \rangle \).

(β) Note that by Clause (1) of the lemma, \( p^r \leq_{\beta_0} a \upharpoonright 1 \), hence \( p^0 \leq_{\beta_0} a \upharpoonright \beta_0 \).

Let \( q' \in W_n(p') \). Again by Clause (1), \( q' \leq_{\beta_0} g(r_{\xi}) \upharpoonright 1 \), where \( \xi \) is the unique index such that \( r_{\xi} = w(a \upharpoonright 1, q' \upharpoonright 1) \).

Finally, combining Lemma 3.5(5) and Lemma 3.6 we have
\[
\check{\eta}_{\beta_0,1}(p^0)(q') \leq_{\beta_0} \check{\eta}_{\beta_0,1}(a \upharpoonright \beta_0)(g(r_{\xi}) \upharpoonright 1) = g(r_{\xi}) \upharpoonright 1 \upharpoonright \emptyset_{\beta_0} = g_0(r^0_{\xi}) \upharpoonright \beta_0,
\]
where the above equalities follow from \( \beta_0 < \min(B) \).

Altogether, \( (a \upharpoonright \gamma_0, r^0, p^0, g_0, \tau_{\gamma_0}) \) witnesses Clauses (1)–(5) of Definition 2.25. Thus, appealing to Lemma 3.6, we obtain \( b \in P_{\gamma_0} \) such that \( b \upharpoonright \beta_0 = p^0 \) and \( b \subseteq_n a \upharpoonright \gamma_0 \) that witnesses the conclusion of Definition 2.25. Clearly, \( b_0 := b \) and \( p^0 \) are as wanted. \( \square \)

Suppose now \( \tau < \theta \), and that \( \langle (b_{\tau'}, p^{\tau'}) \mid \tau' < \tau \rangle \) has been constructed maintaining (I)–(III). Set \( b^* := \bigcup_{\tau' < \tau} b_{\tau'} \) and \( \delta := \text{dom}(b^*) \). Note that \( \delta \leq \beta_\tau \), as \( \gamma_\tau \in \text{nacc}(\mu^+) \). Also, using (I) and (II) of the induction, \( b^* \in P_\delta \) and \( \pi_{\delta,1}(b^*) = p^r \).

**Claim 3.10.3.** There is a pair \( (b_r, p^r) \) satisfying Clauses (I)–(III).

**Proof.** We commence checking (III). As in the previous claim, it suffices to show that \( p^r := \check{\eta}_{\beta_r, \delta}(a \upharpoonright \beta_r)(b^*) \) diagonalizes \( \langle g_r(r^r_{\xi}) \upharpoonright \beta_r \mid \xi < \chi \rangle \).

Once again, we want to appeal to Proposition 2.22, but this time regarded with respect to \( (\check{\eta}_{\beta_r,1}, \pi_{\beta_r,1}) \), \( a \upharpoonright \beta_r, r^r, \langle g_r(r^r_{\xi}) \upharpoonright \beta_r \mid \xi < \chi \rangle, p^r \) and \( p^r \).

(a) The verification is exactly the same as in Claim 3.10.2.
(β) By (II) and (III) of the induction hypothesis, \( b^* \leq_\delta a \mid \delta \) and \( b^* \mid 1 = p' \).

Hence, \( p^\tau \in P_{b^*} \), \( p^\tau \leq_\beta a \mid \beta \) and \( p^\tau \mid 1 = p' \).

Let \( q' \in W_{\xi}(p') \). Our aim is to show that

\[
\hat{\eta}_{b^*,1}(p^\tau)(q') \leq_\beta g_{\tau}(r^\xi_{\xi}) \mid \beta \tau, 
\]

for the unique index \( \xi \) such that \( r^\xi_{\xi} = w(a \mid 1, q') \).

By virtue of Lemma 3.5(5), \( B_{h_{b^*,1}(p^\tau)}(q') = B_{p^\tau} = B_{b^*} \). Hence, it will be enough to check that \( \hat{\eta}_{b^*,1}(p^\tau)(q') \mid \delta \leq_\delta g_{\tau}(r^\xi_{\xi}) \mid \delta \).

For each \( \tau' < \tau \), combining (II) of the induction hypothesis with Clauses (1) and (3) of Lemma 3.5 we have

\[
\hat{\eta}_{b^*,1}(p^\tau)(q') \mid \gamma_{\tau'} = \hat{\eta}_{\gamma_{\tau},1}(b_{\tau'})(q') = \hat{\eta}_{\gamma_{\tau},1}(b_{\tau'})(s_{\tau'}), 
\]

where \( s_{\tau'} := \hat{\eta}_{b^*,1}(b_{\tau'} \mid \beta_{\tau'})(q') \).

Thus, by (III) of our induction hypothesis,

\[
\hat{\eta}_{b^*,1}(p^\tau)(q') \mid \gamma_{\tau'} = \hat{\eta}_{\gamma_{\tau},1}(b_{\tau'})(s_{\tau'}) \leq_\gamma_{\tau'} g_{\tau'}(r^\xi_{\tau'}) \leq_\gamma_{\tau'} g_{\tau'}(r^\xi_{\tau'}),
\]

where \( \xi \) is the unique index such that \( r^\xi_{\tau'} = w(a \mid \beta_{\tau'}, s_{\tau'}) \).

Since \( g_{\tau'}(r^\xi_{\tau'}) \mid \gamma_{\tau'} = g_{\tau'}(r^\xi_{\tau'}) \), the above expression actually yields \( \hat{\eta}_{b^*,1}(p^\tau)(q') \mid \gamma_{\tau'} \leq_\gamma_{\tau'} g_{\tau'}(r^\xi_{\tau'}) \mid \gamma_{\tau'} \). Altogether, we have shown that

\[
\hat{\eta}_{b^*,1}(p^\tau)(q') \mid \delta \leq_\delta g_{\tau}(r^\xi_{\xi}) \mid \delta. 
\]

Finally, note that

\[
r_{\xi} = r^\tau_{\xi} \mid 1 = w(a \mid \beta_{\tau'}, s_{\tau'}) \mid 1 = w(a \mid 1, q'), 
\]

where the last equality follows from Lemma 3.6 and \( s_{\tau'} \mid 1 = q' \).

The above shows that \( (a \mid \gamma_{\tau}, r^\tau_{\tau'}, p^\tau, g_{\tau}, i_{\gamma_{\tau}}) \) fulfills the assumptions of Definition 2.25 with respect the pair \( (\hat{\eta}_{\gamma_{\tau},1}, p_{\tau}, i_{\gamma_{\tau}}) \). Appealing to Lemma 3.6 and Definition 3.7 we obtain \( b_{\tau} \subseteq a \mid \gamma_{\tau} \) with \( b_{\tau} \mid \beta_{\tau} = p^\tau \). It is clear that the pair \( (b_{\tau}, p^\tau) \) witnesses (II).

Let us now show that \( (b_{\tau}, p^\tau) \) satisfies (I) and (II). By (II) of the induction hypothesis and the definition of \( p^\tau \), for each \( \tau' < \tau \),

\[
b_{\tau} \mid \gamma_{\tau'} = \hat{p} \mid \delta \mid \gamma_{\tau'} = b^* \mid \gamma_{\tau'} = b_{\tau}'. 
\]

Similarly, by (I) of the induction hypothesis, \( b_{\tau} \mid 1 = b_{\tau'} \mid 1 = p' \).

The above completes the induction and yields the lemma.

The following technical lemma yields a sufficient condition for the pair \( (P_\alpha, \ell_\alpha) \) to have property \( \mathcal{D} \) in case \( \alpha \in \text{acc}(\mu^+ + 1) \).

---

14 For this latter equality, see equation (**) above.
Lemma 3.11. Let $\alpha \in \text{acc}(\mu^++1)$, $a \in P_\alpha$, $n < \omega$ and $\bar{s} = \langle s_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$. Set $l := \ell_\alpha(a)$. If $D \subseteq (\mathbb{P}_\alpha)_{l+n}$ forms a dense subposet of $(\mathbb{P}_\alpha)_{l+n}$, then $I$ has a winning strategy for the game $\mathcal{D}_{\mathbb{P}_\alpha}(a, \bar{s}, D)$ such that by using this strategy, for any outcome $\langle (a_\xi, b_\xi) \mid \xi < \chi \rangle$ of the game, there will be $b \in P_\alpha$ such that:

- $b$ diagonalizes $\langle b_\xi \mid \xi < \chi \rangle$;
- for all $\gamma \in B_\alpha$, $\pi_{\alpha,\gamma}(b) \lessdot^\eta \pi_{\alpha,\gamma}(a)$.

Proof. Set $p := \pi_{\alpha,1}(a)$ and $r_\xi := \pi_{\alpha,1}(s_\xi)$ for each $\xi < \chi$. By Clauses (4) and (5) of Definition 2.13, $\bar{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a good enumeration of $W_n(p)$.

We now describe our strategy for $I$. Suppose that $\xi < \chi$ and that $\langle (a_\eta, b_\eta) \mid \eta < \xi \rangle$ is an initial play of the game $\mathcal{D}_{\mathbb{P}_\alpha}(a, \bar{s}, D)$; we need to define $a_\xi$.

- If $\xi = 0$, then let $p_0$ be the $0^{th}$-move of $I$ according to some winning strategy in $\mathcal{D}_{\mathbb{P}_1}(p, \bar{r})$, which is available by virtue of Building Block I. Also, let $t_0$ be a condition in $\mathbb{P}_1$ such that $t_0 \leq_1 p_0, r_0$.

If $B_\alpha$ is empty, then let $a_0 := \pi_{\alpha,1}(a)(p_0)$ and $z_0 := \pi_{\alpha,1}(a)(t_0)$. By Lemma 3.6 $z_0 \leq_0 a_0, s_0$, hence $a_0$ is a legitimate move for $I$.

Suppose now that $B_\alpha$ is nonempty, and let $\langle \gamma_\tau \mid \tau \leq \theta \rangle$ be the increasing enumeration of closure of $B_\alpha$. For every $\tau \in \text{nacc}(\theta + 1)$, $\gamma_\tau$ is a successor ordinal, so we let $\beta_\tau$ denote its predecessor. By recursion on $\tau \leq \theta$, we shall define a coherent sequence $\langle (a^n_\tau, z^n_\tau) \mid \tau \leq \theta \rangle \subseteq \prod_{\tau \leq \theta} (P_{\gamma_\tau} \times P_{\beta_\tau})$, and then we shall let $a_\xi := \pi_{\alpha, \gamma_\xi}(a)(a^n_0)$ and $z_\xi := \pi_{\alpha, \gamma_\xi}(a)(z^n_0)$.

The idea is to craft the sequence $\langle a^n_\tau \mid \tau \leq \theta \rangle$ so that for all $\gamma \in B_\alpha$, $a_0 \upharpoonright \gamma$ satisfies (2)–(4) of Lemma 3.10. On the other hand, $\langle z^n_\tau \mid \tau \leq \theta \rangle$ will provide a sequence of auxiliary conditions witnessing that $z^n_\tau \leq_{\gamma_\tau} a^n_\tau, s_0 \upharpoonright \gamma_\tau$. This will ensure at the end that $a_0$ is a legitimate move for $I$.

Set $g_0^n := \text{dom}(\text{tp}_{\gamma_0^n}(a \upharpoonright \gamma_0^n)) + \omega + 1$, and then let

- $a^n_0 := \pi_{\gamma_0^n,1}(a \upharpoonright \gamma_0^n)(p_0)^{\check{g}_0^n}$,
- $z_0^n := \pi_{\gamma_0^n,1}(a \upharpoonright \gamma_0^n)(t_0)^{\check{g}_0^n}$,

where the $\check{}$ operation is provided by Definition 2.23(5) with respect to the type $\text{tp}_{\gamma_0^n}$ over the forking projection $\langle \pi_{\gamma_0^n, \beta_0}, \pi_{\gamma_0^n, \beta_0} \rangle$.

Since $p_0 \leq_0 a \uparrow 1$, $a_0^n \in P_{\gamma_0^n}$ and also $a_0^n \leq_0 a \upharpoonright \gamma_0$. Similarly, $z_0^n \in P_{\gamma_0^n}$.

Claim 3.11.1. $z_0^n \leq_0 a_0^n, s_0 \upharpoonright \gamma_0$.

Proof. Combining Clause (5)(a) of Definition 2.23 with Lemma 3.6,

\[ z^n_0 \leq_{\gamma_0} \pi_{\gamma_0,1}(a \upharpoonright \gamma_0)(t_0) \leq_{\gamma_0} \pi_{\gamma_0,1}(a \upharpoonright \gamma_0)(t_0) = s_0 \upharpoonright \gamma_0. \]

On the other hand, $\pi_{\gamma_0,1}(a \upharpoonright \gamma_0)(t_0) \leq_{\gamma_0} \pi_{\gamma_0,1}(a \upharpoonright \gamma_0)(p_0)$ and

\[ \text{dom}(\text{tp}_{\gamma_0^n}(\pi_{\gamma_0,1}(a \upharpoonright \gamma_0)(t_0))) = \text{dom}(\text{tp}_{\gamma_0^n}(\pi_{\gamma_0,1}(a \upharpoonright \gamma_0)(p_0))), \]

\[ 15\text{Recall that this is part of the rules of } \mathcal{D}_{\mathbb{P}_1}(p, \bar{r}) \text{ (see Definition 2.10).} \]
\[ 16\text{Namely, for each } \tau', \bar{s}' = \bar{s} \upharpoonright \tau' = \bar{a}' \upharpoonright \tau' = \bar{z}' \upharpoonright \tau'. \]
\[ 17\text{Note that } \pi_{\gamma_0, \beta_0}(a \upharpoonright \gamma_0)(p_0) = \pi_{\gamma_0, \beta_0}(a \upharpoonright \gamma_0)(\pi_{\gamma_0, \beta_0}(a \upharpoonright \gamma_0)(p_0)) \text{ (see (*) at Page } 16).} \]
where this last equality follows from Clause (3) of Definition 2.23.

Combining this with Definition 2.23(6) we get \( z_0 \leq \gamma_0 \), as desired. \( \square \)

**Claim 3.11.2.** For all \( i \in [\text{dom}(tp_{\gamma_0}(a \upharpoonright \gamma_0)), \text{dom}(tp_{\gamma_0}(a_0))] \),

\[
\text{tp}_{\gamma_0}(a_0)(i) \leq \text{mtp}_{\gamma_0}(a \upharpoonright \gamma_0).
\]

**Proof.** Let \( i \) be as above. By Definition 2.23(3), \( \text{dom}(tp_{\gamma_0}(a \upharpoonright \gamma_0)(p_0)) = \text{dom}(tp_{\gamma_0}(a \upharpoonright \gamma_0)) \). So, combining Clauses (2) and (5) of Definition 2.23

\[
\text{tp}_{\gamma_0}(a_0)(i) \leq \text{mtp}_{\gamma_0}(a \upharpoonright \gamma_0) \leq \text{mtp}_{\gamma_0}(a \upharpoonright \gamma_0),
\]
as desired. \( \square \)

\( \blacktriangleright\blacktriangleright \) For every \( \tau < \theta \) such that both \( a_0^\tau \) and \( z_0^\tau \) have already been defined, set \( a_0^{\tau+1} := \text{dom}(tp_{\gamma_{\tau+1}}(a \upharpoonright \gamma_{\tau+1}))) + \omega + 1 \), and then let

\[
a_0^{\tau+1} := n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau) \mathop \cdot \gamma_{\tau+1},
\]

\[
z_0^{\tau+1} := n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(z_0^\tau) \mathop \cdot \gamma_{\tau+1}.
\]

where the \( \mathop \cdot \gamma_{\tau+1} \) operation is with respect to the type \( tp_{\gamma_{\tau+1}} \).

**Claim 3.11.3.** For all \( \tau' \leq \tau \), \( a_0^{\tau+1} \upharpoonright \gamma_{\tau'} = a_0^{\tau'} \) and \( z_0^{\tau+1} \upharpoonright \gamma_{\tau'} = z_0^{\tau'} \).

**Proof.** Let \( \tau' \leq \tau \). By Clause (5)(a) of Definition 2.23 and Lemma 3.5(1),

\[
a_0^{\tau+1} \upharpoonright \beta_{\tau+1} = n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau) \upharpoonright \beta_{\tau+1} = n_{\beta_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \beta_{\tau+1})(a_0^\tau).
\]

Hence, Lemma 3.5(5) yields \( a_0^{\tau+1} \upharpoonright \gamma_{\tau} = a_0^\tau \). Using the induction hypothesis, we get \( a_0^{\tau+1} \upharpoonright \gamma_{\tau'} = a_0^{\tau'} \). The argument for \( z_0^{\tau+1} \) is the same. \( \square \)

**Claim 3.11.4.** \( z_0^{\tau+1} \leq_{\gamma_{\tau+1}} a_0^{\tau+1}, s_0 \upharpoonright \gamma_{\tau+1} \).

**Proof.** Recall that by the induction hypothesis, \( z_0^\tau \leq_{\gamma_{\tau}} a_0^\tau, s_0 \upharpoonright \gamma_{\tau} \). Thus, Clause (5) of Definition 2.23 and Lemma 3.6 combined yield

\[
z_0^{\tau+1} \leq_{\gamma_{\tau+1}} n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(z_0^\tau) \leq_{\gamma_{\tau+1}} n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(s_0 \upharpoonright \gamma_{\tau}) = s_0 \upharpoonright \gamma_{\tau+1}.
\]

Similarly, Lemma 3.6 yields

\[
n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(z_0^\tau) \leq_{\gamma_{\tau+1}} n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau).
\]

Also, by Clause (3) of Definition 2.23 and the remark made at Footnote 18

\[
\text{dom}(tp_{\gamma_{\tau+1}}(n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(z_0^\tau))) = \text{dom}(tp_{\gamma_{\tau+1}}(n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau))).
\]

Therefore, Definition 2.23(6) yields \( z_0^{\tau+1} \leq_{\gamma_{\tau+1}} a_0^{\tau+1} \), as desired. \( \square \)

Finally, the following can be proved exactly as in Claim 3.11.2:

**Claim 3.11.5.** For all \( i \in [\text{dom}(tp_{\gamma_{\tau+1}}(a \upharpoonright \gamma_{\tau+1})), \text{dom}(tp_{\gamma_{\tau+1}}(a_0^{\tau+1}))],

\[
\text{tp}_{\gamma_{\tau+1}}(a_0^{\tau+1})(i) \leq \text{mtp}_{\gamma_{\tau+1}}(a \upharpoonright \gamma_{\tau+1}).
\]

\( \text{Footnote 18:} \) Note that \( n_{\gamma_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau) = n_{\gamma_{\tau+1}, \beta_{\tau+1}}(a \upharpoonright \gamma_{\tau+1})(n_{\beta_{\tau+1}, \gamma_{\tau}}(a \upharpoonright \beta_{\tau+1})(a_0^\tau)). \)
For every \( \tau \in \text{acc}(\theta + 1) \), let \( a_0^\tau := \bigcup_{\tau' < \tau} a_0^{\tau'} \) and \( z_0^\tau := \bigcup_{\tau' < \tau} z_0^{\tau'} \). Thanks to the induction hypothesis, \( \langle (a_0^{\tau'}, z_0^{\tau'}) | \tau' \leq \tau \rangle \) is clearly coherent. Clearly, \( z_0^\tau \leq \gamma_\tau, a_0^\tau \) and arguing as in Claim 3.11.4 we have \( z_0^\tau \leq \gamma_\tau, s_0 | \gamma_\tau \).

At the end of the recursion, we define \( a_0 \) and \( z_0 \) as mentioned before. Note that by our construction \( z_0 \) witnesses that \( a_0 \) is a legitimate move for \( I \) so, in response, \( II \) plays a condition \( b_0 \) in \( D \) extending \( a_0 \) and satisfying \( b_0 \leq a_0 \). Finally, note that by our construction \( \gamma_\tau, s_0 \) and arguing as in Claim 3.11.4 we have \( z_0^\tau \leq \gamma_\tau, s_0 | \gamma_\tau \).

Also, for all \( i \in \text{dom}(\text{tp}_\gamma(a \upharpoonright \gamma)) \), \( \text{dom}(\text{tp}_\gamma(a_0 \upharpoonright \gamma)) \),

\[
\text{tp}_\gamma(a_0 \upharpoonright \gamma)(i) \leq \text{mt}_{\gamma}(a \upharpoonright \gamma).
\]

Suppose that \( 0 < \xi < \chi \). Recall that \( \langle (a_\eta, b_\eta) | \eta < \xi \rangle \) is an initial play of the game and that we want to define \( a_\xi \). To that effect, let \( p_\xi \) the \( \xi^{th} \)-move of \( I \) in the game \( \mathcal{D}_{\mathcal{P}_1}(p, \vec{r}) \), provided \( \langle (a_\eta \upharpoonright 1, b_\eta \upharpoonright 1) | \eta < \xi \rangle \) gathers the previous ones. Let \( t_\xi \) be such that \( t_\xi \leq p_\xi, s_\xi \) and set \( B_\xi := \bigcup_{\eta < \xi} B_{b_\eta} \).

If \( B_\xi \) is empty then again set \( a_\xi := \mathcal{m}_{a,1}(a)(p_\xi) \) and \( z_\xi := \mathcal{m}_{a,1}(a)(t_\xi) \) and argue as in the case \( \xi = 0 \). Otherwise, \( B_\xi \) is nonempty and we let \( \langle \gamma_\tau | \tau \leq \theta \rangle \) be the increasing enumeration of the closure of \( B_\xi \). By recursion on \( \tau \leq \theta \), we define a coherent sequence \( \langle (a_\xi^\tau, z_\xi^\tau) | \tau \leq \theta \rangle \in \prod_{\tau \leq \theta}(P_{\gamma_\tau} \times P_{\eta_\tau}) \), and then we shall let \( a_\xi := \mathcal{m}_{a,\gamma_\xi}(a)(a_\xi^\theta) \) and \( z_\xi := \mathcal{m}_{a,\gamma_\xi}(a)(z_\xi^\theta) \). The construction and the subsequent verifications are the same as in the case \( \xi = 0 \), so we skip them. The only difference now is that, for each \( \tau \in \text{nacc}(\theta + 1) \), we set \( a_\xi^\tau := (\sup_{\eta < \xi} \text{dom}(\text{tp}_\gamma(b_\eta \upharpoonright \gamma_\tau))) + \omega + 1 \).

Thereby, we get a condition \( a_\xi \) which is a legitimate move for \( I \) and, in response, \( II \) plays a condition \( b_\xi \) in \( D \) extending \( a_\xi \) and satisfying \( b_\xi \leq a_\xi \). Once again, \( a_\xi \upharpoonright 1 = p_\xi, B_\xi \subseteq B_{a_\xi} \subseteq B_{b_\xi} \) and for all \( \gamma \in B_\xi \),

\[
\langle \sup_{\eta < \xi} \text{dom}(\text{tp}_\gamma(b_\eta \upharpoonright \gamma)) \rangle \uplus 1 < \text{dom}(\text{tp}_\gamma(a_\xi \upharpoonright \gamma)).
\]

Also, for all \( i \in \text{dom}(\text{tp}_\gamma(a \upharpoonright \gamma)), \sup_{\eta < \xi} \text{dom}(\text{tp}_\gamma(b_\eta \upharpoonright \gamma)) \),

\[
\text{tp}_\gamma(a_\xi \upharpoonright \gamma)(i) \leq \text{mt}_{\gamma}(a \upharpoonright \gamma).
\]

At the end we obtain a sequence \( \langle (a_\xi, b_\xi) | \xi < \chi \rangle \) which is a play in the game \( \mathcal{D}_{\mathcal{P}_1}(a, \vec{s}, D) \). By construction, for each \( \xi < \chi, a_\xi \upharpoonright 1 = p_\xi \), so that \( \langle b_\xi \upharpoonright 1 | \xi < \chi \rangle \) is diagonalizable with respect to \( \vec{r} \). Let \( p' \leq \xi \mathcal{P}_{a,1}(a) \) be a witness for this latter fact.

Our next task is to show that \( \langle b_\xi \ | \xi < \chi \rangle \) is diagonalizable and that the corresponding witness \( b \) fulfils the requirements of the lemma.

Claim 3.11.6. The tuple \( (a, \vec{r}, p', g, B_\chi) \) meets the requirements of Lemma 3.10, where \( g : W_\alpha(\pi_{a,1}(a)) \to P_{\alpha,2} \alpha \upharpoonright a \) is defined via \( g(\xi_\tau) := b_\xi \).

\footnote{The inclusion \( B_{a_0} \subseteq B_{b_0} \) is obvious. For the other we use Clause (4) of Definition 2.23, noting that for all \( \gamma \in B_{a_0} \), \( \text{dom}(\text{tp}_\gamma(a_0 \upharpoonright \gamma)) \neq 0 \).}

\footnote{For details about the verification of \( \langle \upharpoonright \upharpoonright \rangle \), see Claim 3.11.2.}
Proof. Let us go over the clauses of Lemma 3.10: Clause (0) holds by the construction of \( \langle B_{b_{\xi}} \mid \xi < \chi \rangle \), Clause (1) is obvious and Clause (2) follows from the discussion of the previous paragraph. So, let us address the rest.

For each \( \gamma \in B_{\chi} \), denote \( \omega_{\gamma} := \min\{\xi < \chi \mid \gamma \in B_{b_{\xi}}\} \).

(3): Let \( \gamma \in B_{\chi} \) and \( \xi < \chi \):
- If \( \xi < \omega_{\gamma} \) then \( \gamma \notin B_{b_{\xi}} \) and so \( b_{\xi} \upharpoonright \gamma = [b_{\xi} \upharpoonright \beta]^{P_{\gamma}} \), where \( \gamma = \beta + 1 \).
  Thus, Lemma 3.6 and Definition 2.23(4) yield \( \text{dom}(\text{tp}_{\gamma}(b_{\xi} \upharpoonright \gamma)) = 0 \).
- If \( \xi = \omega_{\gamma} \), then \( \gamma \in B_{b_{\xi}} \) and so \( b_{\xi} \upharpoonright \gamma \neq [b_{\xi} \upharpoonright \beta]^{P_{\gamma}} \), where \( \gamma = \beta + 1 \).
  Again, Lemma 3.6 and Definition 2.23(4) yield \( \text{dom}(\text{tp}_{\gamma}(b_{\xi} \upharpoonright \gamma)) \geq 1 \).
- If \( \xi > \omega_{\gamma} \), then \( \gamma \in B_{b_{\gamma}} \subseteq B_{\xi} \).\(^{21}\) Combining (\( \dagger \)) above with \( b_{\xi} \leq_{\alpha} a_{\xi} \) and Clause (2) of Definition 2.23 we get
  \[
  (\sup_{\eta < \xi} \text{dom}(\text{tp}_{\eta}(b_{\eta} \upharpoonright \gamma))) + 1 < \text{dom}(\text{tp}_{\gamma}(b_{\xi} \upharpoonright \gamma)).
  \]

(4): Let \( \gamma \in B_{\chi} \), \( \omega_{\gamma} < \xi < \chi \) and \( i \) be as in Clause (4) of Lemma 3.10. By definition, \( \gamma \in B_{b_{\eta_{\xi}}} \subseteq B_{\xi} \), hence (\( \dagger \dagger \)) yields \( \text{tp}_{\gamma}(a_{\xi} \upharpoonright \gamma)(i) \leq \text{mtp}_{\gamma}(a \upharpoonright \gamma) \).
  Combining this with Definition 2.23(3) and \( b_{\xi} \leq_{\alpha} a_{\xi} \) we arrive at
  \[
  \text{tp}_{\gamma}(b_{\xi} \upharpoonright \gamma)(i) \leq \text{tp}_{\gamma}(a_{\xi} \upharpoonright \gamma)(i) \leq \text{mtp}_{\gamma}(a \upharpoonright \gamma).
  \]

(5): Let \( \gamma \in B \). For all \( \xi \) such that \( \omega_{\gamma} \leq \xi < \chi \), then \( \gamma \in B_{b_{\xi}} \). Since for all such \( \xi \)'s, \( b_{\xi} \) is a condition in \( D \subseteq \langle P_{\alpha} \rangle_{i+n} \) then \( \text{mtp}_{\gamma}(b_{\xi} \upharpoonright \gamma) = 0 \) (see Definition 3.8). Thus, clearly, \( \sup_{\omega_{\gamma} \leq \xi < \chi} \text{mtp}_{\gamma}(b_{\xi} \upharpoonright \gamma) < \omega \).
  The above completes the verification of the claim. \( \square \)

Combining Claim 3.11.6 with Lemma 3.10 we get a condition \( b \) witnessing Clauses (a)-(c) of the latter. Note that thanks to (a) and (c) we can appeal to Proposition 2.22 with respect to \( \langle i_{\alpha,1}, i_{\alpha}, \alpha \rangle, a, s, \langle b_{\xi} \mid \xi < \chi \rangle, P_{i} \) and \( b \) and conclude that \( b \) diagonalizes \( \langle b_{\xi} \mid \xi < \chi \rangle \). In effect, \( b \) is as desired. \( \square \)

Corollary 3.12. For every \( \alpha \in \text{acc}(\mu^{+}+1) \), if \( \langle \overline{P}_{\alpha} \rangle_{n} \) forms a dense subposet of \( \langle P_{\alpha} \rangle_{n} \) for every \( n < \omega \), then \( \langle P_{\alpha}, \ell_{\alpha} \rangle \) has property \( D \).

Proof. By Lemmas 3.11 and 2.11. \( \square \)

We are now ready to address Clause (iv) of Goal 3.2.

Lemma 3.13. Assume that for each \( \alpha \in \text{acc}(\mu^{+}+1) \) and every \( n < \omega \), \( \langle \overline{P}_{\alpha} \rangle_{n} \) is a dense subposet of \( \langle P_{\alpha} \rangle_{n} \). Then, for all nonzero \( \alpha \leq \mu^{+} \), \( \langle P_{\alpha}, \ell_{\alpha}, c_{\alpha} \rangle \) is a \( \Sigma \)-Prikry triple having property \( D \) with a greatest element \( \emptyset_{\alpha} \), \( \ell_{\alpha} = \ell_{1} \circ \pi_{\alpha,1} \), and \( \emptyset_{\alpha} \vDash P_{\alpha} \hat{\mu} = \kappa^{+} \).

Proof. We argue by induction on \( \alpha \). The base case \( \alpha = 1 \) follows from the fact that \( P_{1} \) is isomorphic to \( Q \) given by Building Block I. The successor step \( \alpha = \beta + 1 \) follows from the fact that \( P_{\beta+1} \) was obtained by invoking Building Block II.

\(^{21}\)Recall that \( B_{\xi} := \bigcup_{n<\xi} B_{b_{n}} \).
Next, suppose that $\alpha \in \text{acc}(\mu^+ + 1)$ is such that the conclusion of the lemma holds below $\alpha$. In particular, the hypothesis of Lemma 3.6 are satisfied, so that, for all nonzero $\beta \leq \gamma \leq \alpha$, $(\mathfrak{m}_{\gamma, \beta}, \pi_{\gamma, \beta})$ is a forking projection from $(\mathbb{P}_\gamma, \ell_\gamma)$ to $(\mathbb{P}_\beta, \ell_\beta)$. By Corollary 3.12, $(\mathbb{P}_\alpha, \ell_\alpha)$ has property $D$. We now go over the clauses of Definition 2.3:

(1) The first bullet of Definition 2.1 follows from the fact that $\ell_\alpha = \ell_1 \circ \pi_{\alpha, 1}$. Next, let $p \in P_n$ be arbitrary. Denote $\bar{p} := \pi_{\alpha, 1}(p)$. Since $(\mathbb{P}_1, \ell_1, c_1)$ is $\Sigma$-Prikry, we may pick $p' \subseteq 1$ with $\ell_1(p') = \ell_1(\bar{p}) + 1$. As $(\mathfrak{m}_{\alpha, 1}, \pi_{\alpha, 1})$ is a forking projection from $(\mathbb{P}_\alpha, \ell_\alpha)$ to $(\mathbb{P}_1, \ell_1)$, Fact 2.19(2) implies that $\mathfrak{m}_{\alpha, 1}(p)(p')$ is an element of $(\mathbb{P}_\alpha)_n^\beta$.

(2) For each $n < \omega$, by our assumption, $(\mathbb{P}_\alpha)_n$ is a dense subset of $(\mathbb{P}_\alpha)_n$. Thus, we just need to show that $(\mathbb{P}_\alpha)_n$ is $\kappa_n$-directed-closed. For future purpose, we shall prove the following stronger claim.

**Claim 3.13.1.** Let $n < \omega$. For every $\gamma \in [2, \alpha]$ and every directed set $D$ of conditions in $(\mathbb{P}_\gamma)_n$ of size $< \kappa_n$, there is $q \in (\mathbb{P}_\gamma)_n$ such that $q$ is a lower bound of $D$ and $B_q = \bigcup_{p \in D} B_p$.

**Proof.** We argue by induction on $\gamma \in [2, \alpha]$. The base case $\gamma = 2$ can be proved similarly to the successor case below. So, we assume by induction that the statement holds for all $\beta < \gamma$ and prove it for $\gamma$.

Fix an arbitrary directed family $D \subseteq (\mathbb{P}_\gamma)_n$ of size $< \kappa_n$.

- Suppose $\gamma = \beta + 1$. Then $\{ \pi_{\gamma, \beta}(p) \mid p \in D \}$ is a directed subset of $(\mathbb{P}_\beta)_n$ of size $< \kappa_n$, so that the inductive assumption yields a lower bound $p' \in (\mathbb{P}_\beta)_n$ such that $B_{p'} := \bigcup_{p \in D} B_{\pi_{\gamma, \beta}(p)}$. Set $\bar{D} := \{ \mathfrak{m}_{\gamma, \beta}(p)(p') \mid p \in D \}$, and note that $|\bar{D}| \leq |D| < \kappa_n$. By Lemma 3.6, $(\mathfrak{m}_{\gamma, \beta}, \pi_{\gamma, \beta})$ is a forking projection from $(\mathbb{P}_\gamma, \ell_\gamma)$ to $(\mathbb{P}_\beta, \ell_\beta)$. So, by Definition 2.13(7) together with Remark 2.24, $\bar{D}$ is a directed subset of $(\mathbb{P}_\gamma)_n^\pi_{\gamma, \beta}$.

Recalling that $(\mathbb{P}_\gamma)_n^\pi_{\gamma, \beta}$ is isomorphic to the $\kappa_n$-directed-closed poset $\mathbb{A}_n^\beta$ given by Building Block II, we pick a lower bound $q \in (\mathbb{P}_\gamma)_n^\pi_{\gamma, \beta}$ for $D$ such that $\pi_{\gamma, \beta}(q) = p'$. It is clear that $q$ is the desired lower bound.

- Suppose $\gamma$ is limit. Let $C := \text{cl}(\bigcup_{p \in D} B_p) \cup \{1, \gamma\}$. We shall define a $\subseteq$-increasing sequence $\{ p_\beta \mid \beta \in C \} \subseteq \prod_{\beta \in C} (\mathbb{P}_\beta)_n$ such that, for all $\beta \in C$, $p_\beta$ is a lower bound for $\{ \pi_{\alpha, \beta}(p) \mid p \in D \}$ with $B_{p_\beta} = \bigcup_{p \in D} B_{\pi_{\alpha, \beta}(p)}$. Note that for each $\beta \in C$, Lemma 3.9 yields $\{ \pi_{\alpha, \beta}(p) \mid p \in D \} \subseteq (\mathbb{P}_\beta)_n$. We define the sequence $\langle p_\beta \mid \beta \in C \rangle$ by recursion on $\beta \in C$:

- For $\beta = 1$, $\{ \pi_{\alpha, 1}(p) \mid p \in D \}$ is a directed subset of $(\mathbb{P}_1)_n$ of size $< \kappa_n$. By Building Block I, $(\mathbb{P}_1, \ell_1, c_1)$ is $\Sigma$-Prikry, and hence we may find a lower bound $p_1 \in (\mathbb{P}_1)_n$ for the set under consideration.

- Suppose $\beta > 1$ is a non-accumulation point of $C \cap \alpha$. Set $\beta := \tau + 1$ and $\varepsilon = \sup(C \cap \beta)$. Clearly, $\varepsilon \leq \tau$, so that Lemma 3.5(5) yields

$$\mathfrak{m}_{\tau, \varepsilon}(\pi_{\alpha, \tau}(p))(p_\varepsilon) = p_\varepsilon * \emptyset_{\tau},$$
Suppose \( \gamma < \mu \) is nonempty.

From now on, assume \( \alpha < \mu^+ \). Then, \( (\mathbb{P}_\alpha)^\mu_0 \cap (\mathbb{P}_\alpha)^{p'}_0 \) is nonempty.

Proof. If \( \alpha < \mu^+ \), then since \((\mathbb{P}_{\alpha,1}, \pi_{\alpha,1})\) is a forking projection from \((\mathbb{P}_{\alpha}, \ell_{\alpha, c_{\alpha}})\) to \((\mathbb{P}_1, \ell_{1, c_1})\), we get from Clause (8) of Definition 2.13 that \( c_1(p \restriction 1) = c_1(p' \restriction 1) \), and then by Clause (3) of Definition 2.3, we may pick \( r \in (\mathcal{P}_1)^{0}[1] \cap (\mathcal{P}_1)^{p'}[1] \). In effect, Clause (8) of Definition 2.13 entails \( \hat{\mathfrak{m}}_{\alpha,1}(p)(r) = \hat{\mathfrak{m}}_{\alpha,1}(p')(r) \). Finally, Fact 2.19(2) implies that \( \hat{\mathfrak{m}}_{\alpha,1}(p)(r) \) is in \((\mathcal{P}_\alpha)^{p}_0\) and that \( \hat{\mathfrak{m}}_{\alpha,1}(p')(r) \) is in \((\mathcal{P}_\alpha)^{p'}_0\). In particular, \((\mathbb{P}_\alpha)^{p}_0 \cap (\mathbb{P}_\alpha)^{p'}_0\) is nonempty.

From now on, assume \( \alpha = \mu^+ \). In particular, for all nonzero \( \beta < \gamma < \mu^+ \), \((\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)\) is a \( \Sigma \)-Prikry triple admitting a forking projection to \((\mathbb{P}_\beta, \ell_\beta, c_\beta)\) as witnessed by \((\hat{\mathfrak{m}}_{\gamma, \beta}, \pi_{\gamma, \beta})\). To avoid trivialities, assume also
that $|\{1_{\mu^+}, p, p'\}| = 3$. For each $q \in \{p, p'\}$, let $C_q := \text{cl}(B_q)$ and define a function $e_q : C_q \to H_\mu$ via
\[
e_q(\gamma) := (\phi_\gamma[C_q \cap \gamma], c_\gamma(q \upharpoonright \gamma)).
\]

Write $i$ for the common value of $e_{\mu}(p)$ and $e_{\mu}(p')$. It follows that, for every $\gamma \in C_p \cap C_{p'}$, $e_p(\gamma) = e_{p'}(\gamma)$, so that $\phi_\gamma[C_p \cap \gamma] = \phi_\gamma[C_{p'} \cap \gamma]$ and hence $C_p \cap \gamma = C_{p'} \cap \gamma$. Consequently, $R := C_p \cap C_{p'}$ is an initial segment of $C_p$ and an initial segment of $C_{p'}$.

Let $\zeta := \max(C_p \cup C_{p'})$, so that $p = (p \upharpoonright \zeta) * \emptyset_{\mu^+}$ and $p' = (p' \upharpoonright \zeta) * \emptyset_{\mu^+}$. Set $\gamma_0 := \max(\{0\} \cup R)$. By the above analysis, $C_p \cap (\gamma_0, \zeta]$ and $C_{p'} \cap (\gamma_0, \zeta]$ are two disjoint closed sets.

If $\gamma_0 = \zeta$, then $e_p(\zeta) = e_{p'}(\zeta)$, so that $c_\zeta(p \upharpoonright \zeta) = c_\zeta(p' \upharpoonright \zeta)$, and hence $(P_0)^{P|\gamma_0} \cap (P_0)^{P'\gamma_0}$ is nonempty. Pick $r$ in that intersection. Then $r * \emptyset_{\mu^+}$ is an element of $(P_{\mu^+})_0 \cap (P_{\mu^+})_{0}'$.

Next, suppose that $\gamma_0 < \zeta$. Consequently, there exists a finite increasing sequence $(\gamma_{j+1} | j \leq k)$ of ordinals from $C_p \cup C_{p'}$ such that $\gamma_{k+1} = \zeta$ and, for all $j \leq k$:

(i) if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_p$;

(ii) if $\gamma_{j+1} \notin C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_{p'}$.

We now define a sequence $(r_j | j \leq k+1)$ in $\prod_{j=0}^{k+1} (P_{\gamma_j})_0^{P|\gamma_j} \cap (P_{\gamma_j})_0^{P'\gamma_j}$, as follows.

- For $j = 0$, if $\gamma_0 \in C_p \cap C_{p'}$, then $e_p(\gamma_0) = e_{p'}(\gamma_0)$, so that $c_\gamma(p \upharpoonright \gamma_0) = c_\gamma(p' \upharpoonright \gamma_0)$, and we may indeed pick $r_0 \in (P_{\gamma_0})_0^{P|\gamma_0} \cap (P_{\gamma_0})_0^{P'\gamma_0}$. If $\gamma_0 \notin C_p \cap C_{p'}$, then $\gamma_0 = 0$, and we simply let $r_0 := \emptyset$.

- Suppose that $j < k + 1$, where $r_j$ has already been defined. Let $q := m_{\gamma_{j+1}, \gamma_j}(p \upharpoonright \gamma_{j+1}) (r_j)$ and $q' := m_{\gamma_{j+1}, \gamma_j}(p' \upharpoonright \gamma_{j+1}) (r_j)$. By Lemma 3.5(2), $B_q = (B_p \cap \gamma_{j+1}] \cup B_{q'}$ and $B_{q'} = (B_{p'} \cap \gamma_{j+1}] \cup B_{q'}$. In particular, if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (B_q \cup B_{q'}) \subseteq B_q$, so that $q' = r_j * \emptyset_{\gamma_{j+1}}$ and $q \leq \gamma_{j+1}$, $q'$ by Clauses (5) and (6) of Lemma 3.5, respectively. Likewise, if $\gamma_{j+1} \notin C_p$, then $q = r_j * \emptyset_{\gamma_{j+1}}$ so that $q' \leq q$. Thus, $(q, q') \cap (P_{\gamma_j})_0^{P|\gamma_j} \cap (P_{\gamma_j})_0^{P'\gamma_j}$ is nonempty, and we may let $r_{j+1}$ be an element of that set.

Evidently, $r_{k+1} * \emptyset_{\mu^+}$ is an element of $(P_{\mu^+})_0 \cap (P_{\mu^+})_0'$.

(4) Let $p \in P_\alpha$, $n, m < \omega$ and $q \in (P_\alpha^n)_m$ be arbitrary. Recalling that $(\hat{n}_{\alpha, 1}, \pi_{\alpha, 1})$ is a forcing projection from $(P_\alpha, \epsilon_{\alpha})$ to $(P_1, \epsilon_1)$, we infer from Clause (4) of Definition 2.13 that $\hat{n}_{\alpha, 1}(p)(m | p \upharpoonright 1, q \upharpoonright 1)$ is the greatest element of $\{r \leq n \mid p | q \leq n \upharpoonright r\}$.

(5) Recalling that $(P_1, \epsilon_1, c_1)$ is $\Sigma$-Prikry, and that $(\hat{n}_{\alpha, 1}, \pi_{\alpha, 1})$ is a forcing projection from $(P_\alpha, \epsilon_{\alpha})$ to $(P_1, \epsilon_1)$, we infer from Fact 2.19(1) that, for every $p \in P_\alpha$, $|W(p)| = |W(p \upharpoonright 1)| < \mu$.

(6) Let $p', p \in P_\alpha$ with $p' \leq \alpha p$. Let $q \in W(p')$ be arbitrary. For all $\gamma < \alpha$, the pair $(\hat{n}_{\alpha, \gamma}, \pi_{\alpha, \gamma})$ is a forcing projection from $(P_\alpha, \epsilon_{\alpha})$ to
(\(\mathcal{P}_\gamma, \ell_\gamma\)), so that by the special case \(m = 0\) of Clause (4) of Definition 2.13,

\[
w(p, q) = \Pi_{\alpha, \gamma}(p)(w(p \upharpoonright \gamma, q \upharpoonright \gamma)).
\]

Now, for all \(q' \leq \alpha q\), the induction hypothesis implies that, for all \(\gamma < \alpha\), \(w(p \upharpoonright \gamma, q' \upharpoonright \gamma) \leq \gamma w(p \upharpoonright \gamma, q \upharpoonright \gamma)\). Together with Clause (5) of Definition 2.13, it follows that, for all \(\gamma < \alpha\),

\[
w(p, q') \upharpoonright \gamma = w(p \upharpoonright \gamma, q' \upharpoonright \gamma) \leq \gamma w(p \upharpoonright \gamma, q \upharpoonright \gamma) = w(p, q) \upharpoonright \gamma.
\]

So, by the definition of \(\leq \alpha\), \(w(p, q') \leq \alpha w(p, q)\), as desired.

(7) By our assumptions, \((\Pi_{\alpha,1}, \pi_{\alpha,1})\) is a forking projection from \((\mathcal{P}_\alpha, \ell_\alpha)\) to \((\mathcal{P}_1, \ell_1)\), \((\mathcal{P}_1, \ell_1, \ell_1)\) is \(\Sigma\)-Prikry, and \((\mathcal{P}_\alpha, \ell_\alpha)\) has property \(D\). It thus follows from Lemma 2.21 that \((\mathcal{P}_\alpha, \ell_\alpha)\) has the CPP.

To complete our proof we shall need the following claim.

**Claim 3.13.** For each \(\alpha\) with \(1 \leq \alpha \leq \mu^+\), \(1_{\mathcal{P}_\alpha} \Vdash \mu = \kappa^+\).

**Proof.** The case \(\alpha = 1\) is given by Building Block I. Towards a contradiction, suppose that \(1 < \alpha \leq \mu^+\) and that \(1_{\mathcal{P}_\alpha} \nabla \mathcal{P}_\alpha \mu = \kappa^+\). As \(1_{\mathcal{P}_1} \Vdash \mu = \kappa^+\) and \(\mathcal{P}_\alpha\) projects to \(\mathcal{P}_1\), this means that there exists \(p \in \mathcal{P}_\alpha\) such that \(p \Vdash \mathcal{P}_\alpha | \mu| \leq |\kappa|\). Since \(\mathcal{P}_1\) is isomorphic to the poset \(\mathcal{Q}\) of Building Block I, and since \(1_{\mathcal{Q}} \Vdash \kappa\) is singular,” \(22\) \(1_{\mathcal{P}_1} \Vdash \mathcal{P}_1 \kappa\) is singular”. As \(\mathcal{P}_\alpha\) projects to \(\mathcal{P}_1\), in fact \(p \Vdash \mathcal{P}_\alpha \text{cf}(\mu) < \kappa\). Thus, Lemma 2.7(2) yields a condition \(p' \leq \alpha p\) with \(|W(p')| \geq \mu\), contradicting Clause (5) above.

This completes the proof of Lemma 3.13. \(\square\)

4. **An application**

In this section, we present the first application of our iteration scheme. We will be constructing a model of finite simultaneous reflection at a successor of a singular strong limit cardinal \(\kappa\) in the presence of \(\neg \text{SCH}_\kappa\).

**Definition 4.1.** For cardinals \(\theta < \mu = \text{cf}(\mu)\) and stationary subsets \(S, \Gamma\) of \(\mu\), \(\text{Refl}(<\theta, S, \Gamma)\) stands for the following assertion. For every collection \(\mathcal{S}\) of stationary subsets of \(S\), with \(|S| < \theta\) and \(\sup\{|\text{cf}(\alpha)\mid \alpha \in \bigcup \mathcal{S}\} < \mu\), there exists \(\delta \in \Gamma \cap E^\mathcal{S}_{\omega, \omega}\) such that, for every \(\mathcal{S} \in \mathcal{S}\), \(S \cap \delta\) is stationary in \(\delta\).

We write \(\text{Refl}(<\theta, S, \mu)\) for \(\text{Refl}(<\theta, S, \mu)\).

A proof of the following folklore fact may be found in [PRS20, §4].

**Fact 4.2.** If \(\kappa\) is a singular strong limit cardinal admitting a stationary subset \(S \subseteq \kappa^+\) for which \(\text{Refl}(<\text{cf}(\kappa)^+, S)\) holds, then \(2^\kappa = \kappa^+\).

In particular, if \(\kappa\) is a singular strong limit cardinal of countable cofinality for which \(\text{SCH}_\kappa\) fails, and \(\text{Refl}(<\theta, \kappa^+)\) holds, then \(\theta \leq \omega\). We shall soon show that \(\theta := \omega\) is indeed feasible.

---

\(22\)This is the sole part of the whole proof to make use of the fact that the poset given by Building Block I forces \(\kappa\) to be singular.
The following general statement about simultaneous reflection will be useful in our verification later on.

**Proposition 4.3.** Suppose that $\mu$ is non-Mahlo cardinal, and $\theta \leq \text{cf}(\mu)$. For stationary subsets $T, \Gamma, R$ of $\mu$, $\text{Refl}(<2, T, \Gamma) + \text{Refl}(<\theta, \Gamma, R)$ entails $\text{Refl}(<\theta, T \cup \Gamma, R)$.

**Proof.** Given a collection $S$ of stationary subsets of $T \cup \Gamma$, with $|S| < \theta$ and $\sup(|\text{cf}(\alpha) | \alpha \in \bigcup S|) < \mu$, we shall first attach to any set $S \in S$, a stationary subset $S'$ of $\Gamma$, as follows.

- If $S \cap \Gamma$ is stationary, then let $S' := S \cap \Gamma$.
- If $S \cap \Gamma$ is nonstationary, then for every (sufficiently thin) club $C \subseteq \mu$, $S \cap C$ is a stationary subset of $T$, and so by $\text{Refl}(<2, T, \Gamma)$, there exists $\alpha \in \Gamma \cap E^\mu_{<\omega}$ such that $(S \cap C) \cap \alpha$ is stationary in $\alpha$, and in particular, $\alpha \in C$. So, the set $\{\alpha \in \Gamma | S \cap \alpha$ is stationary $\}$ is stationary, and, as $\mu$ is non-Mahlo, we may pick $S'$ which is a stationary subset of it and all of its points consists of the same cofinality.

Next, as $|S| < \text{cf}(\mu)$, we have $\sup(|\text{cf}(\alpha) | \alpha \in S', S \in S|) < \mu$, and so, from $\text{Refl}(<\theta, \Gamma, R)$, we find some $\alpha \in R$ such that $S' \cap \alpha$ is stationary for all $S \in S$.

**Claim 4.3.1.** Let $S \in S$. Then $S \cap \alpha$ is stationary in $\alpha$.

**Proof.** If $S' = S$, then $S \cap \alpha = S' \cap \alpha$ is stationary in $\alpha$, and we are done. Next, assume $S' \neq S$, and let $c$ be an arbitrary club in $\alpha$. As $S' \cap \alpha$ is stationary in $\alpha$, we may pick $\delta \in \text{acc}(c) \cap S'$. As $\delta \in S' \subseteq E^\mu_{<\omega}$, $c \cap \delta$ is a club in $\delta$, and as $\delta \in S'$, $S \cap \delta$ is stationary, so $S \cap c \cap \delta \neq \emptyset$. In particular, $S \cap c \neq \emptyset$. □

This completes the proof. □

4.1. **About Building Block II.** In this subsection, we describe Building Block II that we will be feeding to the iteration scheme of the preceding section. We were originally planning to use the functor given by [PRS20, §6], but unfortunately we found a gap in the proof of the mixing property [PRS20, Lemma 6.16]. To mitigate this gap, we shall relax Clause (4) of [PRS20, Definition 6.2] and prove that the outcome is a functor satisfying the weak mixing property (Lemma 4.16 below). The upshot of this subsection is encapsulated by Corollary 4.18. We commence by describing our setup.

**Setup 4.** Suppose that $(\mathbb{P}, \ell, c)$ is a given $\Sigma$-Prikry notion of forcing and that $(\mathbb{P}, \ell)$ has property $D$. Denote $\mathbb{P} = (P, \leq)$ and $\Sigma = \langle \kappa_n | n < \omega \rangle$. Also, define $\kappa$ and $\mu$ as in Definition 2.3, and assume that $1_P \Vdash \kappa$ is singular” and that $\mu^{<\mu} = \mu$. Recall that for each $n < \omega$, we denote by $\mathbb{P}_n$ a dense $\kappa_n$-directed-closed subset of $P_n$. Our universe of sets is denoted by $V$, and we assume that, for all $n < \omega$, $V^{\mathbb{P}_n} \models \text{Refl}(1, E^\mu_{<\kappa_n})$. Write $\Gamma := \{\alpha < \mu \mid \omega < \text{cf}^V(\alpha) < \kappa\}$. Suppose $\ast \in P$ forces that $\dot{T}$ is a $\mathbb{P}$-name for a stationary
subset $T$ of $(E^n_\omega)^V$ that does not reflect in $\Gamma$ and that $R$ is the binary relation:

$$R := \{(\alpha, q) \in \mu \times P \mid q \leq r^* & \forall r \leq q[\ell(r) \in I \rightarrow r \forces_{p_{\ell(r)}} \dot{\alpha} \in \dot{C}_{\ell(r)}]\},$$

where $I := \omega \setminus \ell(r^*)$.

A moment’s reflection makes it clear that, for all $(\alpha, q) \in R$, $q \forces_p \dot{\alpha} \notin \dot{T}$. Also, if $(\alpha, q) \in R$ and $q' \leq q$ then $(\alpha, q') \in R$, as well.

**Remark 4.4.** The previous setup assumptions can be relaxed to encompass the case where $T$ itself is nonstationary in $V$. Indeed, for each $n < \omega$, set $\dot{T}_n := \{(\dot{\alpha}, p) \mid (\alpha, p) \in \mu \times P_n & p \forces \dot{\alpha} \in \dot{T}\}$ and note that $p \forces_{\dot{T}_n} \dot{T}_n$ is nonstationary$^*$ for every $p \in P_n$. In effect, there is a sequence $\langle \dot{C}_n \mid n < \omega \rangle$, such that, for each $n < \omega$, $\dot{C}_n$ is a $P_n$-name for a club and $p \forces_{\dot{T}_n} \dot{T}_n \cap \dot{C}_n = \emptyset$, for all $p \in P_n$. Now, derive relation $R$ as above taking $r^* = \mathbb{1}_p$ and $I := \omega$.

**Definition 4.5** (relaxed form of [PRS20, Definition 6.2]). Suppose $p \in P$. A labeled $p$-tree is a function $S : W(p) \rightarrow [\mu]^{<\mu}$ such that for all $q \in W(p)$:

1. $S(q)$ is a closed bounded subset of $\mu$;
2. $S(q') \supseteq S(q)$ whenever $q' \leq q$;
3. $q \forces_p S(q) \cap \dot{T} = \emptyset$;
4. there is a natural number $m$ such that for any pair $q' \leq q$ of elements of $W(p)$, if $S(q') \neq \emptyset$ and $\ell(q') \geq \ell(p) + m$, then $(\max(S(q')) \setminus q) \in R$.

The least such $m$ is denoted by $m(S)$.

**Remark 4.6.** Note that for $m$ given by Clause (4), if $q \in W_{\geq m}(p)$ is incompatible with $r^*$, then $S(q') = \emptyset$ for all $q' \leq q$ in $W(p)$.

**Definition 4.7** ([PRS20, Definition 6.3]). For $p \in P$, we say that $\vec{S} = \langle S_i \mid i \leq \alpha \rangle$ is a $p$-strategy iff all of the following hold:

1. $\alpha < \mu$;
2. $S_i$ is a labeled $p$-tree for all $i \leq \alpha$;
3. for every $i < \alpha$ and $q \in W(p)$, $S_i(q) \subseteq S_{i+1}(q)$;
4. for every $i < \alpha$ and a pair $q' \leq q$ in $W(p)$, $(S_{i+1}(q) \setminus S_i(q)) \subseteq (S_{i+1}(q') \setminus S_i(q'))$;
5. for every limit $i \leq \alpha$ and $q \in W(p)$, $S_i(q)$ is the ordinal closure of $\bigcup_{j<i} S_j(q)$. In particular, $S_0(q) = \emptyset$ for all $q \in W(p)$.

Now, we are ready to describe our functor.

**Definition 4.8** ([PRS20, Definition 6.4]). Let $\mathbb{A}(\mathbb{P}, \dot{T})$ be the notion of forcing $\mathbb{A} := (\mathbb{A}, \leq)$, where:

1. $(p, \vec{S}) \in A$ iff $p \in P$, and $\vec{S}$ is either the empty sequence, or a $p$-strategy;
2. $(p', \vec{S}') \leq (p, \vec{S})$ iff:

\[\langle \dot{C}_n \mid n \geq \ell(r^*) \rangle\] denotes a sequence of $\mathbb{P}$-names for club subsets of $\mu$ such that, for each $q \leq r^*$, $q \forces_{\ell(q)} \dot{C}_{\ell(q)}$ is a club subset of $\dot{\mu}$ and $q \forces_{\ell(q)} \dot{C}_{\ell(q)} \cap \dot{T}_{\ell(q)} = \emptyset$. For more details, we refer the reader to [PRS20, §5 and §6].
of elements of $W$ over the clauses of Definition 4.5. To this end, let
and we shall verify that $S_i(p,q)$ for all $i \in \text{dom}(\vec{S})$ and $q \in W(p')$.

For all $p \in P$, denote $[p]^\lambda := (p, \emptyset)$.

**Definition 4.9** ([PRS20, Definitions 6.10 and 6.11]).

- Define $c_\lambda : A \to H_\mu$ by letting, for all $(p, \vec{S}) \in A$,
  
  $$c_\lambda(p, \vec{S}) := (c(p), \{(i, c(q), S_i(q)) \mid i \in \text{dom}(\vec{S}), q \in W(p')\}).$$

- Define $\pi : A \to P$ by stipulating $\pi(p, \vec{S}) := p$ and $\ell_\lambda := \ell \circ \pi$.

- Given $a = (p, \vec{S})$ in $A$, define $\hat{\cap}(a) : P \downarrow p \to A$ by letting for each $p' \leq p$, $\hat{\cap}(a)(p') := (p', T')$, where $T'$ is the sequence $\langle S'_i : W(p') \to [\mu]^{<\kappa} \mid i < \text{dom}(\vec{S}) \rangle$ satisfying:

  $$(*) \quad S'_i(q) := S_i(w(p, q)) \text{ for all } i \in \text{dom}(\vec{S}) \text{ and } q \in W(p').$$

Even after relaxing Clause (4) of [PRS20, Definition 6.2] to that of Definition 4.5, the following remains valid, with essentially the same proofs.

**Fact 4.10** ([PRS20, Corollary 4.13, Lemma 6.6, Theorem 6.8]).

1. $\models A \models \dot{\mu} = \dot{\kappa}^+$;
2. For every $\nu \geq \mu$, if $P$ is a subset of $H_\nu$, then so is $A$;
3. $[r^*]^{\dot{A}} \models A \models \dot{T}$ is nonstationary$^{1}$

**Lemma 4.11.** $(\cap, \pi)$ is a forking projection from $(A, \ell_\lambda, c_\lambda)$ to $(P, \ell, c)$.

**Proof.** The proof of [PRS20, Lemma 6.13] goes through, so we only focus on Clause (2) of Definition 2.13. Let $a \in A$ and $p' \leq \pi(a)$; we shall show that $\hat{\cap}(a)(p') \in A$ and $\hat{\pi}(a)(p') \subseteq a$.

Write $a$ as $(p, \vec{S})$. If $S = \emptyset$, then $\hat{\cap}(a)(p') = [p']^{A}$, and we are done.

Next, suppose that $\text{dom}(\vec{S}) = \alpha + 1$. Let $(p', \vec{S}) := \hat{\cap}(a)(p')$. Let $i \leq \alpha$ and we shall verify that $S'_i$ is a $p'$-labeled tree with $m(S'_i) \leq m(S_i)$. We go over the clauses of Definition 4.5. To this end, let $q' \leq q$ be arbitrary pair of elements of $W(p')$.

1. By Definition 2.3(6), we have $w(p, q') \leq w(p, q)$, so that $S'_i(q') = S_i(w(p, q')) \supseteq S_i(w(p, q)) = S'_i(q)$.
2. As $q \leq w(p, q)$, $w(p, q) \models P S_i(w(p, q)) \cap \dot{T} = \emptyset$, so that, since $S'_i(q) = S_i(w(p, q))$, we clearly have $q \models P S'_i(q) \cap \dot{T} = \emptyset$.
3. To avoid trivialities, Suppose that $S'_i(q') \neq \emptyset$ and $\ell(q) \geq m(S_i)$. Write $\gamma := \max(S'_i(q'))$. As $\ell(w(p, q)) = \ell(q) \geq m(S_i)$ and $\gamma = \max(S_i(w(p, q')))$, we infer that $(\gamma, w(p, q)) \in R$. In addition, $q \leq w(p, q)$, so by the definition of $R$ it follows that $(\gamma, q) \in R$. Recalling that $\max(S'_i(q)) = \gamma$, we are done.

---

$^{1}$Here, Claim 4.15.1 below plays the role of [PRS20, Lemma 6.7]. Also, note that this is trivial when $\dot{T}$ is a $P$-name for a nonstationary subset of $\mu$ in $V$. 
To prove that \((p', \vec{S}')\) is a condition in \(A\) it now remains to argue that \(\vec{S}'\) fulfills the requirements described in Clauses (3) and (5) of Definition 4.7 but this already follows from the definition of \(\vec{S}'\) and the fact that \(\vec{S}\) is a \(p\)-strategy. Finally \(\cap(a)(p') = (p', \vec{S}') \leq (p, \vec{S}) = a\) by the very choice of \(p'\) and by Definition 4.9. □

We now introduce a sequence of orderings \(\langle \subseteq^n | n < \omega \rangle \) of \(A\) and a type \(tp\) over \((\cap, \pi)\) that will witness together the weak mixing property of \((\cap, \pi)\).

**Definition 4.12.** For two conditions \(a = (p, \langle S_i | i < \alpha \rangle)\) and \(b = (p', \langle T_i | i < \alpha' \rangle)\), and \(n < \omega\), we let \(b \subseteq^n a\) if \(b \leq^n a\) and, if \(\alpha \neq 0\), then for all \(i < \alpha'\) and \(q \in W(p')\) with \(\ell(q) < \ell(p') + n\), \(T_i(q) = S_{\min(i, \alpha)}(w(p, q))\).

**Definition 4.13.** Define a map \(tp : A \to <^\mu \omega\), as follows.

Given \(a = (p, \vec{S})\) in \(A\), write \(\vec{S}\) as \(\langle S_i | i < \beta \rangle\), and then let

\[
\text{tp}(a) := \langle m(S_i) \mid i < \beta \rangle.
\]

We shall soon verify that \(\text{tp}\) is a type, but will use the mtp notation of Definition 2.23 from the outset. In particular, we will have \(\hat{A} = (A, \subseteq)\), with \(A := \{a \in A \mid \pi(a) \in P(\pi(a)) \& \text{mtp}(a) = 0\}\). Note that the supercollection \(\{a \in A \mid \text{mtp}(a) = 0\}\) coincides with the set \(A\) from [PRS20, Definition 6.4]. In particular, the proof of [PRS20, Lemma 6.15] goes through, yielding the following fact.

**Fact 4.14.** For all \(n < \omega\), \(\hat{\mathbb{A}}^n_\pi\) is \(\mu\)-directed-closed. □

**Lemma 4.15.** The map \(\text{tp}\) is a type over \((\cap, \pi)\).

**Proof.** We go over the clauses of Definition 2.23:

1. This follows from the mere definition of \(\text{tp}\).
2. Write \(b = (p', \vec{S}')\) and \(a = (p, \vec{S})\). By Definitions 4.8 and 4.13, \(\text{dom}(\text{tp}(b)) = \text{dom}(\vec{S}')\geq \text{dom}(\vec{S}) = \text{dom}(\text{tp}(a))\). Fix \(i \in \text{dom}(\text{tp}(a))\) and let us show that \(\text{tp}(b)(i) \leq \text{tp}(a)(i)\), i.e., that \(m(S'_i) \leq m(S_i)\).

Let \(q' \leq q\) be a pair of elements in \(W(p')\) with \(S'_i(q') \neq \emptyset\) and \(\ell(q) \geq m(S_i)\).

By Definition 4.8(2c), \(S'_i(q') = S_i(w(p, q'))\), hence \(w(p, q') \leq w(p, q)\) is a pair of elements in \(W(p)\) with \(S_i(w(p, q')) \neq \emptyset\). Set \(\gamma := \max(S_i(w(p, q')))\). By Definition 4.5(4), \((\gamma, w(p, q)) \in R\) hence the definition of \(R\) yields \((\gamma, q) \in R\).

Noting that \(\gamma = \max(S'_i(q'))\) it finally follows that \(m(S'_i) \leq m(S_i)\).

3. This follows from Definition 4.9(*)

4. Let \(a \in A\). If \(a = [\pi(a)]^\mathbb{A}\) then \(a = (\pi(a), \emptyset)\), and so \(\text{tp}([\pi(a)]^\mathbb{A})\) is the empty sequence. Conversely, if \(\text{tp}(a)\) is the empty sequence then Definition 4.13 implies that \(a\) takes the form \((\pi(a), \emptyset)\), hence \(a = [\pi(a)]^\mathbb{A}\).

5. Write \(a\) as \((p, \langle S_i \mid i < \text{dom}(\text{tp}(a)) \rangle)\) and let \(\alpha \in \mu \setminus \text{dom}(\text{tp}(a))\). There are two cases to consider:
   - If \(\text{dom}(\text{tp}(a)) = 0\), then let \(a^\cap \alpha := (p, \langle T_i \mid i \leq \alpha \rangle)\), where \(T_i : W(p) \to \{\emptyset\}\) is constant for every \(i \leq \alpha\).
Lemma 4.16.\(^{\triangleright}\) Otherwise, say \(\text{dom}(\text{tp}(a)) = \beta + 1\), let \(a^{\alpha} := (p, \langle T_i \mid i \leq \alpha \rangle)\), where \(T_i := S_{\min\{i, \beta\}}\) for every \(i \leq \alpha\).

It is routine to check that \(a^{\alpha}\) is as desired.

(6) Write \(b = (p', S')\) and \(a = (p, S)\) and set \(\gamma := \text{dom}(\text{tp}(b))\). If \(\gamma = 0\) \(b^{\alpha} \leq a^{\alpha}\) follows simply from \(p' \leq p\). Otherwise, \(\gamma\) takes the form \(\beta + 1\) and the above clause yields \(b^{\alpha} = (p', T')\), where \(T_i' := \langle T_i \mid i \leq \alpha \rangle\) and \(T_i' := S'_{\min\{i, \beta\}}\). Similarly, \(a^{\alpha} = (p, T)\), where \(T := \langle T_i \mid i \leq \alpha \rangle\) and \(T_i := S_{\min\{i, \beta\}}\). Using that \(b \leq a\), Definition 4.8 yields \(b^{\alpha} \leq a^{\alpha}\), as wanted.

(7) Let \(a = (p, S) \in A\). To avoid trivialities, let us assume that \(\bar{S} \neq \emptyset\).

\(\triangleright\) Suppose \(p\) is incompatible with \(r^*\). Then, by Remark 4.6, for all \(i < \text{dom}(\text{tp}(a))\) and all \(q \in W(p), S_i(q) = \emptyset\). Therefore, \(\text{mtp}(a) = 0\). Using Definition 2.3(2) find \(p' \leq^0 p \in \bar{P}\) and set \(b := \langle \langle a \rangle \rangle(p')\). Combining Clauses (2) and (3) above with the fact that \(\text{mtp}(a) = 0\) it easily follows that \(\text{mtp}(b) = 0\). Also, \(\tau(b) = p' \in P_p\). Thus, \(b \in A_p \perp a\), as wanted.

\(\triangleright\) Suppose \(p \leq r^*\). The following claim will give us the desired condition.

Claim 4.15.1. For every \(\epsilon < \mu\), there exist \(\alpha > \epsilon\) and \(b = (q, T) \leq^0 a\) such that \(b \in \bar{A}\), \(\text{dom}(\bar{T}) = \alpha + 1\), and for all \(r \in W(q), \text{max}(T_{\alpha}(r)) = \alpha\).

Proof. Let \(\epsilon < \mu\) be arbitrary. Since \((\bar{P}, \ell, c)\) is \(\Sigma\)-Priky, we infer from Definition 2.3(5) that \(|W(p)| < \mu\). Thus, by possibly extending \(\epsilon\), we may assume that \(S_i(q) \subseteq c\), for all \(q \in W(p)\) and \(i \in \text{dom}(\text{tp}(a))\). By Clause (5), we may also assume that \(\text{dom}(\text{tp}(a))\) is a successor ordinal, say, it is \(\delta + 1\).

As \(p \leq r^*\), by the very same proof of [PRS20, Claim 5.6.2(1)] and using Clause (2) of Definition 2.3, we may fix \((\alpha, q) \in R\) with \(\alpha > \delta + \epsilon\), \(q \leq^0 p\), and \(q \in P_p\). Define \(\bar{T} = \langle T_i : W(q) \rightarrow [\mu]^{<\mu} \mid i \leq \alpha \rangle\) by letting for all \(r \in W(q)\) and \(i \in \text{dom}(\bar{T})\):

\[
T_i(r) := \begin{cases} 
S_i(w(p, r)), & \text{if } i \leq \delta; \\
S_i(w(p, r)) \cup \{\alpha\}, & \text{otherwise.}
\end{cases}
\]

It is easy to see that \(T_i\) is a labeled \(q\)-tree for each \(i \leq \alpha\). By Definitions 4.7, 4.8 and 4.9, we also have that \(b = (q, \bar{T})\) is a condition in \(\bar{A}\) with \(b \leq^0 a\) and \(\tau(b) = q \in P_p\). As \((\alpha, q) \in R\), then \((\alpha, r) \in R\) for all \(r \in W(q)\), hence \(\text{mtp}(b) = 0\). Therefore, \(b\) is a condition in \(\bar{A}\) with the desired properties. \(\square\)

This completes the proof. \(\square\)

Lemma 4.16 (Weak Mixing Property). For all \(a \in A, n < \omega, \bar{r}\), and \(p' \leq^0 \pi(a)\), and for every function \(g : W_n(\pi(a)) \rightarrow A \perp a\), if there exists an ordinal \(i\) such that all of the following hold:

- (1) \(\bar{r} = (r_{\xi} \mid \xi < \chi)\) is a good enumeration of \(W_n(\pi(a))\);
- (2) \((\pi(g(r_{\xi})) \mid \xi < \chi)\) is diagonalizable with respect to \(\bar{r}\), as witnessed by \(p'\);
- (3) for every \(\xi < \chi\):
  - if \(\xi < i\), then \(\text{dom}(\text{tp}(g(r_{\xi}))) = 0\);
• if \( \xi = \iota \), then \( \text{dom}(\text{tp}(g(\xi))) \geq 1 \);
• if \( \xi > \iota \), then \( (\sup_{\eta < \xi} \text{dom}(\text{tp}(g(\eta)))) + 1 < \text{dom}(\text{tp}(g(\xi))) \);

(4) for all \( \xi \in (\iota, \chi) \) and \( \iota \in [\text{dom}(\text{tp}(a)), \sup_{\eta < \xi} \text{dom}(\text{tp}(g(\eta))))] \),

\[ \text{tp}(g(\xi))(i) \leq \text{mt}(a), \]

(5) \( \sup_{\eta \leq \chi} \text{mt}(g(\xi)) < \omega \),

then there exists \( \eta \in \mathbb{N} \) with \( \pi(\eta) = p' \) and \( \text{mt}(\eta) \leq n + \sup_{\eta \leq \chi} \text{mt}(g(\xi)) \), such that for all \( q' \in W_n(p') \),

\[ \pi(q') \leq g(\pi(a), q'). \]

**Proof.** Let \( a := (p, S^k) \). For each \( \xi < \chi \), set \( (p_\xi, S^k) := g(\xi). \)

**Claim 4.16.1.** If \( \iota \geq \chi \) then there is \( b \in A \) as in the lemma.

**Proof.** If \( \iota \geq \chi \) then \( \text{Clause (3)} \) yields \( \text{dom}(\text{tp}(g(\xi))) = 0 \) for all \( \xi < \chi \). Hence, \( \text{Clause (4)} \) of Definition 2.23 yields \( g(\xi) = [p_\xi]^{\alpha_\xi} \) for all \( \xi < \chi \). In particular also \( a = [p_\xi]^{\alpha_\xi} \). Set \( b := [p']^{\alpha_\xi} \), where \( p' \) is given by \( \text{Clause (2)} \).

Clearly, \( \pi(b) = p' \) and \( b \leq 0 \), hence also \( b \in \mathbb{N} \). Let \( q' \in W_n(p') \). By \( \text{Clause (2)} \) above, \( q' \leq 0 \), \( p_\xi \), where \( \xi \) is such that \( r_\xi = w(p, q') \). Finally, Definition 2.13(6) yields \( \pi(b)(q') = [q']^{\alpha_\xi} \leq g(\pi(a), q') \), as desired. \( \square \)

Hereafter let us assume that \( \iota < \chi \). For each \( \xi \in [\iota, \chi) \), \( \text{Clause (3)} \) and Definition 4.13 together imply that \( \text{dom}(S^k) = \alpha_\xi + 1 \) for some \( \alpha_\xi < \mu \). Moreover, \( \text{Clause (3)} \) yields \( \sup_{\eta \leq \xi} \alpha_\eta < \alpha_\xi \) for all \( \xi \in (\iota, \chi) \). Likewise, the same clause implies that \( g(\xi) = [p_\xi]^{\alpha_\xi} \), hence \( \overline{S}^k = \emptyset \), for all \( \xi < \iota \).

Let \( \langle s_\tau \mid \tau < \theta \rangle \) be a good enumeration \( W_n(p'). \) By Fact 2.19, \( \theta < \mu \). For each \( \tau < \theta \), set \( r_\tau := w(p, s_\tau) \). By \( \text{Clause (1)} \) above, for each \( \tau < \theta \),

\[ s_\tau \leq 0 \, \pi(g(w(p, s_\tau))) = \pi(g(r_\xi, \iota)) = p_\xi. \]

Set \( \alpha' := \sup_{\xi \leq \chi} \alpha_\xi \) and \( \alpha := \sup(\text{dom}(S)) \).\(^{25}\) By regularity of \( \mu \) and \( \text{Clause (3)} \) above it follows that \( \alpha < \alpha' < \mu \). Our goal is to define a sequence \( \overline{T} = \langle T_i : W(p') \rightarrow [\mu]^{<\mu} \mid i \leq \alpha' \rangle \) for which \( b := (p', \overline{T}) \) is a condition satisfying the conclusion of the lemma.

As \( \langle s_\tau \mid \tau < \theta \rangle \) is a good enumeration of the \( n^{\text{th}} \)-level of the \( p' \)-tree \( W(p') \), Fact 2.6 entails that, for each \( q \in W(p') \), there is a unique ordinal \( r_q < \theta \), such that \( q \) is comparable with \( s_{r_q} \). It thus follows from Fact 2.6(3) that, for all \( q \in W(p') \), \( \ell(q) - \ell(p') \geq n \) iff \( q \in W(s_{r_q}) \). Moreover, for each \( q \in W_{>n}(p') \), \( q \leq s_{r_q} \leq g(p_{r_q}, q) \), hence \( w(p_{r_q}, q) \) is well-defined.

Now, for all \( i \leq \alpha' \) and \( q \in W(p') \), let:

\[ T_i(q) := \begin{cases} \left. S_{\min(i, \alpha_{r_q})}^k(w(p_{r_q}, q)) \middle| w(p_{r_q}, q) \right. & \text{if } q \in W(s_{r_q}) \land i \leq \xi_{r_q}; \\ S_{\min(i, \alpha_{r_q})}^k(w(p, q)) & \text{if } q \notin W(s_{r_q}) \land \alpha > 0; \\ \emptyset & \text{otherwise}. \end{cases} \]

\(^{25}\)Note that \( a \) might be \( [p]^{\alpha_\xi} \), so we are allowing \( \alpha = 0 \).
Claim 4.16.2. Let \( i \leq \alpha' \). Then \( T_i \) is a labeled \( p' \)-tree.

Proof. Fix \( q \in W(p') \) and let us go over the Clauses of Definition 4.5. The verification of (1), (2) and (3) are similar to that of [PRS20, Claim 6.16.1], so we just provide details for the new Clause (4).

For each \( i < \alpha' \), set
\[
\xi(i) := \min\{\xi \in [i, \chi) \mid i \leq \alpha_\xi\}.
\]

Subclaim 4.16.2.1. If \( i < \alpha' \), then
\[
m(T_i) \leq n + \max\{\text{mtp}(a), \sup_{t \leq n < \xi(i)} \text{mtp}(g(r_t)), \text{tp}(g(r_{\xi(i)}))(i)\}.
\]

Proof. Let \( q' \leq q \) be in \( W(p') \) with \( q \in W_k(p') \), where
\[
k \geq n + \max\{\text{mtp}(a), \sup_{t \leq n < \xi(i)} \text{mtp}(g(r_t)), \text{tp}(g(r_{\xi(i)}))(i)\}.
\]
Suppose that \( T_i(q') \neq \emptyset \). Denote \( \tau := \tau_{q'} \) and \( \delta := \max(T_i(q')) \). Since \( \ell(q) \geq \ell(p') + n \), note that \( q, q' \in W(s_\tau) \). Also, \( i \leq \xi_\tau \), as otherwise \( T_i(q') = \emptyset \). Therefore, we fall into the first option of the casuistic getting
\[
T_i(q') = S_{\min\{i, \alpha_\xi\}}^\xi(w(p_\xi, q')).
\]

\( \blacktriangleright \) Assume that \( \xi_\tau < \xi(i) \). Then, \( \alpha_\xi_\tau < i \) and so
\[
T_i(q') = S_{\alpha_\xi_\tau}^\xi(w(p_\xi, q')).
\]
We have that \( w(p_\xi, q') \leq w(p_\xi, q) \) is a pair in \( W_{k-n}(p_\xi) \) and that the set \( S_{\alpha_\xi_\tau}^\xi(w(p_\xi, q')) \) is non-empty. Also, \( k - n \geq \text{mtp}(g(r_\xi)) = m(S_{\alpha_\xi_\tau}^\xi) \). So, by Clause (4) for \( S_{\alpha_\xi_\tau}^\xi \), we have that \( (\delta, w(p_\xi, q)) \in R \), and thus \( (\delta, q) \in R \).

\( \blacktriangleright \) Assume that \( \xi(i) \leq \xi_\tau \). Then \( i \leq \alpha_{\xi(i)} \leq \alpha_\xi_\tau \), and thus
\[
T_i(q') = S_{\alpha_\xi_\tau}^\xi(w(p_\xi, q')).
\]

If \( \text{dom}(\text{tp}(a)) \leq i \leq \sup_{t \leq n < \xi(i)} \alpha_\eta \), by Clause (4) above,
\[
\text{tp}(g(r_{\xi(i)}))(i) \leq \text{mtp}(a).
\]
Otherwise, if \( \sup_{t \leq n < \xi(i)} \alpha_\eta < i \leq \alpha_{\xi(i)} \), again by Clause (4) above
\[
\text{tp}(g(r_{\xi(i)}))(i) \leq \max\{\text{mtp}(a), \text{tp}(g(r_{\xi(i)}))(i)\}.
\]
In either case, \( w(p_\xi, q) \in W_{k-n}(p_\xi) \) and \( k - n \geq \text{tp}(g(r_\xi))(i) = m(S_{\alpha_\xi_\tau}^\xi) \). So by Clause (4) of \( S_i^\xi \) we get that \( (\delta, w(p_\xi, q)) \in R \), hence \( (\delta, q) \in R \). \( \square \)

Subclaim 4.16.2.2. \( m(T_{\alpha'}) \leq n + \sup_{t \leq \xi < \chi} \text{mtp}(g(r_\xi)) \).

Proof. Let \( q' \leq q \) be in \( W(p') \) with \( q \in W_k(p') \) and \( k \geq n + \sup_{t \leq \xi < \chi} \text{mtp}(g(r_\xi)) \), and suppose that \( T_{\alpha'}(q') \neq \emptyset \). Denote \( \tau := \tau_{q'} \) and \( \delta := \max(T_{\alpha'}(q')) \).

Since \( k \geq n \), \( q, q' \in W(s_\tau) \). Also, \( i \leq \xi_\tau \), as otherwise \( T_{\alpha'}(q') = \emptyset \). Hence, \( T_{\alpha'}(q') = S_{\alpha_\xi_\tau}^\xi(w(p_\xi, q')). \) Then \( w(p_\xi, q') \leq w(p_\xi, q) \) is a pair in \( W_{k-n}(p_\xi) \) with \( k - n \geq \text{mtp}(g(r_\xi)) = m(S_{\alpha_\xi_\tau}^\xi) \). So, by Definition 4.5 (4) regarded with respect to \( S_{\alpha_\xi_\tau}^\xi \), it follows that \( (\delta, w(p_\xi, q)) \in R \). Thus, \( (\delta, q) \in R \), as wanted. \( \square \)
The combination of the above subclaims yield Clause (4) for \( T_i \).

**Claim 4.16.3.** The sequence \( \vec{T} = \langle T_i : W(p') \to [\mu]^\langle i \leq \alpha' \rangle \rangle \) is a \( p' \)-strategy.

**Proof.** We need to go over the clauses of Definition 4.7. However, Clause (1) is trivial, Clause (2) is established in the preceding claim, and Clauses (3) and (5) follow from the corresponding features of \( \vec{S} \) and the \( \vec{S}_r \)'s. Finally, Clause (4) can be proved similarly to [PRS20, Claim 6.16.2], noting that if \( \alpha > 0 \) then \( \iota = 0 \).

Thus, we have established that \( b := (p', \vec{T}) \) is a legitimate condition in \( A \), such that \( \text{mtp}(b) \leq n + \sup_{\xi < \chi} \text{mtp}(g(r_\xi)) \).

The next claims take care of the second bullet concerning \( b \).

**Claim 4.16.4.** Let \( \tau < \theta \). For each \( q \in W_\alpha(s_\tau) \), \( w(p', q) = w(s_\tau, q) = q \).

**Proof.** The first equality can be proved exactly as in [PRS20, Claim 6.16.4]. For the second, notice that \( q \) and \( w(s_\tau, q) \) are conditions in \( W(s_\tau) \) with the same length. Hence, Fact 2.6(2) yields \( q = w(s_\tau, q) \), as wanted.

**Claim 4.16.5.** \( \pi(b) = p' \) and \( b \vdash 0^\alpha a \).

**Proof.** The proof of this can be found in [PRS20, Claim 6.16.3].

**Claim 4.16.6.** For each \( \tau < \theta \), \( \hat{\eta}(b)(s_\tau) \leq^0 g(r_\xi) \).

**Proof.** Let \( \tau < \theta \) and \( \vec{T}_r \) be denote the \( s_\tau \)-strategy such that \( \hat{\eta}(b)(s_\tau) = (s_\tau, \vec{T}_r) \). By Corollary 4.11 we have that \( \pi(\hat{\eta}(b)(s_\tau)) = s_\tau \leq^0 p_\xi \).

If \( \xi < \iota \), then \( \hat{\eta}(b)(s_\tau) \leq^0 p_\xi \) and we are done.

So, let us assume that \( \iota \leq \xi \). Let \( i \leq \alpha_\xi \) and \( q \in W(s_\tau) \). By Definition 4.9(*), \( T_i^r(q) = T_i(w(p', q)) \) and by one of the preceding claims, \( w(p', q) = w(s_\tau, q) = q \), hence \( T_i^r(q) = T_i(q) \). Also \( r_\xi \tau = w(p, s_\tau) = w(p, s_\tau) = r_\xi \tau \), where the second last equality follows from \( q \in W(s_\tau) \). Therefore,

\[
T_i^r(q) = S_{i, \alpha_\xi}^r(w(p_\xi, q)) = S_i^r(w(p_\xi, q)).
\]

Altogether, \( \hat{\eta}(b)(s_\tau) \leq^0 g(r_\xi) \), as wanted.

The above claims yield the proof of the lemma.

Combining Lemmas 4.11 and 4.16 we arrive at:

**Corollary 4.17.** \( (\hat{\eta}, \pi) \) is a forking projection from \( (A, \ell_A, c_A) \) to \( (P, \ell, c) \) having the weak mixing property.

\[\text{26Recall that } \langle s_\tau | \tau < \theta \rangle \text{ was a good enumeration of } W_\alpha(p')\]
Now we take advantage of this latter corollary to establish that \((A, \ell, A, c, A)\) is \(\Sigma\)-Prikry and that \((A, \ell, A)\) has property \(D\). On this respect, note that the latter statement follows combining Corollary 4.17, Lemma 2.28 and property \(D\) of \((\mathbb{P}, \ell)\) (Setup 4). For the former let us go over the clauses of Definition 2.3: Clauses (1),(3),(4),(5) and (6) follow from lemmas 4.5, 4.7, 4.8 and 4.9 of [PRS20], respectively. Clause (7) follows combining property \(D\) of \((P, \ell)\) with Corollary 4.17 and Corollary 2.29. Also, by [PRS20, Corollary 4.13], \(1 \forces_{\mathbb{P}} \check{\mu} = \kappa^+\). Finally, note that Clause (2) follows from Lemma 2.30 together with Corollary 4.17 and Fact 4.14.

Altogether, we arrive at the main result of this section:

**Corollary 4.18.** Suppose:

(i) \((P, \ell, c)\) is a \(\Sigma\)-Prikry notion of forcing such that the pair \((P, \ell)\) has property \(D\);
(ii) \(1 \forces_{\mathbb{P}} \check{\mu} = \kappa^+\);
(iii) \(\mathbb{P} = (P, \leq)\) is a subset of \(H_{\mu^+}\);
(iv) \(r^* \in P\) forces that \(z\) is a \(\mathbb{P}\)-name for a stationary subset of \((E^V_{\mu^+})^V\) if it does not reflect in \(\{\alpha < \mu \mid \omega < cf^V(\alpha) < \kappa\}\), or \(z\) is a \(\mathbb{P}\)-name for a nonstationary subset of \(\mu\) in \(V\).

Then, there exists a \(\Sigma\)-Prikry triple \((A, \ell, A, c)\) such that \((A, \ell, A)\) has property \(D\) and for which the following are true:

1. \((A, \ell, A, c)\) admits a forking projection \((\check{\eta}, \pi)\) to \((\mathbb{P}, \ell, c)\) that has the weak mixing property;
2. for each \(n < \omega\), \(A^\mathbb{P}_n\) is \(\mu\)-directed-closed;
3. \(1 \forces_{\mathbb{A}} \check{\mu} = \kappa^+\);
4. \(A = (A, \subseteq)\) is a subset of \(H_{\mu^+}\);
5. \([r^*]^A\) forces that \(z\) is nonstationary.

**Proof.** Item (1) and the assertion that \((A, \ell, A, c)\) is \(\Sigma\)-Prikry and that \((A, \ell, A)\) has property \(D\) follow from our previous arguments. Item (2) follows from Fact 4.14 and items (3), (4) and (5) already appeared in Fact 4.10. □

### 4.2. Connecting the dots

For the rest of this section, we make the following assumptions:

- \(\Sigma = \langle \kappa_n \mid n < \omega \rangle\) is an increasing sequence of Laver-indestructible supercompact cardinals;
- \(\kappa := \sup_{n < \omega} \kappa_n\), \(\mu := \kappa^+\) and \(\lambda := \kappa^{++}\);
- \(2^\kappa = \kappa^+\) and \(2^\mu = \mu^+\);
- \(\Gamma := \{\alpha < \mu \mid \omega < cf^V(\alpha) < \kappa\}\).

Under these assumptions, [PRS20, Corollary 5.11] reads as follows:

**Fact 4.19.** If \((\mathbb{P}, \ell, c)\) is a \(\Sigma\)-Prikry notion of forcing such that \(1 \forces_{\mathbb{P}} \check{\mu} = \kappa^+\), then \(V^\mathbb{P} \models \text{Refl}(\langle \omega, \Gamma \rangle)\).

We now want to appeal to the iteration scheme of the previous section. For this, we need to introduce our three building blocks of choice.
**Building Block I.** Let $Q$ be the Extender Based Prikry Forcing (EBPF) for blowing up $2^\kappa$ to $\kappa^{++}$. By results in [Pov20, Ch. 10, §2.5], the EBPF can be regarded as a $\Sigma$-Prikry triple $(Q, \ell, c)$ and $(Q, \ell)$ has property $D$. Also, $Q$ is a subset of $H_\mu^+$ and $I_Q \Vdash \kappa = \kappa^+$. In addition, $\kappa$ is singular, so that $I_Q \Vdash \kappa$ is singular”.

**Building Block II.** For every $\Sigma$-Prikry triple $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ having property $D$ such that $\mathbb{P} = (P, \leq)$ is a subset of $H_\mu^+$ and $I_\mathbb{P} \Vdash \mu = \kappa^+$, every $r^* \in P$, and every $\mathbb{P}$-name $z \in H_\mu^+$, we are given a corresponding $\Sigma$-Prikry triple $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ having property $D$ such that:

- (a) $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ admits a forking projection $(\mathcal{A}, \pi)$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ that has the weak mixing property;
- (b) for each $n < \omega$, $\mathcal{A}^n_\mathbb{A}$ is $\kappa_n$-directed-closed;\(^{27}\)
- (c) $I_\mathbb{A} \Vdash A = \kappa^+$;
- (d) $\mathbb{A} = (\mathbb{A}, \leq_\mathbb{A})$ is a subset of $H_\mu^+$;
- (e) each element of $A$ is a pair $(x, y)$ with $\pi(x, y) = x$;
- (f) if $r^* \in P$ forces that $z$ is a $\mathbb{P}$-name for a stationary subset of $(E_\mu^\mathbb{P})^V$ that does not reflect in $\Gamma$, then

$$[r^*]_\mathbb{A} \Vdash \text{ “ } z \text{ is nonstationary” }.$$  

**Remark 4.20.** The above block is obtained as follows:

- If $r^* \in P$ forces that $z$ is a $\mathbb{P}$-name for a stationary subset of $(E_\mu^\mathbb{P})^V$ that does not reflect in $\Gamma$, then we invoke Corollary 4.18.
- Otherwise, appeal to Corollary 4.18 but this time regarding Clause (iv) with respect to the pair $(I_\mathbb{P}, \emptyset)$.

**Building Block III.** As $2^\mu = \mu^+$, we fix a surjection $\psi : \mu^+ \rightarrow H_\mu^+$ such that the preimage of any singleton is cofinal in $\mu^+$.

We would like now to appeal to the iteration scheme of Section 3 with these building blocks. However, Lemma 3.13 bears on the extra assumption that for each nonzero limit ordinal $\alpha \leq \mu^+$ and $n < \omega$, $(\mathbb{P}_n)$ is dense in $(\mathbb{P}_\alpha)_n$. Our next task will be checking that the iteration defined using the previous building blocks has this feature. Once we are done we will prove Theorem 4.24, which yields the very first application of our iteration scheme.

Fix $\alpha \in \text{acc}(\mu^++1)$. Let $2 \leq \gamma < \alpha$ and assume that the triple $(\mathbb{P}_\gamma, \ell_\mathbb{P}, c_\mathbb{P})$ satisfies all requirements in Building Block II. Let $r^*_\gamma \in P_\gamma$ and $z_\gamma$ a $\mathbb{P}_\gamma$-name such that $(r^*_\gamma, z_\gamma, \mathbb{P}_\gamma)$ witness together Clause (f) of Building Block II.

By replacing $z_\gamma$ for a more appropriate $\mathbb{P}_\gamma$-name $z^*_\gamma$ we may assume that $(\emptyset, z^*_\gamma, \mathbb{P}_\gamma)$ witness together Clause (f) of Building Block II.\(^{28}\) Thus, when

\(^{27}\)Recall Footnote 11 in Page 15.

\(^{28}\)Let $\sigma_\gamma$ be a $\mathbb{P}_\gamma$-nice name for a subset of $\mu$ such that $\emptyset, \|_{\mathbb{P}_\gamma} \sigma_\gamma = z_\gamma$. In particular, $r^*_\gamma \|_{\mathbb{P}_\gamma} \sigma_\gamma$ is a stationary subset of $(E_\mu^\mathbb{P})^V$ that does not reflect in $\Gamma^\mathbb{P}$. Setting $\gamma^* := \{(a, p) \in \sigma_\gamma | p \text{ is compatible with } r^*_\gamma\}$ we infer that $(\emptyset, z^*_\gamma, \mathbb{P}_\gamma)$ is as desired.
invoking Building Block II we will assume that \((r^*_γ, z_γ)\) is of the form \((∅, z_γ^*)\). Let \(\{C^n_α \mid n < ω\}\) and \(R_γ\) be as defined in Setup 4 with respect to \((∅, z_γ^*)\).

**Definition 4.21.** Let \(a ∈ P_α\). For each \(β + 1 ∈ B_a\) and \(q ∈ W(π_α,β(a))\), set
\[
σ^{β+1}(π_α,β+1(a), q) := \max(Sγ(q)),
\]
where \(π_α,β+1(a) := π_α,β(\overrightarrow{S})\) and \(\overrightarrow{S} = (S_i \mid i ≤ γ)\). Denote
\[
σ(a) := \sup\{σ^{β+1}(π_α,β+1(a), q) \mid β + 1 ∈ B_a \& q ∈ W(π_α,β(a))\}.
\]

**Lemma 4.22.** Let \(a ∈ P_α\) and \(β + 1 ∈ B_a\).

1. The map \(σ^{β+1}(π_α,β+1(a), \cdot): W(π_α,β(a)) → μ\) is order-preserving;
2. For each \(r, s ∈ W(β)\),
\[
σ^{β+1}(π_α,β, r) = σ^{β+1}(π_α,β, s);\]
3. For each \(b ≤ β + 1 π_α,β+1(a)\) and \(q ∈ W(b)\),
\[
σ^{β+1}(π_α,β+1(a), w(π_α,β(a), q)) ≤ σ^{β+1}(b, q).
\]

**Proof.** (1) By Definition 4.5(2).
(2) By Definition 4.9(*).
(3) By Definition 4.7(3) and Definition 4.9. \(\square\)

**Lemma 4.23.** For all \(α ∈ [2, μ^+]\), \(n < ω\), and \(ε < μ\),
\[
D^ε_{α,n} := \{b ∈ (P_α)n \mid ∀β + 1 ∈ B_b [ε < σ^{β+1}(π_α,β+1(b), π_α,β(b))]\}
\]
is dense in \((P_α)n\).

In particular, for all \(α ∈ [2, μ^+]\) and \(n < ω\), \((P_α)n\) is dense in \((P_α)n\).

**Proof.** We proceed by induction on \(α ≥ 2\). The base case \(α = 2\) can be derived from Claim 4.15.1.\(^{29}\) So, let us suppose that we are given \(α ∈ [2, μ^+]\) such that for all \(β ∈ [2, α)\), \(n < ω\) and \(ε < μ\), \(D^ε_{β,n}\) is dense in \((P_β)n\).

**Case 1:** Suppose that \(α = β + 1\) is a successor ordinal. Let \(a ∈ P_α\) and \(ε < μ\). Fix \(S\) such that \(a = (a \upharpoonright β) \overrightarrow{S}\). Appealing to Claim 4.15.1, we find \(a' ≤_β (a \upharpoonright β)\) and \(ε > max\{ε, σ(a)\}\) with \((a', ε) ∈ R_β\). By the inductive hypothesis, we may now pick \(b' ∈ D^ε_{β,a'(a)}\) extending \(a'\). Note that \((b', ε) ∈ R_β\), as well.

If \(\overrightarrow{S} = ∅\) then it is not hard to check that \(b := [b']^{P_α}\) is as desired. So, suppose \(\overrightarrow{S} ≠ ∅\) and set \(b := b'\overrightarrow{S}\) \((S_i^\prime \mid i ≤ dom(\overrightarrow{S}) + 1)\), where for each \(q ∈ W(b')\), \(S_i^\prime(q)\) is defined as follows:
\[
S_i^\prime(q) := \begin{cases} S_i(w(a \upharpoonright β, q)), & \text{if } i ≤ dom(\overrightarrow{S}); \\ S_i(w(a \upharpoonright β, q) \cup \{ε\}), & \text{otherwise}. \end{cases}
\]

By our choice of \(ε\), it follows that \(b ∈ P_α\) and that \(b ≤_α a\). Moreover, \(ntp_{β+1}(b) = 0\). Thus, since \(b' ∈ P_β\), it follows that \(b ∈ P_α\). Finally, since \(B_b = B_β \cup \{α\}\), our choice of \(b'\) implies that \(b\) is in \(D^ε_{α,a,α}(a)\).

\(^{29}\)For more details, see the discussion of the successor step below.
Case 2: Suppose that \( \text{cf}(\alpha) > \kappa \). Let \( a \in P_\alpha \) and \( \epsilon < \mu \). Then \( B_\alpha \) is bounded in \( \alpha \). Fix \( \gamma < \alpha \) such that \( a = (a \upharpoonright \gamma) \star \emptyset_\alpha \). By the inductive hypothesis, we find \( a' \in D^\epsilon_{\gamma, \text{cf}(a)} \) extending \( a \upharpoonright \gamma \). Set \( b := a' \star \emptyset_\alpha \), so that \( B_b = B_{a'} \). Then \( b \in D^\epsilon_{\alpha, \text{cf}(a)} \) extends \( a \), as desired.

Case 3: Suppose that \( 1 < \text{cf}(\alpha) \leq \kappa \). As \( \kappa \) is the limit of the strictly increasing sequence \( \langle \kappa_n \mid n < \omega \rangle \) (see Building Block I), we may let \( n < \omega \) be the least such that \( \text{cf}(\alpha) < \kappa_n \).

Claim 4.23.1. For all \( l \geq n \) and \( \epsilon < \mu \), \( D^\epsilon_{\alpha,l} \) is dense in \( (P_\alpha) \).

Proof. Let \( a \in P_\alpha \) and \( \epsilon < \mu \) such that \( l := \ell_\alpha(a) \) is greater or equal to \( n \). By the proof of Case 2, we may assume that \( B_\alpha \) is unbounded in \( \alpha \). Let \( \langle \gamma_\tau \mid \tau < \text{cf}(\alpha) \rangle \) be the increasing enumeration of some cofinal subsets of \( B_\alpha \) of size \( \text{cf}(\alpha) \). For every \( \tau < \text{cf}(\alpha) \), \( \gamma_\tau \) is a successor ordinal, so we let \( \beta_\tau \) denote its predecessor. We shall construct a sequence \( \langle b_\tau \mid \tau < \text{cf}(\alpha) \rangle \) such that, for each \( \tau < \text{cf}(\alpha) \), the following hold:

1. \( b_\tau \in \bar{P}_{\beta_\tau} \) and \( b_\tau \leq_{\gamma_\tau} a \upharpoonright \gamma_\tau \);
2. if \( \tau \in \text{nacc}(\text{cf}(\alpha)) \), then for all \( \beta + 1 \in B_{b_\tau} \), \( \epsilon < \sigma^{\beta+1}(b_\tau, b_\tau \upharpoonright \beta_\tau) \);
3. \( b_\tau \upharpoonright \gamma_\eta \leq_{\gamma_\eta} b_\eta \) for all \( \eta \leq \tau \).

Let \( \tilde{S} \) be such that \( a \upharpoonright \gamma_0 = (a \upharpoonright \beta_0) \smallsetminus \tilde{S} \). By Claim 4.15.1, we may find \( b_0 \leq_{\beta_0} (a \upharpoonright \beta_0) \) and \( \varepsilon > \max\{\epsilon, \sigma(a)\} \) such that \( (b_0, \varepsilon) \in R_{\beta_0} \). Also, by the inductive assumption, we may assume that \( b_0 \in \tilde{P}_{\beta_0} \). Now set

\[
\tilde{b}_0 := b_0 \smallsetminus \langle S_i \mid i \leq \text{dom}(\tilde{S}) + 1 \rangle
\]

where the latter is defined as in Case 1.

Once again, \( \tilde{S} \) is a \( b_0 \)-strategy, and so \( b_0 \leq_{\gamma_0} (a \upharpoonright \gamma_0) \). Also, thanks to our choice of \( \varepsilon \), \( \text{mtp}_{\beta_0+1}(b_0) = 0 \), hence \( b_0 \in \tilde{P}_{\beta_0+1} \). Note also that \( \sigma^{\beta_0+1}(b_0, b_0) = \varepsilon > \varepsilon \), so that (2) holds as well. Finally, (3) is obvious at this stage.

Next, let us assume that \( \tau < \text{cf}(\alpha) \) and we have already successfully defined \( \langle b_\eta \mid \eta < \tau \rangle \). Then we distinguish the following two cases:

- If \( \tau = \eta + 1 \), then set \( c := \tilde{n}_{\gamma_\tau, \gamma_\eta}(a \upharpoonright \gamma_\tau)(b_\eta) \). By Clause (1), \( c \in P_{\gamma_\tau} \).

Hence we may appeal to Case 1 above and find \( b_\tau \in \bar{P}_{\gamma_\tau} \) with \( b_\tau \leq_{\gamma_\tau} c \), witnessing (2). Clearly, \( b_\tau \) satisfies (1)–(3).

- Otherwise, Lemma 3.9 and (3) of the inductive assumption imply that \( (b_\eta \star \emptyset_{\gamma_\tau} \mid \eta < \tau) \) is a \( \leq_{\gamma_\tau} \)-decreasing sequence in \( (\bar{P}_{\gamma_\tau}) \). Thus, since \( \tau < \kappa_\ell \), by Lemma 3.13, it admits a lower bound \( b_\tau^* \in (\bar{P}_{\gamma_\tau}) \).

Thereby, a sequence \( \langle b_\tau \mid \tau < \text{cf}(\alpha) \rangle \) witnessing (1)–(3) has been constructed. Repeating the argument of the limit case above we find a lower bound \( b \in (P_\alpha)_0 \) for this sequence. By virtue of Claim 3.13.1 we may assume that \( B_b = \bigcup_{\tau < \text{cf}(\alpha)} B_{b_\tau} \). Clearly, \( b \leq_{\alpha} a \). We claim that \( b \in D^\epsilon_{\alpha,l} \). To see this, fix \( \beta + 1 \in B_b \). Let \( \tau \in \text{nacc}(\text{cf}(\alpha)) \) be such that \( \beta + 1 \in B_{b_\tau} \). Since
Clause (2) above imply \( \epsilon < \sigma \). Let \( n \) be such that we are done with the proof of the lemma. So, let us suppose that \( n \geq 1 \), and let us show how to bring this down to \( \| \).

**Claim 4.23.2.** Let \( l < \omega \) be such that \( \| \) holds. Then \( \| \) holds, as well.

**Proof.** Let \( \epsilon < \mu \) and \( a \in P_\alpha \) such that \( \ell_\alpha(a) = l \). We need to find a condition \( b \in D'^{\alpha}_{\omega,l} \) extending \( a \).

Set \( a_0 := a \) and \( c_0 := \epsilon \). Since \( \| \) holds, \( D'^{\alpha}_{\omega,l+1} \subseteq \langle \| \rangle \) is a dense subset of \( \langle \| \rangle \). Letting \( s \) be a good enumeration of \( W_1(a_0) \) we can play \( \preceq_{\pi}(a_0, s, D^{\alpha}_{\omega,l+1}) \) and produce a sequence \( \langle b_\xi \mid \xi < \chi \rangle \) corresponding to the moves of \( \Pi \). Appealing to Lemma 3.11, there is a condition \( a_1 \) in \( P_\alpha \) which diagonalizes \( \langle b_\xi \mid \xi < \chi \rangle \) and such that, for all \( \beta + 1 \in B_{\alpha,0} \),

\[
\pi_{\alpha,\beta+1}(a_1) \subseteq \pi_{\alpha,\beta+1}(a_0).
\]

Recalling Definition 4.12, this means the following.

(I) For each \( \beta + 1 \in B_{\alpha,0} \), and \( i \leq \operatorname{max}(\operatorname{dom}(S^{\beta,a_1})), \)

\[
S^{\beta,a_1}(a_1 \mid \beta) = S_{\min\{i,a_0\}}^{\beta,a_0}(a_0 \mid \beta),
\]

where for each \( i \in \{0, 1\} \), and \( \beta + 1 \in B_{\alpha,1} \), we set

\[
\pi_{\alpha,\beta+1}(a_i) := (a_1 \mid \beta)^\beta, \quad \tilde{S}^{\beta,a_1} := \langle S^{\beta,a_1} \mid j \leq a_1 \rangle.
\]

By 0-extending \( a_1 \), we may suppose that \( a_1 \models (P_\alpha), \epsilon_1 \in \bigcap_{\beta+1 \in B_{\alpha,0}} \dot{C}^{\beta}, \)

where \( \epsilon_1 \) is some ordinal such that \( \sup_{\epsilon < \chi} \sigma(b_\xi) < \epsilon_1 < \mu \). Therefore:

(II) For \( \beta + 1 \in B_{\alpha,0} \), \( (a_1 \mid \beta) \models (P_\beta), \epsilon_1 \in \dot{C}^{\beta} \). Also, \( \sigma(a_0) < \epsilon_1 \). \( \tag{31} \)

The whole point of diagonalizing conditions in \( D'^{\alpha}_{\omega,l+1} \) is (III) and (IV):

**Subclaim 4.23.2.1.** (III) For each \( \beta + 1 \in B_{\alpha,0} \) and \( q \in W_1(a_1) \),

\[
\epsilon_0 < \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(a_1), q).
\]

**Proof.** Let \( \beta + 1 \) and \( q \) be as in the statement. Defining \( c := \pi_\alpha(\beta)(q) \) we have that \( c \in W_1(a_1) \), and so there is \( \xi < \chi \) such that \( c \leq a_0 \). Hence, \( b_\xi \leq a_0 \), then \( B_{\alpha,0} \subseteq B_{\beta} \), and so \( \beta + 1 \in B_{\beta} \). Therefore,

\[
\epsilon_0 < \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), \pi_{\alpha,\beta}(b_\xi)) \leq \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(c), q),
\]

where the second inequality follows from \( \pi_{\alpha,\beta+1}(c) \leq a_0 \) and \( \pi_{\alpha,\beta}(c) = q \). Also, by Lemma 4.22(3),

\[
\sigma^{\beta+1}(\pi_{\alpha,\beta+1}(c), q) = \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(a_1), q),
\]

\( \tag{30} \)

Since \( a_1 \models (P_\alpha) \) note that \( w(a_0 \mid \beta_0, a_1 \mid \beta_0) = a_0 \mid \beta_0 \).

\( \tag{31} \) Note that the last assertion follows from \( b_\xi \leq a_0 \) and our choice of \( \epsilon_1 \).
which yields the desired inequality.

Subclaim 4.23.2.2.

(IV) For all \( c \in W_{\geq 1}(a) \) and \( \beta + 1 \in B_{a_1} \),
\[ \pi_{\alpha,\beta}(c) \models (P_{\beta+1})_{\alpha(c)} \bigcirc_{\ell_{\alpha(c)}} (d, \ell_1) \neq \emptyset. \]

Proof. Fix \( c \in W_{\geq 1}(a) \) and \( \beta + 1 \in B_{a_1} \). Appealing to Fact 2.6, let \( \tilde{c} \) be the unique condition in \( W_1(a_1) \) such that \( c \leq_{\alpha} \tilde{c} \).

Since \( a_1 \) diagonalizes \( \{ \xi \mid \xi < \chi \} \), there is \( \xi < \chi \) such that \( \tilde{c} \leq_{\alpha} b_\xi \). Recall that \( b_\xi \in \check{P}_\alpha \), hence Lemma 3.9 yields \( \pi_{\alpha,\beta+1}(b_\xi) \in \check{P}_{\beta+1} \). In particular, we have that \( mtp_{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi)) = 0 \) or, equivalently,
\[ (\pi_{\alpha,\beta}(b_\xi), \sigma_{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), q)) \in R_\beta \text{ for all } q \in W(\pi_{\alpha,\beta}(b_\xi)). \]

Since \( \pi_{\alpha,\beta}(c) \leq_{\beta} \pi_{\alpha,\beta}(\tilde{c}) \leq_{\beta} \pi_{\alpha,\beta}(b_\xi) \),
\[ \pi_{\alpha,\beta}(c) \models (P_{\beta+1})_{\alpha(c)} \bigcirc_{\ell_{\alpha(c)}} (d, \ell_1) \in \check{C}_{\ell_{\alpha(c)}}. \]

Finally, as \( \sigma_{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), \pi_{\alpha,\beta}(b_\xi)) \in (\epsilon_0, \epsilon_1) \), the claim follows. \( \square \)

Repeating the above argument \( \omega \)-many times we obtain a \( \leq_\alpha \)-decreasing sequence \( \langle a_n \mid n < \omega \rangle \) and an increasing sequence of ordinals \( \langle \epsilon_n \mid n < \omega \rangle \) such that, for each \( n < \omega \), \( \langle a_n, a_{n+1}, \epsilon_n, \epsilon_{n+1} \rangle \) witnesses together (I), (II), (III) and (IV). Note that any \( \leq_\alpha \)-lower bound \( b \in P_\alpha \) for this sequence would give a condition in \( D'_{\alpha,l} \) extending \( a \).

Subclaim 4.23.2.3. There is \( b \in \check{P}_\alpha \) such that \( b \leq_{\alpha} a_n \) for all \( n < \omega \).

In particular, there is \( b \in D'_{\alpha,l} \) such that \( b \leq_\alpha a \).

Proof. Fix an increasing enumeration \( \langle b_\tau + 1 \mid \tau < \theta \rangle \) of \( \bigcup_{n<\omega} B_{a_n} \). The condition \( b \) will be defined as the limit of a sequence \( \langle b_\tau \mid \tau < \theta \rangle \) such that
\begin{enumerate}
  \item \( b_\tau \in (\check{P}_{\beta_{\tau}+1}) \text{ and } B_{b_\tau} = \{ b_\theta + 1 \mid \theta \leq \tau \}; \)
  \item \( b_\tau \leq_{\beta_{\tau}+1} \pi_{\alpha,\beta_{\tau}+1}(a_n) \text{ for all } n < \omega; \)
  \item \( \pi_{\beta_{\tau}+1}(b_\tau) = b_0 \text{ for all } \tau \leq \theta. \)
\end{enumerate}

We construct this sequence by recursion on \( \tau < \theta \).

\begin{itemize}
  \item Suppose \( \tau = 0 \). Since \( \langle \pi_{\alpha,1}(a_n) \mid n < \omega \rangle \) is a decreasing sequence in \( (\check{P}_1)_l = (\check{P}_1)_l \), we can appeal to Clause (2) of Definition 2.3 and let \( p \in (P_1)_l \) be a lower bound for it.

  For each \( n < \omega \), set
  \[ c_n := \ominus_{\beta_0,\beta_0}(\pi_{\alpha,\gamma_0}(a_n))([p]^{\check{P}_0}). \]

  For each \( n < \omega \), \( [p]^{\check{P}_0} \leq \beta_0 \omega_{\beta_0} = \pi_{\alpha,\beta_0}(a_n) \), hence \( c_n \in (P_{\beta_0})_{\alpha(n)} \). Actually, \( \tilde{c} := \langle c_n \mid n < \omega \rangle \) is \( \leq_{\gamma_0} \)-decreasing, hence, as in the proof of [PRS20, Lemma 6.15], \( (\tilde{c}, \leq_{\gamma_0}) \) is order-isomorphic to \( (\omega, \exists) \).
\end{itemize}

\( \square \)

\( \square \) See Definition 4.5(4) and Definition 4.13.

\( \square \) Recall that \( \epsilon_1 > \sup_{\xi < \chi} \sigma(b_\xi) \).
Let \( n_0 < \omega \) denote the least such that \( \gamma_0 \in B_{\alpha_n} \) for each \( n \geq n_0 \). For each \( n \geq n_0 \), set \( c_n := [p]^B_{\alpha_n} \cap S^n \). By Clause (5) of Definition 2.23
\[
\alpha_n = \max(\text{dom}(S^{3\alpha_n,a_n})) = \max(\text{dom}(S^n))\).
\]
Set \( \alpha := \sup_{n<\omega} \alpha_n \). Note that for every \( n \geq n_0 \) and \( i \leq \alpha_n \) we have
\[
S^n_i([p]^B_{\alpha_n}) = S^{\alpha_n}(w(\pi_{\alpha_n}(a_n), [p]^B_{\alpha_n})) = S^{\alpha_n}(\pi_{\alpha_n}(a_n)),
\]
where the right-most equality follows from \([p]^B_{\alpha_n} \leq \alpha_n \pi_{\alpha_n}(a_n)\).

Combining this with Clause (I) we have
\[
(*) \quad S^n_i([p]^B_{\alpha_n}) = S^{\alpha_n}(\pi_{\alpha_n}(a_n)).
\]

Let us now define a \( P_{\gamma_0} \)-lower bound for \( (c_n \mid n < \omega) \).

Let \( b_0 := [p]^B_{\alpha_n} \cap S \), where \( S := (S_i \mid i \leq \alpha) \) is the sequence defined according to the following casuistic:

\begin{itemize}
  \item For \( i < \alpha \), \( S_i(q) \) is defined as the unique element of \( \{ S^n_i(q) \mid n \geq n_0, \alpha_n \geq i \} \).
  
  \item At stage \( \alpha \), we distinguish two cases:
    \begin{itemize}
      \item If \( S_i(q) = \emptyset \) for all \( i < \alpha \), then we continue and let \( S_\alpha(q) := \emptyset \).
      \item Otherwise, let
      \[
      (\dagger) \quad S_\alpha(q) := \begin{cases}
        \bigcup_{i<\alpha} S_i(q) \cup \{ \epsilon \}, & \text{if } \ell_\alpha(q) \geq 1, \\
        S^{\alpha_n}(\pi_{\alpha_n}(a_n_0)) \cup \{ \epsilon \}, & \text{otherwise}.
      \end{cases}
      \]
    \end{itemize}
\end{itemize}

**Subsubclaim 4.23.2.3.1.** For each \( q \in W([p]^B_{\alpha_n}) \), \( \max(S_\alpha(q)) = \epsilon \).

**Proof.** In case \( q = [p]^B_{\alpha_n} \) we appeal to Clause (II), noting that \( \sigma(a_{n_0}) < \epsilon_{n_0+1} < \epsilon \). Otherwise, it is enough to show that
\[
\epsilon = \sup\{ \max(S^{n}_{\alpha_n}) \mid S^{n}_{\alpha_n} \neq \emptyset, n_0 \leq n < \omega \},
\]
which follows combining Clauses (II) and (III).

**Subsubclaim 4.23.2.3.2.** \( b_0 \) witnesses (1)--(3).

**Proof.** A moment’s reflection makes it clear that we only need to verify (1).

Considering Fact 4.23.2.3.1 and that \( S^n \) were \( [p]^B_{\alpha_n} \)-strategies, it is clear that we only need to verify that \( S_\alpha \) is a labeled \( [p]^B_{\alpha_n} \)-tree with \( m(S_\alpha) = 0 \). For this it is enough to check Clauses (3) and (4) of Definition 4.5:

\begin{itemize}
  \item (3) To avoid trivialities, assume that \( S_\alpha(q) \neq \emptyset \). We already know that
    \( q \models_{P_{\gamma_0}} S^n_{\alpha_n}(q) \cap \bar{T}^\beta = \emptyset \), for each \( n \geq n_0 \). Thus, to establish the clause it suffices to show that \( (\epsilon, [p]^B_{\alpha_n}) \in R_{\beta_0} \). First note that
    \( [p]^B_{\alpha_n} \leq \beta_0 r_{\beta_0} \). Also, for each \( n < \omega \), \( [p]^B_{\alpha_n} \leq \beta_0 \pi_{\alpha_n}(a_n) \), hence
    Clause (II) yields \( [p]^B_{\alpha_n} \models_{(P_{\alpha_n})t} \epsilon_n \in \mathcal{C}_t^{\beta_0} \), for all \( n \geq n_0 \). Thus,
    \[
    [p]^B_{\alpha_n} \models_{(P_{\alpha_n})t} \epsilon \in \mathcal{C}_t^{\beta_0}.
    \]
\end{itemize}
Thus, \( b_{\beta_0} \) to construct \( \lceil \cdot \rceil \).

Subsubclaim 4.23.2.3.3. where, \( B \) and \( \sum \) to check that for all \( \beta \)
follow automatically from (2) and (3) of our induction hypothesis.

Proof. Note that once we have established the first assertion the second will
yield automatically from (2) and (3) of our induction hypothesis. \( \square \)

For (1) and (2) one proceeds exactly as in Subsubclaims 4.23.2.3.1
and 4.23.2.3.2, complementing the argument with Subsubclaim 4.23.2.3.3
above. Finally, Clause (3) follows from our induction hypothesis. \( \square \)

The above completes the inductive construction of a sequence \( \langle b_\tau \mid \tau < \theta \rangle \)
witnessing (1)-(3) above. Set \( b := (\bigcup_{\tau < \theta} b_\tau) \ast \emptyset_a \).
Subsubclaim 4.23.2.3.5. \( b \in \mathcal{P}_\alpha \) and \( b \leq_\alpha^0 a_\alpha \) for all \( n < \omega \).

Proof. The latter assertion is an outright consequence of (1) and (2). As for the former, let \( \beta + 1 \in B_b \). By Clause (1), \( B_b = \{ \beta_\tau : \tau < \theta \} \), hence \( \beta + 1 \in B_{\beta_\tau} \) for some \( \tau < \theta \). Combining (1) and (3) we have

\[
mtp_{\beta+1}(\pi_{\alpha,\beta+1}(b)) = mtp_{\beta+1}(\pi_{\beta+1,\beta+1}(b_\tau)) = 0.
\]

Thus, \( b \in \mathcal{P}_\alpha \), as wanted. \( \square \)

The above fact completes the proof of the subclaim. \( \square \)

Subclaim 4.23.2.3 yields a condition \( b \in D^\alpha_{\alpha,l} \) such that \( b \leq_\alpha^0 a \). Thereby, for every \( \epsilon < \mu \) the set \( D^\epsilon_{\alpha,l} \) is dense in \( (\mathbb{P}_\alpha)_l \), and thus \( \dag \) holds. \( \square \)

Appealing to Claim 4.23 iteratively we have that \( \dag \) holds. Namely, the moreover part of the lemma is satisfied. This completes the proof. \( \square \)

Thanks to Lemma 4.23 we can now appeal to the iteration scheme of Section 3 with respect to the building blocks of this section and obtain, in return, a \( \Sigma \)-Prikry triple \( (\mathbb{P}_\mu^+, \ell_\mu^+, c_\mu^+) \).

Theorem 4.24. In \( V^{P_{\mu^+}} \) all of the following hold true:

1. Any cardinal in \( V \) remains a cardinal and retains its cofinality;
2. \( \kappa \) is a singular strong limit of countable cofinality;
3. \( 2^\kappa = \kappa^{++} \);
4. \( \text{Refl}(\omega, \kappa^+) \).

Proof. (1) By Fact 2.7(1), no cardinal \( \leq \kappa \) changes its cofinality; by Fact 2.7(3), \( \kappa^+ \) is not collapsed, and by Definition 2.3(3), no cardinal \( \kappa^+ \) changes its cofinality.

(2) In \( V \), \( \kappa \) is a singular strong limit of countable cofinality, and so by Fact 2.7(1), this remains valid in \( V^{P_{\mu^+}} \).

(3) In \( V \), we have that \( 2^\kappa = \kappa^+ \). In addition, by Remark 3.3(1), \( P_{\mu^+} \) is isomorphic to a subset of \( H_{\mu^+} \), so that, from \( |H_{\mu^+}| = \kappa^{++} \), we infer that \( V^{P_{\mu^+}} \models 2^\kappa \leq \kappa^{++} \). Finally, as \( P_{\mu^+} \) projects to \( P_1 \) which is isomorphic to \( Q \), we get that \( V^{P_{\mu^+}} \models 2^\kappa \geq \kappa^{++} \). Altogether, \( V^{P_{\mu^+}} \models 2^\kappa = \kappa^{++} \).

(4) As \( \kappa^+ = \mu \) and \( \kappa \) is singular, \( \text{Refl}(\omega, \kappa^+) \) is equivalent to \( \text{Refl}(\omega, E^{\mu}_{\leq \kappa}) \). By Fact 4.19, we already know that \( V^{P_{\mu^+}} \models \text{Refl}(\omega, \Gamma) \). So, by Proposition 4.3, it suffices to verify that \( \text{Refl}(2, (E^{\mu}_{\omega})^V, \Gamma) \) holds in \( V^{P_{\mu^+}} \).

Let \( G \) be \( P_{\mu^+} \)-generic over \( V \) and hereafter work within \( V[G] \). Towards a contradiction, suppose that there exists a subset \( T \) of \( (E^{\mu}_{\omega})^V \) that does not reflect in \( \Gamma \). Fix \( r^* \in G \) and a \( P_{\mu^+} \)-name \( \tau \) such that \( \tau_G \) is equal to such a \( T \) and such that \( r^* \) forces \( \tau \) to be a stationary subset of \( (E^{\mu}_{\omega})^V \) that does not reflect in \( \Gamma \). Furthermore, we may require that \( \tau \) be a nice name, i.e., each element of \( \tau \) is a pair \( (\xi, p) \) where \( (\xi, p) \in (E^{\mu}_{\omega})^V \times P_{\mu^+} \), and, for all \( \xi \in (E^{\mu}_{\omega})^V \), the set \( \{ p \mid (\xi, p) \in \tau \} \) is an antichain.
As $\mathbb{P}_{\mu^+}$ satisfies Clause (3) of Definition 2.3, $\mathbb{P}_{\mu^+}$ has the $\mu^+$-cc. Consequently, there exists a large enough $\beta < \mu^+$ such that

$$B_{r^*} \cup \bigcup \{ B_\mu \mid (\xi, p) \in \tau \} \subseteq \beta.$$ 

Let $r := r^* \upharpoonright \beta$ and set

$$\sigma := \{ (\xi, p \upharpoonright \beta) \mid (\xi, p) \in \tau \}.$$ 

From the choice of Building Block III, we may find a large enough $\alpha < \mu^+$ with $\alpha > \beta$ such that $\psi(\alpha) = (\beta, r, \sigma)$. As $\beta < \alpha$, $r \in P_\beta$ and $\sigma$ is a $\mathbb{P}_\beta$-name, the definition of our iteration at step $\alpha + 1$ involves appealing to Building Block II with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$, $r^* := r \upharpoonright \emptyset_\alpha$ and $z := i_\beta^\emptyset(\sigma)$. For any ordinal $\eta < \mu^+$, denote $G_\eta := \pi_{\mu^+, \eta}[G]$. By the choice of $\beta$, and as $\alpha > \beta$, we have

$$\tau = \{ ((\xi, p \upharpoonright \emptyset_{\mu^+}) \mid (\xi, p) \in \sigma \} = \{ ((\xi, p \upharpoonright \emptyset_{\mu^+}) \mid (\xi, p) \in z \},$$

so that, in $V[G]$,

$$T = \tau_G = \sigma_{G_{\beta}} = z_{G_\alpha}.$$ 

In addition, $r^* = r^* \upharpoonright \emptyset_{\mu^+}$.

Finally, as $r^*$ forces $\tau$ is a stationary subset of $(E_{\omega_1}^\mu)^V$ that does not reflect in $\Gamma$, $r^*$ forces that $z$ is a stationary subset of $(E_{\omega_1}^\mu)^V$ that does not reflect in $\Gamma$. So, since $\pi_{\mu^+, \alpha + 1}(r^*) = r^* \upharpoonright \emptyset_{\alpha + 1} = [r^*]^{\mathcal{P}_{\alpha + 1}}$ is in $G_{\alpha + 1}$, Clause (f) of Building Block II entails that, in $V[G_{\alpha + 1}]$, there exists a club in $\mu$ which is disjoint from $T$. In particular, $T$ is nonstationary in $V[G]$, contradicting its very choice. \hfill \Box

Thus, we arrive at the following strengthening of the theorem announced by Sharon in [Sha05].

**Corollary 4.25.** Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, converging to a cardinal $\kappa$. Then there exists a forcing extension where the following properties hold:

1. $\kappa$ is a singular strong limit cardinal of countable cofinality;
2. $2^\kappa = \kappa^{++}$, hence SCH$_\kappa$ fails;
3. Refl$(<\omega, \kappa^+)$ holds.

**Proof.** Let $\mathbb{L}$ be the inverse limit of the iteration $(\mathbb{L}_n : \check{Q}_n \mid n < \omega)$, where $\mathbb{L}_0$ is the trivial forcing and for positive integer $n$, if $\mathbb{L}_n \models \kappa_{n-1}$ is supercompact, then $\mathbb{L}_n \models \kappa_{n}$ is a Laver preparation for $\kappa_n$ above $\kappa_{n-1}$”. After forcing with $\mathbb{L}$, each $\kappa_n$ remains supercompact and, moreover, becomes indestructible under $\kappa_n$-directed-closed forcing. Also, the cardinals and cofinalities of interest are preserved.

Working in $V^\mathbb{L}$, set $\mu := \kappa^+$, $\lambda := \kappa^{++}$ and $C := \text{Add}(\lambda, 1)$. Finally, work in $W := V^{L[\mathbb{C}]}$. Since $\kappa$ is singular strong limit of cofinality $\omega < \kappa_0$ and $\kappa_0$ is supercompact, $2^\kappa = \kappa^+$. Also, thanks to the forcing $C$, $2^\mu = \mu^+$. Altogether, in $W$, all the following hold:

34Recall that by Fact 4.2, the extent of reflection obtained is optimal.
\( \langle \kappa_n \mid n < \omega \rangle \) is an increasing sequence of Laver-Indestructible supercompact cardinals;

\( \kappa := \sup_{n<\omega} \kappa_n, \mu := \kappa^+ \) and \( \lambda := \kappa^{++} \);

\( 2^\kappa = \kappa^+ \) and \( 2^\mu = \mu^+ \);

Now, appeal to Theorem 4.24.

\[\square\]

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