SIGMA-PRIKRY FORCING II: ITERATION SCHEME

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Abstract. In Part I of this series [PRS20], we introduced a class of notions of forcing which we call \( \Sigma \)-Prikry, and showed that many of the known Prikry-type notions of forcing that centers around singular cardinals of countable cofinality are \( \Sigma \)-Prikry. We proved that given a \( \Sigma \)-Prikry poset \( \mathbb{P} \) and a \( \mathbb{P} \)-name for a non-reflecting stationary set \( T \), there exists a corresponding \( \Sigma \)-Prikry poset that projects to \( \mathbb{P} \) and kills the stationarity of \( T \). In this paper, we develop a general scheme for iterating \( \Sigma \)-Prikry posets, as well as verify that the Extender Based Prikry Forcing is \( \Sigma \)-Prikry. As an application, we blow up the power of a countable limit of Laver-indestructible supercompact cardinals, and then iteratively kill all non-reflecting stationary subsets of its successor. This yields a model in which the singular cardinal hypothesis fails and simultaneous reflection of finite families of stationary sets holds.

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1. Introduction

In the introduction to Part I of this series [PRS20], we described the need for iteration schemes and the challenges involved in devising such schemes, especially at the level of successor of singular cardinals. The main tool available to obtain consistency results at the level of singular cardinals and their successors is the method of forcing with large cardinals and, in particular, Prikry-type forcings. By Prikry-type forcings one usually means to a poset $\mathbb{P} = (P, \leq)$ having the following property.

**Prikry Property.** There exists an ordering $\leq^*$ on $P$ coarser than $\leq$ (typically, of a better closure degree) satisfying that for every sentence $\varphi$ in the forcing language and every $p \in P$ there exists $q \in P$ with $q \leq^* p$ deciding $\varphi$.

In this paper, we develop an iteration scheme for Prikry-type posets, specifically, for the class of $\Sigma$-Prikry forcings that we introduced in [PRS20] (see Definition 2.3 below). Of course, viable iteration schemes for Prikry-type posets already exists, namely, the Magidor iteration and the Gitik iteration (see [Git10, §6]). In both these cases the ordering $\leq^*$ witnessing the Prikry Property of the iteration can be roughly described as the finite-support iteration of the $\leq^*$-orderings of its components. As the expectation from the final $\leq^*$ is to have an eventually-high closure degree, the two schemes are typically useful in the context where one carries an iteration $\langle P_\alpha; \dot{Q}_\alpha | \alpha < \rho \rangle$ with each $\dot{Q}_\alpha$ being a $P_\alpha$-name for either a trivial forcing, or a Prikry-type forcing concentrating on the combinatorics of the inaccessible cardinal $\alpha$. This should be compared with the iteration to control the power function $\alpha \mapsto 2^\alpha$ below some cardinal $\rho$.

In contrast, in this paper, we are interested in carrying out an iteration of length $\kappa^{++}$, where $\kappa$ is a singular cardinal (or, more generally, forced by the first step of the iteration to become one), and all components of the iteration are Prikry-type forcings that concentrate on the combinatorics of $\kappa$ or its successor. For this, we will need to allow a support of arbitrarily large size below $\kappa$. To be able to lift the Prikry property through an infinite-support iteration, members of the $\Sigma$-Prikry class are thus required to possess the following stronger property, which is inspired by the concepts coming from the study of topological Ramsey spaces [Tod10].

**Complete Prikry Property.** There is a partition of the ordering $\leq$ into countably many relations $\langle \leq^n | n < \omega \rangle$ such that, if we denote $\text{cone}_n(q) := \{r | r \leq^n q\}$, then, for every 0-open $U \subseteq P$ (i.e., $q \in U \implies \text{cone}_0(q) \subseteq U$), every $p \in P$ and every $n < \omega$, there exists $q \leq^* p$ such that $\text{cone}_n(q)$ is either a subset of $U$ or disjoint from $U$.

Another parameter that requires attention when devising an iteration scheme is the chain condition of the components to be used. In view of the goal of solving a problem concerning the combinatorics of $\kappa$ or its successor through an iteration of length $\kappa^{++}$, there is a need to know that all counterexamples to our problem will show up at some intermediate stage of the
iteration, so that we at least have the chance to kill them all. The standard way to secure the latter is to require that the whole iteration $\mathbb{P}_{\kappa^+}$ would have the $\kappa^++$-chain condition ($\kappa^+-cc$). As the $\kappa$-support iteration of $\kappa^+-cc$ posets need not have the $\kappa^+-cc$ (see [Ros18] for an explicit counterexample), members of the $\Sigma$-Prikry class are required to satisfy the following strong form of the $\kappa^+-cc$:

**Linked\(_0\) Property.** There exists a map $c : P \to \kappa^+$ satisfying that for all $p, q \in P$, if $c(p) = c(q)$, then $p$ and $q$ are compatible, and, furthermore, $\text{cone}_0(p) \cap \text{cone}_0(q)$ is nonempty.

In particular, our verification of the chain condition of $\mathbb{P}_{\kappa^+}$ will not go through the $\Delta$-system lemma; rather, we will take advantage of a basic fact concerning the density of box products of topological spaces.

Now that we have a way to ensure that all counterexamples show up at intermediate stages, we fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^+ \rangle$, and shall want that, for any $\alpha < \kappa^+$, $\mathbb{P}_{\alpha+1}$ will amount to forcing over the model $V^{\mathbb{P}_\alpha}$ to solve a problem suggested by $z_\alpha$. The standard approach to achieve this is to set $\mathbb{P}_{\alpha+1} := \mathbb{P}_\alpha * \mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha$ is a $\mathbb{P}_\alpha$-name for a poset that takes care of $z_\alpha$. However, the disadvantage of this approach is that if $\mathbb{P}_1$ is a notion of forcing that blows up $2^{\kappa^+}$, then any typical poset $\mathbb{Q}_1$ in $V^{\mathbb{P}_1}$, which is designed to add a subset of $\kappa^+$ via bounded approximations will fail to have the $\kappa^+-cc$. To work around this, in our scheme, we set $\mathbb{P}_{\alpha+1} := A(\mathbb{P}_\alpha, z_\alpha)$, where $A(\cdot, \cdot)$ is a functor that, to each $\Sigma$-Prikry poset $\mathbb{P}$ and a problem $z$, produces a $\Sigma$-Prikry poset $A(\mathbb{P}, z)$ that projects onto $\mathbb{P}$ and solves the problem $z$. A key feature of this functor is that the projection from $A(\mathbb{P}, z)$ to $\mathbb{P}$ splits, that is, in addition to a projection map $\pi$ from $A(\mathbb{P}, z)$ onto $\mathbb{P}$, there is a map $\mathbb{m}$ that goes in the other direction, and the two maps commute in a very strong sense. The exact details may be found in our definition of forking projection (see Definition 2.13 below).

A special case of the main result of this paper may be roughly stated as follows.

**Main Theorem.** Suppose that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of regular uncountable cardinals, converging to a cardinal $\kappa$. For simplicity, let us say that a notion of forcing $\mathbb{P}$ is nice if it has property $\mathcal{D}$, $\mathbb{P} \subseteq H_{\kappa^+}$ and $\mathbb{P}$ does not collapse $\kappa^+$. Now, suppose that:

- $\mathbb{Q}$ is a nice $\Sigma$-Prikry notion of forcing;
- $A(\cdot, \cdot)$ is a functor that produces for every nice $\Sigma$-Prikry notion of forcing $\mathbb{P}$, and every $z \in H_{\kappa^+}$, a corresponding nice $\Sigma$-Prikry notion of forcing $A(\mathbb{P}, z)$. Moreover, $A(\cdot, \cdot)$ admits a forking projection to $\mathbb{P}$ with the weak mixing property;
- $2^{2^\kappa} = \kappa^+$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^+ \rangle$.

Then there exists a sequence $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^+ \rangle$ of forcings such that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$, $\mathbb{P}_{\alpha+1}$ is isomorphic to $A(\mathbb{P}_\alpha, z_\alpha)$, and, for every pair $\alpha \leq \beta < \kappa^+$, $\mathbb{P}_\beta$ is isomorphic to $\mathbb{P}_\alpha * \mathbb{Q}_\alpha$.
\[ \beta \leq \kappa^{++}, \mathbb{P}_\beta \text{ projects onto } \mathbb{P}_\alpha. \text{ Moreover, if for each nonzero limit ordinal } \alpha \leq \kappa^{++}, \mathbb{P}_\alpha \text{ contains a canonical dense subforcing } \mathbb{P}_\alpha, \text{ then } \langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++} \rangle \text{ consists of } \Sigma\text{-Prikry forcings.} \]

1.1. Organization of this paper. In Section 2, we recall the definitions of the \( \Sigma \)-Prikry class, forking projections, and introduce property D and the weak mixing property.

In Section 3, we present our abstract iteration scheme for \( \Sigma \)-Prikry posets, and prove the Main Theorem of this paper (see Lemmas 3.6 and 3.13).

In Section 4, we present the very first application of our scheme. We carry out an iteration of length \( \kappa^{++} \), where the first step of the iteration is the Extender Based Prikry Forcing (EBPF) due to Gitik and Magidor [GM94, §3] for making \( 2^\kappa = \kappa^{++} \), and all the later steps are obtained by invoking the functor \( \mathcal{A}(P, z) \) from [PRS20, §6] for killing a nonreflecting stationary subset \( z \). This functor is due to Sharon [Sha05, §2], and as a corollary, we obtain a correct proof of the main result of [Sha05, §3]:

**Corollary.** If \( \kappa \) is the limit of a countable increasing sequence of supercompact cardinals, then there exists a cofinality-preserving forcing extension in which \( \kappa \) remains a strong limit, every finite collection of stationary subsets of \( \kappa \) reflects simultaneously, and \( 2^\kappa = \kappa^{++} \).

1.2. Notation and conventions. Our forcing convention is that \( p \leq q \) means that \( p \) extends \( q \). We write \( \mathbb{P} \downarrow q \) for \( \{ p \in \mathbb{P} \mid p \leq q \} \). Denote \( E^\mu_{<\theta} := \{ \alpha < \mu \mid \text{cf}(\alpha) = \theta \} \). The sets \( E^\mu_{>\theta} \) and \( E^\mu_{>\omega} \) are defined in a similar fashion. For a stationary subset \( S \) of a regular uncountable cardinal \( \mu \), we write \( \text{Tr}(S) := \{ \delta \in E^\mu_{>\omega} \mid S \cap \delta \text{ is stationary in } \delta \} \). \( H_\nu \) denotes the collection of all sets of hereditary cardinality less than \( \nu \). For every set of ordinals \( x \), we denote \( \text{cl}(x) := \{ \sup(x \cap \gamma) \mid \gamma \in \text{Ord}, x \cap \gamma \neq \emptyset \} \), \( \text{acc}(x) := \{ \gamma \in x \mid \sup(x \cap \gamma) = \gamma > 0 \} \) and \( \text{nacc}(x) := x \setminus \text{acc}(x) \).

2. The \( \Sigma \)-Prikry class and forking projections

In this section, we recall some definitions and facts from [PRS20, §2] and [PRS20, §4], and then continue developing the theory of forking projections. The reader is not assumed to be familiar with [PRS20].

2.1. The \( \Sigma \)-Prikry class and Property D.

**Definition 2.1.** We say that \( (\mathbb{P}, \ell) \) is a graded poset iff \( \mathbb{P} = (P, \leq) \) is a poset, \( \ell : P \to \omega \) is a surjection, and, for all \( p \in P \):

- For every \( q \leq p \), \( \ell(q) \geq \ell(p) \);
- There exists \( q \leq p \) with \( \ell(q) = \ell(p) + 1 \).

**Convention 2.2.** For a graded poset as above, we denote \( P_n := \{ p \in P \mid \ell(p) = n \} \), \( P_n^p := \{ q \in P \mid q \leq p, \ell(q) = \ell(p) + n \} \), and sometime write \( q \leq^n p \) (and say the \( q \) is an \( n \)-step extension of \( p \)) rather than writing \( q \in P_n^p \).
**Definition 2.3.** Suppose that $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $1$, and that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$. Suppose that $\mu$ is a cardinal such that $1 \Vdash \mu = \kappa^+$. For functions $\ell : P \to \omega$ and $c : P \to \mu$, we say that $(\mathbb{P}, \ell, c)$ is $\Sigma$-Prikry iff all of the following hold:

1. $(\mathbb{P}, \ell)$ is a graded poset;
2. For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{1\}, \leq)$ contains a dense subposet $\mathbb{P}'_n$ which is $\kappa_n$-directed-closed;
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P^q_0 \cap P^p_0$ is non-empty;
4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{ r \leq^n p \mid q \leq^m r \}$ contains a greatest element which we denote by $m(p, q)$.\(^1\) In the special case $m = 0$, we shall write $w(p, q)$ rather than $0(p, q)$.\(^2\)
5. For all $p \in P$, the set $W(p) := \{ w(p, q) \mid q \leq p \}$ has size $< \mu$;
6. For all $p' \leq p$ in $P$, $q \mapsto w(p, q)$ forms an order-preserving map from $W(p')$ to $W(p)$;
7. Suppose that $U \subseteq P$ is a $0$-open set, i.e., $r \in U$ iff $P^r_0 \subseteq U$. Then, for all $p \in P$ and $n < \omega$, there is $q \leq^0 p$, such that, either $P^q_n \cap U = \emptyset$ or $P^{q'}_n \subseteq U$.

**Remark 2.4.**

1. Clause (2) differs from that of [PRS20, Definition 2.3], where we originally required $\mathbb{P}_n$ itself to be $\kappa_n$-directed-closed.
2. Note that Clause (3) is the Introduction’s Linked property. Often, we will want to avoid encodings and opt to define the function $c$ as a map from $P$ to some natural set $\mathfrak{M}$ of size $\leq \mu$, instead of a map to the cardinal $\mu$ itself. In the special case that $\mu^{\leq \mu} = \mu$, we shall simply take $\mathfrak{M}$ to be $H_\mu$.
3. Note that Clause (7) is the Complete Prikry Property (CPP).

**Definition 2.5.** Let $p \in P$. For each $n < \omega$, we write $W_n(p) := \{ w(p, q) \mid q \in P^p_n \}$, and $W_{\geq n}(p) := \{ w(p, q) \mid \exists m \in \omega \setminus n[q \in P^p_m] \}$. The object $W(p) := \bigcup_{n<\omega} W_n(p)$ is called the $p$-tree.

**Fact 2.6** ([PRS20, Lemma 2.8]). Let $p \in P$.

1. For every $n < \omega$, $W_n(p)$ is a maximal antichain in $\mathbb{P} \downarrow p$;
2. Every two compatible elements of $W(p)$ are comparable;
3. For any pair $q' \leq q$ in $W(p)$, $q' \in W(q)$;
4. $c \upharpoonright W(p)$ is injective.

**Fact 2.7** ([PRS20, Lemma 2.10]).

1. $\mathbb{P}$ does not add bounded subsets of $\kappa$;
2. For every regular cardinal $\nu \geq \kappa$, if there exists $p \in P$ for which $p \Vdash \text{cf}(\nu) < \kappa$, then there exists $p' \leq p$ with $|W(p')| \geq \nu$.\(^3\)

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\(^1\)By convention, a greatest element, if exists, is unique.

\(^2\)Note that $w(p, q)$ is the weakest $n$-step extension of $p$ above $q$.

\(^3\)For future reference, we point out that this fact relies only on Clauses (1), (2), (4) and (7) of Definition 2.3. Furthermore, we do not need to know that $1$ decides a value for $\kappa^+$. 
Definition 2.8. We say that $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a good enumeration of a set $A$ iff $\chi$ is a cardinal, $\vec{r}$ is injective, and $\{r_\xi \mid \xi < \chi \} = A$.

Definition 2.9 (Diagonalizability). Given $p \in P$, $n < \omega$, and a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ of $W_n(p)$, we say that $\vec{q} = \langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable (with respect to $\vec{r}$) iff the two hold:

(a) $q_\xi \leq^0 r_\xi$ for every $\xi < \chi$;
(b) there is $p' \leq^0 p$ such that for every $q' \in W_n(p')$, $q' \leq^0 q_\xi$, where $\xi$ is the unique index to satisfy $r_\xi = w(p, q')$.

Definition 2.10 (Diagonalizability game). Given $p \in P$, $n < \omega$, a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ of $W_n(p)$, and a dense subset $D$ of $P_{\ell_\xi(p) + n}$, $\mathcal{D}_P(p, \vec{r}, D)$ is a game of length $\chi$ between two players $I$ and $II$, defined as follows:

- At stage $\xi < \chi$, $I$ plays a condition $p_\xi \leq^0 p$ compatible with $r_\xi$, and then $II$ plays $q_\xi \in D$ such that $q_\xi \leq p_\xi$ and $q_\xi \leq^0 r_\xi$;
- $I$ wins the game iff the resulting sequence $\vec{q} = \langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable.

In the special case that $D$ is all of $P_{\ell_\xi(p) + n}$, we omit it, writing $\mathcal{D}_P(p, \vec{r})$.

The following lemma will be useful later.

Lemma 2.11. Given $p \in P$, $n < \omega$, a good enumeration $\vec{r}$ of $W_n(p)$, and a dense subset $D$ of $P_{\ell_\xi(p) + n}$, $I$ has a winning strategy for $\mathcal{D}_P(p, \vec{r}, D)$ iff it has a winning strategy for $\mathcal{D}_P(p, \vec{r})$.

Proof. Only the forward implication requires an argument. Write $\vec{r}$ as $\langle r_\xi \mid \xi < \chi \rangle$; we shall describe a winning strategy for $I$ in the game $\mathcal{D}_P(p, \vec{r})$ by producing sequences of the form $\langle (p_\eta, q_\eta, q'_\eta) \mid \eta < \xi \rangle$, where $\langle (p_\eta, q_\eta) \mid \eta < \xi \rangle$ is an initial play (consisting of $\xi$ rounds) in the game $\mathcal{D}_P(p, \vec{r})$, and $\langle (p_\eta, q'_\eta) \mid \eta < \xi \rangle$ is an initial play in the game $\mathcal{D}_P(p, \vec{r}, D)$.

Assuming that $I$ has a winning strategy for $\mathcal{D}_P(p, \vec{r}, D)$, here is a description of our winning strategy for $I$ in the game $\mathcal{D}_P(p, \vec{r})$:

- For $\xi = 0$, we play a condition $p_0$ according to the winning strategy of $I$ in the game $\mathcal{D}_P(p, \vec{r}, D)$. Then, $II$ plays $q_0 \leq p_0$ such that $q_0 \leq^0 r_0$. Since $D$ is dense in $P_{\ell_\xi(p) + n}$, we then pick $q'_0 \in D$ with $q'_0 \leq^0 q_0$.

- Suppose that $\xi < \chi$ is nonzero and that $\langle (p_\eta, q_\eta, q'_\eta) \mid \eta < \xi \rangle$ has already been defined. Let $p_\xi$ be given by the winning strategy of $I$ for the game $\mathcal{D}_P(p, \vec{r}, D)$ with respect to the initial play $\langle (p_\eta, q'_\eta) \mid \eta < \xi \rangle$. Then, $II$ plays $q_\xi \leq p_\xi$ such that $q_\xi \leq^0 r_\xi$. Finally, pick $q'_\xi \in D$ such that $q'_\xi \leq^0 q_\xi$.

At the end of the above process, since $\langle (p_\xi, q'_\xi) \mid \xi < \chi \rangle$ is a play in the game $\mathcal{D}_P(p, \vec{r}, D)$ using the winning strategy of $I$, we may fix $p' \leq^0 p$ witnessing that $\langle q'_\xi \mid \xi < \chi \rangle$ is diagonalizable. So, for every $q' \in W_n(p')$, if $\xi$ is the unique index to satisfy $r_\xi = w(p, q')$, then $q' \leq^0 q'_\xi \leq^0 q_\xi$. In particular, $p'$ witnesses that $\langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable, as desired. \qed
Definition 2.12 (Property D). We say that \((\mathbb{P}, \ell_{\mathbb{P}})\) has property D iff for any \(p \in P\), \(n < \omega\) and any good enumeration \(\vec{r} = \langle r_\xi | \xi < \chi \rangle\) of \(W_\nu(p)\), \(I\) has a winning strategy for the game \(\triangleright; (p, \vec{r})\).

2.2. Forking projections. In this and the next subsection, we continue the work started in [PRS20, §4] concerning forking projections. This will play a key role in Section 3, where we deal with iterating \(\Sigma\)-Prikry posets.

Definition 2.13 ([PRS20, Definition 4.1]). Suppose that \((\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})\) is a \(\Sigma\)-Prikry triple, \(A = (A, \leq)\) is a notion of forcing, and \(\ell_A\) and \(c_A\) are functions with \(\text{dom}(\ell_A) = \text{dom}(c_A) = A\).

A pair of functions \((\check{\mu}, \pi)\) is said to be a forking projection from \((A, \ell_A)\) to \((\mathbb{P}, \ell_{\mathbb{P}})\) iff all of the following hold:

1. \(\pi\) is a projection from \(\mathbb{A}\) onto \(\mathbb{P}\), and \(\ell_A = \ell_{\mathbb{P}} \circ \pi\);
2. for all \(a \in A\), \(\check{\mu}(a)\) is an order-preserving function from \((\mathbb{P} \downarrow \pi(a), \leq)\) to \((A, \downarrow, \leq)\);
3. for all \(p \in P\), \(\{a \in A | \pi(a) = p\}\) admits a greatest element, which we denote by \([p]^A\);
4. for all \(n, m < \omega\) and \(b \leq a^m\), \(m(a, b)\) exists and satisfies:
   \[m(a, b) = \check{\mu}(a)(m(\pi(a), \pi(b)))\]
5. for all \(a \in A\) and \(q \leq \pi(a)\), \(\pi(\check{\mu}(a)(q)) = q\);
6. for all \(a \in A\) and \(q \leq \pi(a)\), \(a = [\pi(a)]^A\) iff \(\check{\mu}(a)(q) = [q]^A\);
7. for all \(a \in A\), \(a' \leq a\) and \(r \leq 0\), \(\check{\mu}(a')(r) \leq \check{\mu}(a)(r)\).

The pair \((\check{\mu}, \pi)\) is said to be a forking projection from \((A, \ell_A, c_A)\) to \((\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})\) iff, in addition to all of the above, the following holds:

8. for all \(a, a' \in A\), if \(c_A(a) = c_A(a')\), then \(c_{\mathbb{P}}(\pi(a)) = c_{\mathbb{P}}(\pi(a'))\) and, for all \(r \in P_0\pi(a) \cap P_0\pi(a')\), \(\check{\mu}(a)(r) = \check{\mu}(a')(r)\).

Remark 2.14. Intuitively speaking, \(\check{\mu}(a)\) is an operator that, for each condition \(p \in P \downarrow \pi(a)\), provides the \(\leq\)-greatest condition \(b \leq a\) with \(\pi(b) = p\).

Example 2.15. Suppose that \((\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})\) is a \(\Sigma\)-Prikry triple. Let \(\mu\) denote the cardinal such that \(\mathbb{I} \Vdash \mu = \kappa^+\). We define the following objects:

- \(A = (A, \leq)\), where \(A := P \times \mu\) and \(\langle p, \alpha \rangle \leq (p, \beta)\) iff \(p \leq q\) and \(\alpha \supseteq \beta\);
- \(\ell_A : A \rightarrow \omega\) via \(\ell_A(p, \alpha) := \ell_{\mathbb{P}}(p)\);
- \(c_A : A \rightarrow \mu \times \mu\) via \(c_A(p, \alpha) := (c_{\mathbb{P}}(p), \alpha)\);
- \(\pi : A \rightarrow P\) via \(\pi(p, \alpha) := p\);
- for \(a = (p, \alpha) \in A\), define \(\check{\mu}(a) : \mathbb{P} \downarrow p \rightarrow A\) via \(\check{\mu}(a)(q) := (q, \alpha)\).

Then \((\check{\mu}, \pi)\) is a forking projection from \((A, \ell_A, c_A)\) to \((\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})\).

Definition 2.16. Given two posets \(\mathbb{P} = (P, \leq)\) and \(A := (A, \leq)\), and a projection \(\pi\) from \(A\) to \(\mathbb{P}\), we denote by \(A^\pi\) the poset \((A, \leq^\pi)\), where \(a \leq^\pi b\) iff \(a \leq b\) and \(\pi(a) = \pi(b)\).

For a subposet \(\hat{A} = (\hat{A}, \leq)\) of \(A\), we likewise denote \(\hat{A}^\pi := (\hat{A}, \leq^\pi)\).

\footnote{For future reference we point out that \([a]^A = (\pi(a), 0)\) for all \(a \in A\).}
Lemma 2.17. Suppose that $\langle \mathfrak{p}, \pi \rangle$ is a forking projection from $\langle \mathbb{A}, \ell_{\mathbb{A}} \rangle$ to $\langle \mathbb{P}, \ell_{\mathbb{P}} \rangle$. For every $a \in A$, $\mathfrak{p}(a)(\pi(a)) = a$.

Proof. By Definition 2.13(4), using $(n, m, b) := (0, 0, a)$, we infer that $\mathfrak{p}(a)(\pi(a)) = \mathfrak{p}(a)(w(\pi(a), \pi(a))) = w(a, a) = a$. □

Lemma 2.18 (Canonical form). Suppose that $\langle \mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}} \rangle$ and $\langle \mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}} \rangle$ are both $\Sigma$-Prikry notions of forcing. Denote $\mathbb{P} = (P, \leq)$ and $\mathbb{A} = (A, \leq)$.

If $\langle \mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}} \rangle$ admits a forking projection to $\langle \mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}} \rangle$ as witnessed by a pair $\langle \mathfrak{p}, \pi \rangle$, then we may assume that all of the following hold true:

1. each element of $A$ is a pair $(x, y)$ with $\pi(x, y) = x$;
2. for all $a \in A$, $[\pi(a)]_{\mathbb{A}} = (\pi(a), \emptyset)$;
3. for all $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}([p]_{\mathbb{A}}) = c_{\mathbb{A}}([q]_{\mathbb{A}})$.

Proof. By applying a bijection, we may assume that $A = |A|$ with $\mathbb{1}_{\mathbb{A}} = \emptyset$. To clarify what we are about to do, we agree to say that “$a$ is a lift” iff $a = [\pi(a)]_{\mathbb{A}}$. Now, define $f : A \to P \times A$ via:

$$f(a) := \begin{cases} (\pi(a), \emptyset), & \text{if } a \text{ is a lift;} \\ (\pi(a), a), & \text{otherwise.} \end{cases}$$

Claim 2.18.1. $f$ is injective.

Proof. Suppose $a, a' \in A$ with $f(a) = f(a')$.

- If $a$ is not a lift and $a'$ is not a lift, then from $f(a) = f(a')$ we immediately get that $a = a'$.
- If $a$ is a lift and $a'$ is a lift, then from $f(a) = f(a')$, we infer that $\pi(a) = \pi(a')$, so that $a = [\pi(a)]_{\mathbb{A}} = [\pi(a')]_{\mathbb{A}} = a'$.
- If $a$ is not a lift, but $a'$ is a lift, then from $f(a) = f(a')$, we infer that $a = \emptyset = \mathbb{1}_{\mathbb{A}}$, contradicting the fact that $\mathbb{1}_{\mathbb{A}} = [\mathbb{1}_{\mathbb{P}}]_{\mathbb{A}} = [\pi(\mathbb{1}_{\mathbb{A}})]_{\mathbb{A}}$ is a lift. So this case is void. □

Let $B := \text{Im}(f)$ and $\leq_B := \{(f(a), f(b)) \mid a \leq b\}$, so that $\mathbb{B} := (B, \leq_B)$ is isomorphic to $\mathbb{A}$. Define $\ell_{\mathbb{B}} := \ell_{\mathbb{A}} \circ f^{-1}$ and $\pi_{\mathbb{B}} := \pi \circ f^{-1}$. Also, define $\mathfrak{p}_{\mathbb{B}}$ via $\mathfrak{p}_{\mathbb{B}}(b)(p) := f(\mathfrak{p}(f^{-1}(b))(p))$. It is clear that $b \in B$ is a lift iff $f^{-1}(a)$ is a lift iff $b = (\pi_{\mathbb{B}}(b), \emptyset)$.

Next, define $c_{\mathbb{B}} : B \to \mu \times 2$ by letting for all $b \in B$:

$$c_{\mathbb{B}}(b) := \begin{cases} (\pi_{\mathbb{B}}(\pi_{\mathbb{B}}(b)), 0), & \text{if } b \text{ is a lift;} \\ (c_{\mathbb{A}}(f^{-1}(b)), 1), & \text{otherwise.} \end{cases}$$

Claim 2.18.2. Suppose $b_0, b_1 \in B$ with $c_{\mathbb{B}}(b_0) = c_{\mathbb{B}}(b_1)$. Then $c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_0)) = c_{\mathbb{P}}(\pi_{\mathbb{B}}(b_1))$ and, for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\mathfrak{p}_{\mathbb{B}}(b_0)(r) = \mathfrak{p}_{\mathbb{B}}(b_1)(r)$.

Proof. We focus on verifying that for all $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$, $\mathfrak{p}_{\mathbb{B}}(b_0)(r) = \mathfrak{p}_{\mathbb{B}}(b_1)(r)$. For each $i < 2$, denote $a_i := f^{-1}(b_i)$ and $p_i := \pi_{\mathbb{B}}(b_i)$, so that $\pi(a_i) = p_i$. Suppose $r \in P_0^{\pi_{\mathbb{B}}(b_0)} \cap P_0^{\pi_{\mathbb{B}}(b_1)}$. □
If $b_0$ is a lift, then so are $b_1, a_0, a_1$. Therefore, for each $i < 2$, Definition 2.13(6) implies that $\hat{m}(b_1)(r) = f(\hat{m}(a_i)(r)) = f([r]^A) = [r]_B$. In effect, $\hat{m}(b_0)(r) = \hat{m}(b_1)(r)$, as desired.

Otherwise, $c_A(a_0) = c_A(a_1)$. As $r \in P_0^{p(a_i)} \cap P_0^{p(a_i)}, \hat{m}(b_0)(p) = f(\hat{m}(a_0)(p)) = f(\hat{m}(a_1)(p)) = \hat{m}(b_1)(p)$. □

This completes the proof. □

Setup 2. Throughout the rest of this section, suppose that:

- $P = (\mathbb{P}, \leq)$ is a notion of forcing with a greatest element $1_{\mathbb{P}}$;
- $A = (A, \preceq)$ is a notion of forcing with a greatest element $1_A$;
- $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$, and $\mu$ is a cardinal such that $1_{\mathbb{P}} \forces \mu = \kappa^+$;
- $\ell_{\mathbb{P}}$ and $c_{\mathbb{P}}$ are functions witnessing that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is a $\Sigma$-Prikry;
- $\ell_A$ and $c_A$ are functions with $\text{dom}(\ell_A) = \text{dom}(c_A) = A$;
- $(\hat{m}, \pi)$ is a forking projection from $(A, \ell_A, c_A)$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$.

The next two facts will help verifying Clauses (1) and (3) of Definition 2.3 for the different stages of the iteration of Section 3.

Fact 2.19 ([PRS20, Lemma 4.3]). Suppose that $(\hat{m}, \pi)$ is a forking projection from $(A, \ell_A)$ to $(\mathbb{P}, \ell_{\mathbb{P}})$, or, just a pair of maps satisfying Clauses (1), (2) and (4) of Definition 2.13. For each $a \in A$, the following holds:

1. $\hat{m}(a) | W(\pi(a))$ forms a bijection from $W(\pi(a))$ to $W(a)$;
2. for all $n < \omega$ and $r \in P_n^{p(a)}, \hat{m}(a)(r) \in A_n^a$.

In particular, $(A, \ell_A)$ is a graded poset.

Fact 2.20 ([PRS20, Lemma 4.7]). Suppose that $(\hat{m}, \pi)$ is a forking projection from $(A, \ell_A, c_A)$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$, or, just a pair of maps satisfying Clauses (1), (2), (4), (7) and (8) of Definition 2.13. For all $a, a' \in A$, if $c_A(a) = c_A(a')$, then $A_0^a \cap A_0^a'$ is non-empty. In particular, if $|\text{Im}(c_A)| \leq \mu$, then $(A, \ell_A)$ is $\mu^+$-2-linked.

Lemma 2.21. Suppose that $(A, \ell_A)$ has property $\mathcal{D}$. Then it has the CPP.

Proof. Let $U \subseteq A$ be a 0-open set, $a \in A$ and $n < \omega$. Let $r = \langle r_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$. Let $\langle (a_\xi, b_\xi) \mid \xi < \chi \rangle$ list the rounds of the game $\mathcal{G}_A(a, r)$ in which, in round $\xi$, I plays according to their winning strategy and II plays $b_\xi \leq^n a_\xi$ such that

1. $b_\xi \leq^0 r_\xi$, and
2. if $A_0^{b_\xi} \cap U \neq \emptyset$, then $b_\xi \in U$.

Let $a' \leq^0 a$ be a condition witnessing the diagonalizability of $\langle b_\xi \mid \xi < \chi \rangle$. Set $p := \pi(a)$ and $p' := \pi(a')$. By Fact 2.19, $W(a) = \hat{m}(a) \preceq W(p)$, hence, for each $q \leq^n p$, we may let $\xi(q) < \chi$ be such that $\hat{m}(a)(w(p, q)) = r_{\xi(q)}$. Set $\bar{U} := \{ q \in P_0^n \mid b_{\xi(q)}(q) \in U \}$. As $q' \leq^0 q \leq^n p$ implies $\xi(q') = \xi(q)$, the set $\bar{U}$
is 0-open. Now, since \((\mathbb{P}, \mathcal{F}, c)\) is \(\Sigma\)-Prikry (see Setup 2), applying CPP to \(\bar{U}, p, c\), and \(n\), we find \(\bar{p} \leq^0 p\) such that either \(P_n^\mathbb{P} \subseteq \bar{U}\) or \(P_n^\mathbb{P} \cap \bar{U} = \emptyset\).

Set \(\bar{a} := \cap (a')(\bar{p})\). Since \(\bar{p} \leq^0 p\), \(\bar{a}\) witnesses that \(\bar{a} \leq^0 a\).

**Claim 2.21.1.** Let \(b \in A_n^\mathbb{P}\). Then:

1. \(b \leq^0 b_{\xi(\pi(b))}\);
2. If \(b \in U\), then \(P_n^\mathbb{P} \subseteq \bar{U}\).

**Proof.** Denote \(q := \pi(b)\).

1. Since \(w(a', b) \in W_n(a')\) and \(a'\) is a witness to diagonalizability of \(\langle b\xi \mid \xi < \chi\rangle\), \(b \leq^0 w(a', b) \leq^0 b_{\xi}\), where \(\xi\) is the unique index to satisfy \(\pi_{\xi} = w(a, b)\).

By Clause (4) of Definition 2.13,

\[ r_{\xi} = w(a, b) = \cap (a)(w(p, q)) = r_{\xi(q)} \]

so that \(\xi = \xi(\pi(b))\).

2. Assuming that \(b \in U\), we altogether infer that \(b \in A_{\xi(\pi(b))}^\mathbb{P} \cap U \neq \emptyset\), and then Clause (ii) above implies that \(b_{\xi(\pi(b))} \in U\). By the definition of \(\bar{U}\), then, \(q \in U \cap P_n^\mathbb{P}\). So, by the choice of \(\bar{p}\), furthermore \(P_n^\mathbb{P} \subseteq \bar{U}\).

It thus follows that if \(A_n^\mathbb{P} \cap U \neq \emptyset\), then for every \(b \in A_n^\mathbb{P}\), \(\pi(b) \in P_n^\mathbb{P} \subseteq \bar{U}\), so that \(b_{\pi(\pi(b))} \in U\). By the preceding claim, \(b \leq^0 b_{\pi(\pi(b))}\), so, since \(U\) is 0-open, \(b \in U\). Thus we have shown that if \(A_n^\mathbb{P} \cap U \neq \emptyset\), then \(A_n^\mathbb{P} \subseteq U\). □

**Proposition 2.22.** Let \(a \in A\), \(n < \omega\) and \(\bar{s} = \langle s_{\xi} \mid \xi < \chi\rangle\) be a good enumeration of \(W_n(a)\).

Suppose that \(\langle b_{\xi} \mid \xi < \chi\rangle\) is a sequence of conditions in \(\mathbb{A} \downarrow a\) such that:

(a) \(\langle \pi(b_{\xi}) \mid \xi < \chi\rangle\) is diagonalizable with respect to \(\langle \pi(s_{\xi}) \mid \xi < \chi\rangle\), as witnessed by \(p' \leq^0 \pi(a)\).

(\(\beta\)) \(b\) is a condition in \(\mathbb{A}\) with \(\pi(b) = p'\) such that, for all \(q' \in W_n(p')\),

\[ \cap (a)(q') \leq^0 b_{\xi}, \]

where \(\xi\) is the unique index such that \(\pi(s_{\xi}) = w(\pi(a), q')\).

Then \(b\) witnesses that \(\langle b_{\xi} \mid \xi < \chi\rangle\) is diagonalizable with respect to \(\bar{s}\).

**Proof.** We go over the two clauses of Definition 2.9:

(a) Let \(\xi < \chi\). By Clause (\(\alpha\)) above, \(\pi(b_{\xi}) \leq^0 \pi(s_{\xi})\). Together with Definition 2.3(6), it follows that

\[ w(\pi(a), \pi(b_{\xi})) \leq^0 w(\pi(a), \pi(s_{\xi})) = \pi(s_{\xi}). \]

Finally, Clauses (1),(4) and (5) of Definition 2.13 yield

\[ b_{\xi} \leq^0 w(a, b_{\xi}) = \cap (a)(w(\pi(a), \pi(b_{\xi}))) \leq^0 \cap (a)(\pi(s_{\xi})) = s_{\xi}. \]

\(5\)By Fact 2.19, \(\langle \pi(s_{\xi}) \mid \xi < \chi\rangle\) is a good enumeration of \(W_n(\pi(a))\), hence the clause is well-posed.
(b) Let \( b' \in W_n(b) \), and we shall show that \( b' \leq \eta^\omega_\xi b_\xi \), where \( \xi \) is the unique index to satisfy \( s_\xi = w(a, b') \). Set \( q' := \pi(b') \). As \( \pi(b) = p' \), we infer from Definition 2.13(4) that \( b' = \mathfrak{h}(b)(q') \) and \( q' \in W_n(p') \). Thus, by Clause (β) above \( b' = \mathfrak{h}(b)(q') \leq b_\xi \), where \( \xi \) is the unique index such that \( \pi(s_\xi) = w(\pi(a), q') \). Again by Definition 2.13(4),

\[
\mathfrak{h}(a)(\pi(s_\xi)) = \mathfrak{h}(a)(w(\pi(a), q')) = w(a, b'),
\]
as desired.  

\[\square\]

2.3. Types and the weak mixing property. In this subsection, we will provide a sufficient condition for \((\mathbb{A}, \ell_\mathbb{A})\) to inherit property \(\mathcal{D}\) from \((\mathbb{P}, \ell_\mathbb{P})\).

While reading the next two definitions, the reader may want to have a simple example in mind. Such an example is given by Lemma 2.27 below.

**Definition 2.23** (Types). A type over \((\mathfrak{h}, \pi)\) is a map \( \text{tp} : A \rightarrow \langle^\mu \omega \rangle \) having the following properties:

1. for each \( a \in A \), either \( \text{dom}(\text{tp}(a)) = \alpha + 1 \) for some \( \alpha < \mu \), in which case we define \( \text{mtp}(a) := \text{tp}(a)(\alpha) \), or \( \text{tp}(a) \) is empty, in which case we define \( \text{mtp}(a) := 0 \);
2. for all \( a, b \in A \) with \( b \leq a \), \( \text{dom}(\text{tp}(a)) \leq \text{dom}(\text{tp}(b)) \) and for each \( i \in \text{dom}(\text{tp}(a)) \), \( \text{tp}(b)(i) \leq \text{tp}(a)(i) \);
3. for all \( a \in A \) and \( q \leq \pi(a) \), \( \text{dom}(\text{tp}(\mathfrak{h}(a)(q))) = \text{dom}(\text{tp}(a)) \);
4. for all \( a \in A \), \( \text{tp}(a) = \emptyset \) iff \( a = [\pi(a)]^\mathbb{A} \);
5. for all \( a \in A \) and \( \alpha \in \mu \setminus \text{dom}(\text{tp}(a)) \), there exists a stretch of \( a \) to \( \alpha \), denoted \( a^{\alpha} \), and satisfying the following:
   \( a^{\alpha} \leq \pi a \);
   \( \text{dom}(\text{tp}(a^{\alpha})) = \alpha + 1 \);
   \( \text{tp}(a^{\alpha})(i) \leq \text{mtp}(a) \) whenever \( \text{dom}(\text{tp}(a)) \leq i \leq \alpha \);
6. for all \( a, b \in A \) with \( \text{dom}(\text{tp}(a)) = \text{dom}(\text{tp}(b)) \), for every \( \alpha \in \mu \setminus \text{dom}(\text{tp}(a)) \), if \( b \leq a \), then \( b^{\alpha} \leq a^{\alpha} \);
7. For each \( n < \omega \), the poset \( \mathbb{A}_n \) is dense in \( \mathbb{A}_n \), where \( \mathbb{A}_n := (\mathbb{A}_n, \subseteq) \) and \( \mathbb{A}_n := \{ a \in \mathbb{A}_n \mid \pi(a) \in \mathbb{P}_n \land \text{mtp}(a) = 0 \} \).

**Remark 2.24.** Note that Clauses (2) and (3) imply that for all \( m, n < \omega \), \( a \in \mathbb{A}_m \), and \( q \leq \pi(a) \), if \( q \in \mathbb{P}_n \) then \( \mathfrak{h}(a)(q) \in \mathbb{A}_n \).

The next definition is a weakening of [PRS20, Definition 4.11].

**Definition 2.25** (Weak Mixing property). The forking projection \((\mathfrak{h}, \pi)\) is said to have the weak mixing property if it admits a type \( \text{tp} \) satisfying that for every \( n < \omega \), there exists an ordering \( \underline{n} \subseteq \pi^0 \) such that for all \( a \in A, r' \), and \( p' \leq \pi(a) \), for every function \( g : W_n(\pi(a)) \rightarrow A \downarrow a \), if there exists an ordinal \( \ell \) such that all of the following hold:

1. \( r' = (r_\xi \mid \xi < \chi) \) is a good enumeration of \( W_n(\pi(a)) \);
2. \( \langle \pi(g(r_\xi)) \mid \xi < \chi \rangle \) is diagonalizable with respect to \( r' \), as witnessed by \( p' \).\(^6\)

\(^6\)In particular, \( \ell_\mathbb{A}(g(r_\xi)) = \ell_\mathbb{A}(a) + n \) for every \( \xi < \chi \).
(3) for every $\xi < \chi$:  
- if $\xi < \iota$, then $\text{dom}(\text{tp}(g(\xi))) = 0$;
- if $\xi = \iota$, then $\text{dom}(\text{tp}(g(\xi))) \geq 1$;
- if $\xi > \iota$, then $(\sup_{\eta < \xi} \text{dom}(\text{tp}(g(\eta)))) + 1 < \text{dom}(\text{tp}(g(\xi)))$;

(4) for all $\xi \in (\iota, \chi)$ and $i \in [\text{dom}(\text{tp}(a))], \sup_{\eta < \xi} \text{dom}(\text{tp}(g(\eta)))$,

$$\text{tp}(g(\xi))(i) \leq \text{mtp}(a),$$

(5) $\sup_{\xi < \chi} \text{mtp}(g(\xi)) < \omega$.

Remark 2.26. Note that there may be more than one witness for a forking projection to admit the weak mixing property. For instance, if $(\bar{\emptyset}, \pi)$ has the weak mixing property as witnessed by some type $\text{tp}$ and a sequence of “fusion” orderings $\langle \subseteq_n \mid n < \omega \rangle$, then the weak mixing property is also witnessed by $\text{tp}$ and the constant sequence $\langle \leq 0 \mid n < \omega \rangle$. The more we know about these witnesses, the better form of mixing we obtain. The role of the witnesses will only become apparent when carrying out a transfinite iteration (see Lemma 3.10 and Claim 4.22.2 below). For the main example of these fusion orderings, see Definition 4.11.

**Lemma 2.27.** The forking projection $(\bar{\emptyset}, \pi)$ from Example 2.15 has the weak mixing property.

**Proof.** We attach a type $\text{tp}: A \to <\omega$ as follows. For every $a = (p, \alpha) \in A$, with $\alpha > 0$, let $\text{tp}(a)$ be the constant $(\alpha + 1)$-sequence whose sole value is 0. Otherwise, let $\text{tp}(a) := \emptyset$. We shall verify that $\text{tp}$ and the constant sequence $\langle \leq 0 \mid n < \omega \rangle$ witness the weak mixing property. To this end, suppose that we are given $n < \omega$, $a \in A$, $\bar{r} = \langle r_\xi \mid \xi < \chi \rangle$, $p' \leq 0 \pi(a)$, a function $g: W_n(\pi(a)) \to A \downarrow a$ and an ordinal $\iota$ satisfying Clauses (1)–(4) of Definition 2.25. For each $\xi < \chi$, write $(q_\xi, \alpha_\xi) := g(r_\xi)$. Note that by Definition 2.23(4) and Footnote 4, $\alpha_\xi = 0$ for all $\xi < \iota$.

Set $b := (p', \alpha')$, for $\alpha' := \sup_{\xi \in \chi} \alpha_\xi$. Now, since $p'$ witnesses that $\langle q_\xi \mid \xi < \chi \rangle$ is diagonalizable, for every $q' \in W_n(p')$, if we let $\xi$ denote the unique index to satisfy $r_\xi = w(\pi(a), q')$, then $q' \leq 0 q_\xi$. As $\alpha' \geq \alpha_\xi$, it altogether follows that $(q', \alpha') = \bar{\emptyset}(b)(q') \leq 0 g(w(\pi(a), q')) = (q_\xi, \alpha_\xi)$.

**Lemma 2.28.** Suppose that $(\bar{\emptyset}, \pi)$ has the weak mixing property and that $(P, \bar{\xi})$ has property $\mathcal{D}$. Then $(A, \bar{\xi})$ has property $\mathcal{D}$, as well.

**Proof.** Let $a \in A$, $n < \omega$ and $\bar{s} = \langle s_\xi \mid \xi < \chi \rangle$ be a good enumeration of $W_n(a)$, with $\chi = |W_n(a)|$. By Lemma 2.11 and Definition 2.23(7), it is enough to show that $I$ has a winning strategy in $\mathcal{G}_A(a, s, D)$, where $D := 1 + |A|$. For each $\xi < \chi$, let $r_\xi := \pi(s_\xi)$. By Fact 2.19, $s_\xi = \bar{\emptyset}(r_\xi)$, and $\bar{r} := \langle r_\xi \mid \xi < \chi \rangle$ forms a good enumeration of $W_n(\pi(a))$.

---

7The role of the $\iota$ would be to keep track of the support when we apply the weak mixing lemma in the iteration (see, e.g. Lemma 3.10 and Claim 3.11.6).
Fix any type tp witnessing the weak mixing property of \((\mathring{\eta}, \pi)\). We may assume that \(\subseteq^n\) is nothing but \(\leq^0\). We shall describe a winning strategy for \(I\) in the game \(\mathcal{G}_\Lambda(a, s, D)\) by producing sequences of the form \(\langle p_\eta, a_\eta, b_\eta, q_\eta | \eta < \xi \rangle\), where \(\langle a_\eta, b_\eta | \eta < \xi \rangle\) is an initial play (consisting of \(\xi\) rounds) in the game \(\mathcal{G}_\Lambda(a, s, D)\), and \(\langle p_\eta, q_\eta | \eta < \xi \rangle\) is an initial play in the game \(\mathcal{G}_\mathcal{P}(\pi(a), r)\).

For \(\xi = 0\), we first play a condition \(p_0\) according to the winning strategy for \(I\) in the game \(\mathcal{G}_\mathcal{P}(\pi(a), r)\). In particular, \(p_0 \leq^0 \pi(a)\). As \(p_0\) is compatible with \(r_0\), fix a condition \(r' \leq p_0, r_0\), and note that it follows from Definition 2.13(2) that \(\mathring{\eta}(a)(r') \leq \mathring{\eta}(a)(p_0), \mathring{\eta}(a)(r_0)\). Now set \(\bar{\alpha}_0 := \text{dom}(tp(a)) + 1\) and \(a_0 := \mathring{\eta}(a)(p_0) \ominus \bar{\alpha}_0\). By Definition 2.23(5), \(a_0 \leq^0 a\), and it can also be shown that \(a_0\) is compatible with \(s_0\).

Next, let \(\Pi\) play \(b_0 \in D\) such that \(b_0 \leq a_0\) and \(b_0 \leq^0 s_0\). Finally, let \(q_0 := \pi(b_0)\).

Suppose that \(\xi < \chi\) is nonzero and that \(\langle p_\eta, a_\eta, b_\eta, q_\eta | \eta < \xi \rangle\) has already been defined. Let \(p_\xi\) be given by the winning strategy for \(I\) in the game \(\mathcal{G}_\mathcal{P}(\pi(a), r)\) with respect to the initial play \(\langle p_\eta, q_\eta | \eta < \xi \rangle\). As in the previous case, we may fix a condition \(r'\) such that \(\mathring{\eta}(a)(r') \subseteq \mathring{\eta}(a)(p_\xi), s_\xi\).

Set \(a_\xi := (\sup_{\eta < \xi} \text{dom}(tp(b_\eta))) + 1\). Then, by Clauses (5) and (6) of Definition 2.23, we may let \(a_\xi := \mathring{\eta}(a)(p_\xi) \ominus \bar{\alpha}_\xi\), and infer that:

- \(a_\xi \leq^\pi \mathring{\eta}(a)(p_\xi)\);
- \(\text{dom}(tp(a_\xi)) = \bar{\alpha}_\xi + 1\);
- \(tp(a_\xi)(i) \leq \text{mtp}(\mathring{\eta}(a)(p_\xi)), \text{ whenever dom}(tp(\mathring{\eta}(a)(p_\xi))) \leq i \leq \bar{\alpha}_\xi\);
- \(\mathring{\eta}(a)(r') \ominus \bar{\alpha}_\xi \subseteq \mathring{\eta}(a)(p_\xi) \ominus \bar{\alpha}_\xi = a_\xi\) and \(\mathring{\eta}(a)(r') \ominus \bar{\alpha}_\xi \subseteq \mathring{\eta}(a)(r') \subseteq s_\xi\).

In particular, \(a_\xi \leq^0 a\), \(\pi(a_\xi) = p_\xi\) and \(a_\xi\) is compatible with \(s_\xi\). Next, let \(\Pi\) play \(b_\xi \in D\) such that \(b_\xi \leq a_\xi\) and \(a_\xi \leq^0 s_\xi\). Finally, let \(q_\xi := \pi(b_\xi)\).

At the end of the game, we have produced a sequence \(\langle p_\xi, a_\xi, b_\xi, q_\xi | \xi < \chi \rangle\) of the form \(\mathcal{G}_\mathcal{P}(\pi(a), r)\) in which \(I\) played according to a winning strategy, we may fix \(r' \leq^0 \pi(a)\) witnessing the \(\langle q_\xi | \xi < \chi \rangle\) is diagonalizable.

It follows that if we define a function \(g : W_n(\pi(a)) \rightarrow D\) via \(g(r_\xi) := b_\xi\), then all the requirements of Definition 2.25 are fulfilled with respect to \(\iota = 0.9\). For instance, to see that Clause (4) of Definition 2.25 holds, notice that by Clauses (2) and (3) of Definition 2.23, for all \(\xi < \chi\) and \(i \in \text{dom}(tp(a))\), \(\text{dom}(tp(a_\xi))\),

\[
\text{tp}(b_\xi)(i) \leq \text{tp}(a_\xi)(i) \leq \text{mtp}(\mathring{\eta}(a)(p_\xi)) \leq \text{mtp}(a).
\]

In effect, we may pick \(b \leq^0 a\) with \(\pi(b) = p'\) such that for all \(q' \in W_n(p')\),

\[
\mathring{\eta}(b)(q') \leq^0 g(w(\pi(a), q')).
\]

By definition, for each \(q' \in W_n(p'), g(w(\pi(a), q')) = b_\xi\), where \(\xi\) is the unique index such that \(\pi(s_\xi) = w(\pi(a), q')\). Therefore, invoking Proposition 2.22 we infer that \(b\) diagonalizes \(b_\xi | \xi < \chi\), as desired.

---

8For details see the upcoming argument in the case \(\xi > 0\).

9Note that by our construction, \(\text{dom}(tp(a_\xi)) > 0\) for all \(\xi < \chi\).
**Corollary 2.29.** If \((\mathbb{P}, \ell_\mathbb{P})\) has property \(D\), and \((\hat{n}, \pi)\) has the weak mixing property, then \((A, \ell_A)\) has the CPP.

*Proof.* By Lemmas 2.21 and 2.28. □

**Lemma 2.30.** Suppose that \((\hat{n}, \pi)\) is as in Setup 2 or, just a pair of maps satisfying Clauses (1), (2), (5) and (7) of Definition 2.13.

Let \(n < \omega\). If \((\hat{n}, \pi)\) admits a type, and \(\hat{k}_n\) is defined according to the last clause of Definition 2.23, if \(\hat{k}_n\) is \(\kappa_n\)-directed-closed, then so is \(\hat{k}_n\).

*Proof.* The proof is very similar to that of [PRS20, Lemma 4.6], bearing Remark 2.24 in mind. □

### 3. Iteration Scheme

In this section, we present our iteration scheme for \(\Sigma\)-Prikry posets. Throughout the section, assume that \(\Sigma = \langle \kappa_n \mid n < \omega \rangle\) is a non-decreasing sequence of regular uncountable cardinals. Denote \(\kappa := \sup_{n<\omega} \kappa_n\). Also, assume that \(\mu\) is some cardinal satisfying \(\mu < \mu = \mu\), so that \(|H_\mu| = \mu\).

The following convention will be applied hereafter:

**Convention 3.1.** For all ordinals \(\gamma \leq \alpha \leq \mu^+\):

1. \(\emptyset_\alpha := \alpha \times \{\emptyset\}\) denotes the \(\alpha\)-sequence with constant value \(\emptyset\);
2. For a \(\gamma\)-sequence \(p\) and an \(\alpha\)-sequence \(q\), \(p \star q\) denotes the unique \(\alpha\)-sequence satisfying that for all \(\beta < \alpha\):
   \[
   (p \star q)(\beta) = \begin{cases} 
   q(\beta), & \text{if } \gamma \leq \beta < \alpha; \\
   p(\beta), & \text{otherwise.}
   \end{cases}
   \]
3. Let \(\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha)\) and \(\mathbb{P}_\gamma := (P_\gamma, \leq_\gamma)\) be forcing posets such that \(P_\alpha \subseteq ^\alpha H_\mu^+\) and \(P_\gamma \subseteq ^\gamma H_\mu^+\). Also, assume \(p \mapsto p \upharpoonright \gamma\) defines a projection between \(\mathbb{P}_\alpha\) and \(\mathbb{P}_\gamma\). We denote by \(i_\alpha^{\mathbb{P}_\gamma} : V_\gamma^\mathbb{P}_\gamma \rightarrow V_\alpha^{\mathbb{P}_\alpha}\) the map defined by recursion over the rank of each \(\mathbb{P}_\gamma\)-name \(\sigma\) as follows:
   \[
i_\alpha^{\mathbb{P}_\gamma}(\sigma) := \{(i_\alpha^{\mathbb{P}_\gamma}(\tau), p \star \emptyset_\alpha) \mid (\tau, p) \in \sigma\}.\]

Our iteration scheme requires three building blocks:

**Building Block I.** We are given a \(\Sigma\)-Prikry triple \((Q, \ell, c)\) such that \(Q = (Q, \leq_Q)\) is a subset of \(H_\mu^+\), \(1_Q \models \hat{\mu} = \kappa^+\) and \(1_Q \models \text{“}\kappa\text{ is singular”}\).\(^{10}\)
Additionally, we assume that \((Q, \ell)\) has property \(D\). To streamline the matter, we also require that \(1_Q\) be equal to \(\emptyset\).

**Building Block II.** For every \(\Sigma\)-Prikry triple \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) having property \(D\) such that \(\mathbb{P} = (P, \leq)\) is a subset of \(H_\mu^+\), \(1_\mathbb{P} \models \hat{\mu} = \kappa^+\) and \(1_\mathbb{P} \models \text{“}\kappa\text{ is singular”}\), every \(r^* \in P\), and every \(\mathbb{P}\)-name \(z \in H_\mu^+\), we are given a corresponding \(\Sigma\)-Prikry triple \((A, \ell_A, c_A)\) having property \(D\) such that:

\(^{10}\)At the behest of the referee, we stress that the last hypothesis plays a rather isolated role; see Footnote 20 on page 33.
(a) \((A, \ell_A, c_A)\) admits a forking projection \((\hat{\mathfrak{h}}, \pi)\) to \((\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})\) that has the weak mixing property;
(b) for each \(n < \omega\), \(A^n\) is \(\kappa_n\)-directed-closed;
(c) \(\mathbb{I}_n \Vdash \not\exists \mathbb{A} \subseteq \mathbb{A}_n\) \(\kappa_n^+;\)
(d) \(A = (A, \leq)\) is a subset of \(H_{\mu^+}\).

By Lemma 2.18, we may streamline the matter, and also require that:

(e) each element of \(A\) is a pair \((x, y)\) with \(\pi(x, y) = x;\)
(f) for every \(a \in A\), \([\pi(a)]^A = (\pi(a), \emptyset);\)
(g) for every \(p, q \in P\), if \(c_p(p) = c_p(q)\), then \(c_A([p]^A) = c_A([q]^A)\).

**Building Block III.** We are given a function \(\psi : \mu^+ \to H_{\mu^+}\).

**Goal 3.2.** Our goal is to define a system \((\langle P_\alpha, \ell_\alpha, c_\alpha, (\hat{\mathfrak{h}}_{\alpha, \gamma} \mid \gamma \leq \alpha) \rangle \mid \alpha \leq \mu^+)\) in such a way that for all \(\gamma \leq \alpha \leq \mu^+:\)

(i) \(P_\alpha\) is a poset \((P_\alpha, \leq_\alpha)\), \(P_\alpha \subseteq A\) \(\kappa_\alpha\)-directed-closed, and, for all \(p \in P_\alpha\), \(|B_p| < \mu\), where \(B_p := \{\beta + 1 \mid \beta \in \text{dom}(p) \& p(\beta) \neq \emptyset\};\)
(ii) The map \(\pi_{\alpha, \gamma} : P_\alpha \to P_\gamma\), defined by \(\pi_{\alpha, \gamma}(p) := p\gamma\) forms a projection from \(P_\alpha\) to \(P_\gamma\), and \(\ell_\alpha = \ell_1 \circ \pi_{\alpha, \gamma};\)
(iii) \(P_0\) is a trivial forcing, \(P_1\) is isomorphic to \(Q\) given by Building Block I, and \(P_{\alpha+1}\) is isomorphic to \(A\) given by Building Block II when invoked with \((P_\alpha, \ell_\alpha, c_\alpha)\) and a pair \((r^*, z)\) which is decoded from \(\psi(\alpha)\);
(iv) If \(\alpha > 0\), then \((P_\alpha, \ell_\alpha, c_\alpha)\) is a \(\Sigma\)-Prikry triple having property \(D\) whose greatest element is \(\emptyset_\alpha\), \(\ell_\alpha = \ell_1 \circ \pi_{\alpha, 1}\), and \(\emptyset_\alpha \Vdash \not\exists \mathbb{A} \subseteq A\); \(\kappa_\alpha^+;\)
(v) If \(0 < \gamma < \alpha \leq \mu^+;\) then \((\hat{\mathfrak{h}}_{\alpha, \gamma}, \pi_{\alpha, \gamma})\) is a forking projection from \((P_\alpha, \ell_\alpha)\) to \((P_\gamma, \ell_\gamma);\) in case \(\alpha < \mu^+\), \((\hat{\mathfrak{h}}_{\alpha, \gamma, \pi_{\alpha, \gamma}})\) is furthermore a forking projection from \((P_\alpha, \ell_\alpha, c_\alpha)\) to \((P_\gamma, \ell_\gamma, c_\gamma)\), and in case \(\alpha = \gamma + 1\), \((\hat{\mathfrak{h}}_{\alpha, \gamma, \pi_{\alpha, \gamma}})\) has the weak mixing property;
(vi) If \(0 < \gamma \leq \beta \leq \alpha\), then, for all \(p \in P_\alpha\) and \(r \leq_\gamma p \mid \gamma, \hat{\mathfrak{h}}_{\beta, \gamma}(p\beta)(r) = (\hat{\mathfrak{h}}_{\alpha, \gamma}(p(r))\mid \beta).\)

**Remark 3.3.** Note the asymmetry between the cases \(\alpha < \mu^+\) and \(\alpha = \mu^+:\)

1. By Clause (i), we will have that \(P_\alpha \subseteq H_{\mu^+}\) for all \(\alpha < \mu^+,\) but \(P_{\mu^+} \not\subseteq H_{\mu^+}\). Still, \(P_{\mu^+}\) will nevertheless be isomorphic to a subset of \(H_{\mu^+}\), as we may identify \(P_{\mu^+}\) with \(\{p \mid (\sup(B_p) + 1) \mid p \in P_{\mu^+}\}\).
2. Clause (v) puts a weaker assertion for \(\alpha = \mu^+\). In order to avoid trivialities, let us assume that \(\mu^+\)-many stages in our iteration \(P_{\mu^+}\) are non-trivial. To see the restriction in Clause (v) is necessary note that, by the pigeonhole principle, there must exist two conditions \(p, q \in P_{\mu^+}\) and an ordinal \(\gamma < \mu^+\) for which \(c_{\mu^+}(p) = c_{\mu^+}(q), B_p \subseteq \gamma,\) but \(B_q \not\subseteq \gamma\). Now, towards a contradiction, assume there is a map \(\hat{\mathfrak{h}}\) such that \((\hat{\mathfrak{h}}, \pi_{\mu^+, \gamma})\) forms a forking projection from \((P_{\mu^+}, \ell_{\mu^+, \gamma}, c_{\mu^+})\) to \((P_\gamma, \ell_\gamma, c_\gamma)\). By Definition 2.13(8), then, \(c_\gamma(p \mid \gamma) = c_\gamma(q \mid \gamma)\), so that by Definition 2.3(3), we should be able to pick \(r \in (P_\gamma)^{\mu^+}_0 \cap (P_{\mu^+})^{\mu^+} \cap \not\exists \mathbb{A} \subseteq (\mathbb{A}_n)\).
(P_\gamma)_0^{<\gamma}$, and then by Definition 2.13(8), $\Diamond(p)(r) = \Diamond(q)(r)$. Finally, as $B_p \subseteq \gamma$, $p = [p \upharpoonright \gamma]^{P_{\mu^+}}$, so that, by Definition 2.13(6), $\Diamond(p)(r) = [r]^{P_{\mu^+}}$. But then $\Diamond(q)(r) = [r]^{P_{\mu^+}}$, so that, by Definition 2.13(6), $q = [q \upharpoonright \gamma]^{P_{\mu^+}}$, contradicting the fact that $B_q \not\subseteq \gamma$.

3.1. **Defining the iteration.** For every $\alpha < \mu^+$, fix an injection $\phi_\alpha : \alpha \to \mu$. As $|H_\mu| = \mu$, by the Engelking-Karlowicz theorem, we may also fix a sequence $(e^i | i < \mu)$ of functions from $\mu^+$ to $H_\mu$ such that for every function $e : C \to H_\mu$ with $C \subseteq [\mu^+]^{<\mu}$, there is $i < \mu$ such that $e \subseteq e^i$.

The upcoming definition is by recursion on $\alpha \leq \mu^+$, and we continue as long as we are successful. We shall later verify that the described process is indeed successful.

- Let $P_0 := (\{0\}, \leq 0)$ be the trivial forcing. Let $\ell_0$ and $c_0$ be the constant function $(\{0\}, \emptyset)$, and let $\eta_{0,0}$ be the constant function $(\{0\}, (\{0\}, \emptyset))$, so that $\eta_{0,0}(\emptyset)$ is the identity map.

- Let $P_1 := (P_1, \leq_1)$, where $P_1 := \langle Q \rangle$ and $p \leq_1 p'$ if $p(0) \leq Q p'(0)$. Define $\ell_1$ and $c_1$ by stipulating $\ell_1(p) := \ell(p(0))$ and $c_1(p) := c(p(0))$. For all $p \in P_1$, let $\eta_{1,0}(p) : (\{0\} \to \{p\}$ be the constant function, and let $\eta_{1,1}(p)$ be the identity map.

- Suppose $\alpha < \mu^+$ and that $\langle (P_\beta, \ell_\beta, c_\beta, (\eta_{\beta, \gamma} | \gamma \leq \beta)) | \beta \leq \alpha \rangle$ has already been defined. We now define the triple $(P_{\alpha + 1}, \ell_{\alpha + 1}, c_{\alpha + 1})$ and the sequence of maps $(\eta_{\alpha + 1, \gamma}) | \gamma \leq \alpha + 1)$.

  - If $\psi(\alpha)$ happens to be a triple $(\beta, r, \sigma)$, where $\beta < \alpha$, $r \in P_\beta$ and $\sigma$ is a $P_\beta$-name, then we appeal to Building Block II with $(P_\alpha, \ell_\alpha, c_\alpha)$, $r^* := r \cup \emptyset$ and $z := \eta_{\beta, r}(\sigma)$ to get a corresponding $\Sigma$-Prikry poset $(h, \ell_h, c_h)$.

  - Otherwise, we obtain $(h, \ell_h, c_h)$ by appealing to Building Block II with $(P_\alpha, \ell_\alpha, c_\alpha)$, $r^* := \emptyset$ and $z := \emptyset$.

In both cases, we also obtain a forking projection $(\Diamond, \pi)$ from $(h, \ell_h, c_h)$ to $(P_\alpha, \ell_\alpha, c_\alpha)$. Furthermore, each condition in $h = (A, \leq)$ is a pair $(x, y)$ with $\pi(x, y) = x$, and, for every $p \in P_\alpha$, $\pi^*(p) = (p, \emptyset)$. Now, define $P_{\alpha + 1} := (P_{\alpha + 1}, \leq_{\alpha + 1})$ by letting $P_{\alpha + 1} := \{x^\gamma(y) \upharpoonright (x, y) \in A\}$, and then let $p \leq_{\alpha + 1} p'$ iff $(p \upharpoonright \alpha, p(\alpha)) \subseteq (p' \upharpoonright \alpha, p'(\alpha))$. Put $\ell_{\alpha + 1} := \ell_1 \circ \pi_{\alpha + 1, 1}$ and define $c_{\alpha + 1} : P_{\alpha + 1} \to H_\mu$ via $c_{\alpha + 1}(p) := c_h(p \upharpoonright \alpha, p(\alpha))$.

Next, let $p \in P_{\alpha + 1}$, $\gamma \leq \alpha + 1$ and $r \leq p \upharpoonright \gamma$ be arbitrary; we need to define $\eta_{\alpha + 1, \gamma}(p)(r)$. For $\gamma = \alpha + 1$, let $\eta_{\alpha + 1, \gamma}(p)(r) := r$, and for $\gamma \leq \alpha$, let $(*)$ $\eta_{\alpha + 1, \gamma}(p)(r) := x^\gamma(y)$ if $\Diamond(p \upharpoonright \alpha, p(\alpha))(\eta_{\alpha, \gamma}(p \upharpoonright \alpha)(r)) = (x, y)$.

- Suppose $\alpha \in \text{acc}(\mu^+ + 1)$, and that $\langle (P_\beta, \ell_\beta, c_\beta, (\eta_{\beta, \gamma} | \gamma \leq \beta)) | \beta < \alpha \rangle$ has already been defined. Define $P_\alpha := (P_\alpha, \leq_\alpha)$ by letting $P_\alpha$ be all $\alpha$-sequences $p$ such that $|B_p| < \mu$ and $\forall \beta < \alpha(p \upharpoonright \beta \in P_\beta)$. Let $p \leq_\alpha q$ iff $\forall \beta < \alpha(p \upharpoonright \beta \leq \beta q \upharpoonright \beta)$. Let $\ell_\alpha := \ell_1 \circ \pi_{\alpha, 1}$. Next, we define $c_\alpha : P_\alpha \to H_\mu$, as follows.

\footnote{This is a consequence of the fact that $p = (p \upharpoonright \gamma)*_0 \mu^+ = (p \upharpoonright \gamma)^{P_{\mu^+}}$. See the discussion at the beginning of Lemma 3.6.}
If \( \alpha < \mu^+ \), then, for every \( p \in P_\alpha \), let
\[
c_\alpha(p) := \{(\phi_\alpha(\gamma), c_\gamma(p \upharpoonright \gamma)) \mid \gamma \in B_p]\.
\]
If \( \alpha = \mu^+ \), then, given \( p \in P_\alpha \), first let \( C := \text{cl}(B_p) \), then define a function
\( e: C \rightarrow H_\mu \) by stipulating:
\[
e(\gamma) := (\phi_\gamma[C \cap \gamma], c_\gamma(p \upharpoonright \gamma)),
\]
and then let \( c_\alpha(p) := i \) for the least \( i < \mu \) such that \( e \subseteq e^i \).

Finally, let \( p \in P_\alpha \), \( \gamma \leq \alpha \) and \( r \leq \gamma p \upharpoonright \gamma \) be arbitrary: we need to define \( n_{\alpha,\gamma}(p)(r) \). For \( \gamma = \alpha \), let \( n_{\alpha,\gamma}(p)(r) := r \), and for \( \gamma < \alpha \), let
\[
n_{\alpha,\gamma}(p)(r) := \bigcup \{n_{\beta,\gamma}(p \upharpoonright \beta)(r) \mid \gamma \leq \beta < \alpha\}.
\]

**Convention 3.4.** Even though \((P_0, \ell_0)\) is not a graded poset, in order to smooth up inductive claims that come later, we define \( \leq_0 \) to be \( \leq_0 \), and likewise, for every \( p \in P_0 \), we interpret \((P_0)_0^p\) as \{\( q \in P_0 \mid q \leq_0 p \} \).

### 3.2. Verification

We now verify that for all \( \alpha \leq \mu^+ \), \((P_\alpha, \ell_\alpha, c_\alpha, \langle n_{\alpha,\gamma} \rangle_{\gamma \leq \alpha})\) fulfills requirements (i)–(vi) of Goal 3.2. By the recursive definition given so far, it is obvious that Clauses (i) and (iii) hold, so we focus on the rest. We commence with an expanded version of Clause (vi).

**Lemma 3.5.** For all \( \gamma \leq \alpha \leq \mu^+ \), \( p \in P_\alpha \) and \( r \in P_\gamma \) with \( r \leq \gamma p \upharpoonright \gamma \), if we let \( q := n_{\alpha,\gamma}(p)(r) \), then:

1. \( q \upharpoonright \beta = n_{\beta,\gamma}(p \upharpoonright \beta)(r) \) for all \( \beta \in [\gamma, \alpha] \);
2. \( B_q = B_p \cup B_r \);
3. \( q \upharpoonright \gamma = r \);
4. If \( \gamma = 0 \), then \( q = p \);
5. \( p = (p \upharpoonright \gamma) * \emptyset_\alpha \) iff \( q = r * \emptyset_\alpha \);
6. For all \( p' \leq_{\alpha} p \), if \( r \leq_{\gamma} p' \upharpoonright \gamma \), then \( n_{\alpha,\gamma}(p')(r) \leq_{\alpha} n_{\alpha,\gamma}(p)(r) \).

**Proof.** Clause (3) follows from Clause (1) and the fact that \( n_{\gamma,\gamma}(p \upharpoonright \gamma) \) is the identity function. Clause (5) follows from Clauses (2) and (3).

We now prove Clauses (1), (2), (4) and (6) by induction on \( \alpha \leq \mu^+ \):

- The case \( \alpha = 0 \) is trivial, since, in this case, all the conditions under consideration (and their corresponding \( B \)-sets) are empty, and all the maps under consideration are the identity.
- The case \( \alpha = 1 \) follows from the fact that, by definition, \( n_{1,0}(p)(r) = p \) and \( n_{1,1}(p)(r) = r \).
- Suppose \( \alpha \geq 2 \) is a successor ordinal, say \( \alpha = \alpha' + 1 \), and that the claim holds for \( \alpha' \). Fix arbitrary \( \gamma \leq \alpha \), \( p \in P_\alpha \) and \( r \in P_\gamma \) with \( r \leq \gamma p \upharpoonright \gamma \). Denote \( q := n_{\alpha,\gamma}(p)(r) \). Recall that \( P_\alpha = P_{\alpha' + 1} \) was defined by feeding \((P_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})\) into Building Block II, thus obtaining a \( \Sigma \)-Prikry triple \((A, \ell_A, c_A)\) along with a forking projection \( (\dot{\gamma}, \pi) \), such that each condition in the poset \( A = (A, \leq) \) is a pair \( (x, y) \) with \( \pi(x, y) = x \). Furthermore, by the definition of \( n_{\alpha,\gamma} \), \( q = n_{\alpha,\gamma}(p)(r) \) is equal to \( x^\gamma(y) \), where
\[
(x, y) := (\dot{\gamma}(p \upharpoonright \alpha', p(\alpha')))(\dot{\gamma}(p \upharpoonright \alpha')(r)).
\]
In particular, $q \upharpoonright \alpha' = x = \pi(\Diamond(p \upharpoonright \alpha', p(\alpha'))(\Diamond_{\alpha', \gamma}(p \upharpoonright \alpha') (r)))$, which, by Definition 2.13(5), is equal to $\Diamond_{\alpha', \gamma}(p \upharpoonright \alpha') (r)$.

(1) It follows that, for all $\beta \in [\gamma, \alpha)$,
\[
q \upharpoonright \beta = (q \upharpoonright \alpha') \upharpoonright \beta = \Diamond_{\alpha', \gamma}(p \upharpoonright \alpha') (r) \upharpoonright \beta = \Diamond_{\beta, \gamma}(p \upharpoonright \beta) (r),
\]
where the rightmost equality follows from Lemma 2.17. Altogether, the case $\beta = \alpha$ is trivial.

(2) To avoid trivialities, assume $\gamma < \alpha$. By Clause (1), $q \upharpoonright \alpha' = \Diamond_{\alpha, \gamma}(p \upharpoonright \alpha') (r)$. So, by the induction hypothesis, $B_{q|\alpha'} = B_{p|\alpha'} \cup B_r$, and we are left with showing that $\alpha \in B_q$ iff $\alpha \in B_p$. As $q \leq_p p$, we have $B_q \supseteq B_p$, so the forward implication is clear. Finally, if $\alpha \notin B_p$, then $p(\alpha') = 0$, and hence

\[
(x, y) = \Diamond(p \upharpoonright \alpha', \emptyset)(\Diamond_{\alpha', \gamma}(p \upharpoonright \alpha')(r)).
\]

It thus follows from Clause (f) of Building Block II together with the fact that $\Diamond$ satisfies Clause (6) of Definition 2.13 that $(x, y) = \Diamond_{\alpha', \gamma}(p \upharpoonright \alpha')(r), \emptyset)$. Recalling that $q = x^\frown y$, we conclude that $\alpha \notin B_q$, as desired.

(4) If $\gamma = 0$, then, by the induction hypothesis, $\Diamond_{\alpha, \gamma}(p \upharpoonright \alpha')(r) = p \upharpoonright \alpha'$, so that
\[
(x, y) = \Diamond(p \upharpoonright \alpha', p(\alpha'))(\Diamond_{\alpha, \gamma}(p \upharpoonright \alpha')(r)) = \Diamond(p \upharpoonright \alpha', p(\alpha'))(p \upharpoonright \alpha') = (p \upharpoonright \alpha', p(\alpha')) = (x, y),
\]
where the rightmost equality follows from Lemma 2.17. Altogether, $q = x^\frown y = p$.

(6) To avoid trivialities, assume that $\Diamond_{\alpha, \gamma}(p')(r) \neq \Diamond_{\alpha, \gamma}(p)(r)$, so that $\gamma < \alpha$. By Clause (4), we may also assume that $0 < \gamma$. Fix $p' \leq_{\gamma} p$ with $r \leq_{\gamma} p' \upharpoonright \gamma$. By the definition of $\leq_{\alpha' + 1}$, proving $\Diamond_{\alpha, \gamma}(p')(r) \leq_{\alpha'} \Diamond_{\alpha, \gamma}(p)(r)$ amounts to verifying that $(x', y') \leq (x, y)$, where
\[
(x', y') := \Diamond(p' \upharpoonright \alpha', p'(\alpha'))(\Diamond_{\alpha', \gamma}(p' \upharpoonright \alpha')(r)).
\]

Now, by the induction hypothesis, $\Diamond_{\alpha', \gamma}(p' \upharpoonright \alpha')(r) \leq_{\alpha'} \Diamond_{\alpha, \gamma}(p)(r)$. So, since $\Diamond(p \upharpoonright \alpha', p(\alpha'))$ is order-preserving, it suffices to prove that
\[
(x', y') \leq (p \upharpoonright \alpha', p(\alpha'))(\Diamond_{\alpha', \gamma}(p' \upharpoonright \alpha')(r)).
\]

Denote $a := (p \upharpoonright \alpha', p(\alpha'))$ and $a' := (p' \upharpoonright \alpha', p'(\alpha'))$. Then, by Clause (7) of Definition 2.13, indeed
\[
\Diamond(a)(\Diamond_{\alpha', \gamma}(p' \upharpoonright \alpha')(r)) \leq \Diamond(a)(\Diamond_{\alpha', \gamma}(p' \upharpoonright \alpha')(r)).
\]

\[
\Box
\]

Suppose $\alpha \in \text{acc}(\mu^+ + 1)$ is an ordinal such that, for all $\alpha' < \alpha$, $\beta \in [\gamma, \alpha']$, $p \in P_{\alpha'}$ and $r \in P_{\gamma}$ with $r \leq_{\gamma} p \upharpoonright \gamma$,
\[
\Diamond_{\beta, \gamma}(p \upharpoonright \beta)(r) = (\Diamond_{\alpha', \gamma}(p \upharpoonright \alpha')(r)) \upharpoonright \beta.
\]
Claim 3.6.1. Our next task is to verify Clauses (ii) and (v) of Goal 3.2:

**Lemma 3.6.** Suppose that \( \alpha, \beta < \mu \) and \( p \in \mathcal{P} \) are such that \( \pi_{\alpha, \gamma}(p) = p \) and \( \pi_{\alpha, \gamma}(q) = q \). Then:

- for all nonzero \( \gamma \leq \alpha \), \( \pi_{\alpha, \gamma} \) is a forking projection from \( \mathcal{P} \) to \( \mathcal{P}_{\gamma} \), where \( \pi_{\alpha, \gamma} \) is defined as in Goal 3.2(ii);
- if \( \alpha < \mu \) and \( \pi_{\alpha, \gamma} \) is furthermore a forking projection from \( \mathcal{P} \) to \( \mathcal{P}_{\gamma} \), then \( \pi_{\alpha, \gamma} \) has the weak mixing property.

**Proof.** Let us go over the clauses of Definition 2.13.

Clause (5) is covered by Lemma 3.5(3), and Clause (7) is covered by Lemma 3.5(6). Clause (3) is obvious, since for all nonzero \( \gamma < \alpha \) and \( p \in \mathcal{P}_{\gamma} \), a straight-forward verification makes it clear that \( p \vdash 0_{\alpha} \) is the greatest element of \( \{ q \in \mathcal{P}_\alpha | \pi_{\alpha, \gamma}(q) = p \} \). In effect, Clause (6) follows from Lemma 3.5(5).

Thus, we are left with verifying Clauses (1), (2), (4) and (8). The next claim takes care of the first three.

**Claim 3.6.1.** For all nonzero \( \gamma < \alpha \) and \( p \in \mathcal{P}_{\alpha} \):

1. \( \pi_{\alpha, \gamma} \) forms a projection from \( \mathcal{P}_{\alpha} \) to \( \mathcal{P}_{\gamma} \), and \( \ell_{\alpha} = \ell_{\gamma} \circ \pi_{\alpha, \gamma} \);
2. \( \pi_{\alpha, \gamma}(p) \) is an order-preserving function from \( \langle \mathcal{P}_{\gamma} \downarrow (p \uparrow \gamma), \leq_{\gamma} \rangle \) to \( \langle \mathcal{P}_{\alpha} \downarrow p, \leq_{\alpha} \rangle \);
3. for all \( n, m < \omega \) and \( q \leq_{\alpha, \gamma}^{n+m} p, m(p, q) \) exists and, furthermore,

\[
m(p, q) = \pi_{\alpha, \gamma}(p)(m(p \uparrow \gamma, q \uparrow \gamma)).
\]

**Proof.** We commence by proving (2) and (3) by induction on \( \alpha \leq \mu^+ \):

- The case \( \alpha = 1 \) is trivial, since, in this case, \( \gamma = \alpha \).
- Suppose \( \alpha = \alpha' + 1 \) is a successor ordinal and that the claim holds for \( \alpha' \). Let \( \gamma < \alpha \) be arbitrary. To avoid trivialities, assume \( \gamma < \alpha \). By the induction hypothesis, \( \pi_{\alpha', \gamma}(p \uparrow \alpha') \) is an order-preserving function from \( \mathcal{P}_{\gamma} \downarrow (p \uparrow \gamma) \) to \( \mathcal{P}_{\alpha'} \downarrow (p \uparrow \alpha') \).

Recall that \( \mathcal{P}_{\alpha} = \mathcal{P}_{\alpha' + 1} \) was defined by feeding \( \langle \mathcal{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'} \rangle \) into Building Block II, thus obtaining a \( \Sigma \)-Prikry triple \( \langle \mathcal{P}, \ell_{\alpha'}, c_{\alpha'} \rangle \) along with the pair \( \langle \pi, \pi_{\alpha} \rangle \). Now, as \( \pi(p \uparrow \alpha', p(\alpha')) \) and \( \pi_{\alpha', \gamma}(p \uparrow \alpha') \) are both order-preserving, the very definition of \( \pi_{\alpha, \gamma}(p \uparrow \gamma) \) and \( \leq_{\alpha', 1} \) implies that \( \pi_{\alpha, \gamma}(p \uparrow \gamma) \) is order-preserving. In addition, as \( (x, y) \) is
a condition in $A$ iff $x^\gamma(y) \in P_\alpha$ and as $\eta(p \upharpoonright \alpha', p(\alpha'))$ is an order-preserving function from $P_{\alpha'} \downarrow (p \upharpoonright \alpha')$ to $A \downarrow (p \upharpoonright \alpha', p(\alpha'))$, we infer that, for all $r \leq \gamma p \upharpoonright \gamma$, $\eta_{\alpha,\gamma}(p \upharpoonright \gamma)(r)$ is in $P_\alpha \downarrow p$.

Let $q \leq_n p$ for some $n, m < \omega$. Let

$$(x, y) := m((p \upharpoonright \alpha', p(\alpha')),(q \upharpoonright \alpha', q(\alpha'))).$$

Trivially, $m(p,q)$ exists and is equal to $x^\gamma(y)$. We need to show that

$$m(p,q) = \eta_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma)).$$

By Definition 2.13(4),

$$(x, y) = \eta(p \upharpoonright \alpha', p(\alpha'))(m(p \upharpoonright \alpha', q \upharpoonright \alpha')).$$

By the induction hypothesis,

$$m(p \upharpoonright \alpha', q \upharpoonright \alpha') = \eta_{\alpha',\gamma}(p \upharpoonright \alpha')(m(p \upharpoonright \gamma, q \downharpoonright \gamma)).$$

and so it follows that

$$(x, y) = \eta(p \upharpoonright \alpha', p(\alpha'))(\eta_{\alpha',\gamma}(p \upharpoonright \alpha')(m(p \upharpoonright \gamma, q \downharpoonright \gamma))).$$

Thus, by the definition of $\eta_{\alpha,\gamma}$ and the above equation, we have that $\eta_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \downharpoonright \gamma))$ is indeed equal to $x^\gamma(y)$.

Suppose $\alpha \in \text{acc}(\mu^* + 1)$ is an ordinal for which the claim holds below $\alpha$. Let $\gamma \leq \alpha$ and $p \in P_\alpha$ be arbitrary. To avoid trivialities, assume $\gamma < \alpha$. By Lemma 3.5(1), for every $r \in P_\gamma \downarrow (p \upharpoonright \gamma)$:

$$\eta_{\alpha,\gamma}(p)(r) = \bigcup_{\gamma \leq \gamma' < \alpha} \eta_{\alpha',\gamma}(p \upharpoonright \alpha')(r).$$

As for all $q, q' \in P_\alpha$, $q \leq \alpha q'$ iff $\forall \alpha' < \alpha(q(\alpha') \leq \alpha q'(\alpha'))$, the induction hypothesis implies that $\eta_{\alpha,\gamma}(p)$ is an order-preserving function from $P_\gamma \downarrow (p \upharpoonright \gamma)$ to $P_\alpha \downarrow p$.

Finally, let $q \leq \alpha p$; we shall show that $m(p,q)$ exists and is, in fact, equal to $\eta_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \downharpoonright \gamma))$. By Lemma 3.5(1) and the induction hypothesis,

$$\eta_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \downharpoonright \gamma)) = \bigcup_{\gamma \leq \gamma' < \alpha} m(p \upharpoonright \alpha', q \upharpoonright \alpha'),$$

call it $r$. We shall show that $r$ plays the role of $m(p,q)$.

By the definition of $\leq \alpha$, it is clear that $q \leq^m r \leq^\alpha p$, so it remains to show that it is the greatest condition in $(P_\alpha^p)_\alpha$ to satisfy this. Fix an arbitrary $s \in (P_\alpha^p)_\alpha$ with $q \leq^\alpha s$. For each $\alpha' < \alpha$, $q \upharpoonright \alpha' \leq^\alpha s \upharpoonright \alpha'$, so that $s \upharpoonright \alpha' \leq^\alpha m(p \upharpoonright \alpha', q \upharpoonright \alpha')$, and thus $s \leq \alpha r$. Altogether this shows that $r = m(p,q)$.

This completes the proof of Clauses (2) and (3) above.

We are left to prove (1). The case $\gamma = \alpha$ is trivial, so assume $\gamma < \alpha$. Clearly, $\pi_{\alpha,\gamma}$ is order-preserving and also $\pi_{\alpha,\gamma}(\emptyset_\alpha) = \emptyset_\gamma$. Let $p \in P_\alpha$ and $q \in P_\beta$ be such that $q \leq \alpha \pi_{\alpha,\gamma}(p)$. Set $q^* := \eta_{\alpha,\gamma}(p)(q)$. By Lemma 3.5(3), $\pi_{\alpha,\gamma}(q^*) = q$ and by Clause (2) of this claim, $q^* \leq \alpha p$. Altogether, $\pi_{\alpha,\gamma}$ is indeed a projection. For the second part, recall that, for all $\beta \leq \mu^+$, $\ell_\beta := \ell_1 \circ \pi_{\beta,1}$, hence $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1} = \ell_1 \circ (\pi_{\gamma,1} \circ \pi_{\alpha,\gamma}) = (\ell_1 \circ \pi_{\gamma,1}) \circ \pi_{\alpha,\gamma} = \ell_\gamma \circ \pi_{\alpha,\gamma}$. \[\square\]
We are left with verifying Clause (8) of Definition 2.13 to show that
 \(\langle \pi_{\alpha,\gamma}, \pi_{\alpha,\gamma} \rangle\) is a forking projection from \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)\) to \((\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)\).

**Claim 3.6.2.** Suppose \(\alpha \neq \mu^+\). For all \(p, p' \in P_\alpha\) with \(c_\alpha(p) = c_\alpha(p')\) and all nonzero \(\gamma \leq \alpha:\)

- \(c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)\), and
- \(\pi_{\alpha,\gamma}(p)(r) = \pi_{\alpha,\gamma}(p')(r)\) for every \(r \in (P_\gamma)_0^{\gamma} \cap (P_\gamma)_0^{\gamma}\).

**Proof.** By induction on \(\alpha < \mu^+:\)

- The case \(\alpha = 1\) is trivial, since, in this case, \(\gamma = \alpha\).
- Suppose \(\alpha = \alpha' + 1\) is a successor ordinal and that the claim holds for \(\alpha'.\) Fix an arbitrary pair \(p, p' \in P_\alpha\) with \(c_\alpha(p) = c_\alpha(p')\).

Recall that \(\mathbb{P}_\alpha = \mathbb{P}_{\alpha' + 1}\) was defined by feeding \((\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})\) into Building Block II, thus obtaining a \(\Sigma\)-Prikry triple \((\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})\) along the pair \((\pi, \pi)\). By the definition of \(c_{\alpha' + 1}\), we have

\[c_{\mathbb{A}}(p \upharpoonright \alpha', p(\alpha')) = c_\alpha(p) = c_\alpha(p') = c_{\mathbb{A}}(p' \upharpoonright \alpha', p'(\alpha')).\]

So, as \(\langle \pi, \pi \rangle\) is a forking projection from \((\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})\) to \((\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})\), we have \(c_{\alpha'}(p \upharpoonright \alpha') = c_{\alpha'}(p' \upharpoonright \alpha')\), and, for all \(r \in (P_{\alpha'})_0^{\alpha'} \cap (P_{\alpha'})_0^{\alpha'},\)

\[\pi_{\alpha,\gamma}(p)(r) = \pi_{\alpha,\gamma}(p')(r).\]

Now, as \(c_{\alpha'}(p \upharpoonright \alpha') = c_{\alpha'}(p' \upharpoonright \alpha')\), the induction hypothesis implies that \(c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)\) for all nonzero \(\gamma \leq \alpha'\). In addition, the case \(\gamma = \alpha\) is trivial.

Finally, fix a nonzero \(\gamma \leq \alpha\) and \(r \in (P_{\alpha'})_0^{\alpha'} \cap (P_{\alpha'})_0^{\alpha'},\) and let us prove that \(\pi_{\alpha,\gamma}(p)(r) = \pi_{\alpha,\gamma}(p')(r)\). To avoid trivialities, assume \(\gamma < \alpha.\) It follows from the definition of \(\pi_{\alpha,\gamma}\) that \(\pi_{\alpha,\gamma}(p)(r) = x^{\gamma}(y)\) and \(\pi_{\alpha,\gamma}(p')(r) = x^{\gamma}(y'),\) where:

\[- (x, y) := (p \upharpoonright \alpha', p(\alpha'))(\pi_{\alpha,\gamma}(p \upharpoonright \alpha')(r)),\]

\[- (x', y') := (p' \upharpoonright \alpha', p'(\alpha'))(\pi_{\alpha,\gamma}(p' \upharpoonright \alpha')(r)).\]

But we have already pointed out that the induction hypothesis implies that \(\pi_{\alpha',\gamma}(p(\alpha')(r)) = \pi_{\alpha',\gamma}(p'(\alpha')(r)),\) call it, \(r'.\) So, we just need to prove that \(\pi(p \upharpoonright \alpha', p(\alpha'))(r') = (p' \upharpoonright \alpha', p'(\alpha'))(r')\). But also have \(c_{\alpha'}(p \upharpoonright \alpha', p(\alpha')) = c_{\alpha'}(p) = c_{\alpha'}(p') = c_{\alpha'}(p' \upharpoonright \alpha', p'(\alpha'))\), so, as \(\langle \pi, \pi \rangle\) is a forking projection from \((\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})\) to \((\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})\), Clause (8) of Definition 2.13 implies that \(\pi(p \upharpoonright \alpha', p(\alpha'))(r') = (p' \upharpoonright \alpha', p'(\alpha'))(r')\), as desired.

- Suppose \(\alpha \in \text{acc}(\mu^+)\) is an ordinal for which the claim holds below \(\alpha\). For any condition \(q \in \bigcup_{\alpha' \leq \alpha} P_{\alpha'}\), define a function \(f_q : B_{\mu} \rightarrow H_{\mu}\) via \(f_q(\alpha') := c_{\alpha'}(q \upharpoonright \alpha')\). Now, fix an arbitrary pair \(p, p' \in P_{\alpha}\) with \(c_\alpha(p) = c_\alpha(p')\).

By the definition of \(c_\alpha\) this means that

\[\{(\phi_\alpha(\gamma), c_\gamma(p \upharpoonright \gamma)) : \gamma \in B_{p'}\} = \{(\phi_\alpha(\gamma), c_\gamma(p' \upharpoonright \gamma)) : \gamma \in B_{p'}\}.\]

As \(\phi_\alpha\) is injective, \(f_\alpha = f_{p'}.\) Next, let \(\gamma \leq \alpha\) be nonzero; we need to show that \(c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)\). The case \(\gamma = \alpha\) is trivial, so assume \(\gamma < \alpha.\)
Claim 3.6.3. Once again, recall that property.

\[ \alpha \vdash \rho \] and \( \langle \subseteq \rangle \)

is clear that \( \text{tp} \) be the witnesses to the weak mixing property of \( \langle \subseteq \rangle \) this next.

For each nonzero \( \alpha \), \( \langle \subseteq \rangle \) has the weak mixing property. Let \( \text{tp} \) be a type over \( \langle \subseteq \rangle \) and \( \delta \) of Building Block II, and for a limit ordinal \( \gamma \), this follows from Clauses (f) and (g) of Building Block II, and for a successor ordinal \( \gamma \), this follows from the fact that the injectivity of \( \varphi \) and the equality \( f_{\delta,\gamma} = f_{\delta} = f_{\delta,\gamma} \) implies that \( c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma) \).

Finally, fix a nonzero \( \gamma \leq \alpha \) and \( r \in (P_0)_{\gamma,\alpha} \cap (P_0')_{\gamma,\alpha} \), and let us prove that \( \hat{\varphi}_{\alpha,\gamma}(p)(r) = \hat{\varphi}_{\alpha,\gamma}(p')(r) \). To avoid trivialities, assume \( \gamma < \alpha \). We already know that, for all \( \alpha' \in [\gamma,\alpha), c_{\alpha'}(p \upharpoonright \alpha') = c_{\alpha'}(p' \upharpoonright \alpha') \), and so the induction hypothesis implies that \( \hat{\varphi}_{\alpha,\gamma}(p \upharpoonright \alpha')(r) = \hat{\varphi}_{\alpha',\gamma}(p'(\alpha')(r), \) and then by Lemma 3.5.1:

\[
\hat{\varphi}_{\alpha,\gamma}(p)(r) = \bigcup_{\gamma \leq \alpha' < \alpha} \hat{\varphi}_{\alpha',\gamma}(p \upharpoonright \alpha')(r) = \bigcup_{\gamma \leq \alpha' < \alpha} \hat{\varphi}_{\alpha',\gamma}(p'(\alpha')(r),
\]

as desired.

Claim 3.6.3. If \( 0 < \alpha < \mu^+ \) then \( \langle \hat{\varphi}_{\alpha+1,\alpha}, \pi_{\alpha+1,\alpha} \rangle \) has the weak mixing property.

Proof. Once again, recall that \( P_{\alpha+1} \) was defined by feeding \( \langle P_0, \ell_0, c_0 \rangle \) into Building Block II, thus obtaining a \( \Sigma \)-Prikry triple \( \langle A, \ell_1, c_1 \rangle \), along with a pair \( \langle \hat{\varphi}, \pi \rangle \) having the weak mixing property. Let \( \text{tp} \) be a type over \( \langle \hat{\varphi}, \pi \rangle \) and \( \langle \exists n \mid n < \omega \rangle \) be a sequence of orderings witnessing this. For each \( p \in P_{\alpha+1} \), set \( \text{tp}_{\alpha+1}(p) := \text{tp}(p \upharpoonright \alpha, p(\alpha)) \). Also, for each \( n < \omega \), derive

\[
\exists n_{\alpha+1} := \{(p, q) \in P_{\alpha+1} \times P_{\alpha+1} \mid (p \upharpoonright \alpha, p(\alpha)) \exists n (q \upharpoonright \alpha, q(\alpha))\}.
\]

The canonical isomorphism from \( A \) to \( P_{\alpha+1} \) (i.e., \( (x, y) \mapsto x \upharpoonright (y) \)) makes it clear that \( \text{tp}_{\alpha+1} \) and \( \langle \exists n_{\alpha+1} \mid n < \omega \rangle \) witness together that \( \langle \hat{\varphi}_{\alpha+1,\alpha}, \pi_{\alpha+1,\alpha} \rangle \) has the weak mixing property.

This completes the proof of Lemma 3.6. 

Definition 3.7. For each nonzero \( \alpha < \mu^+ \), we let \( \text{tp}_{\alpha+1} \) and \( \langle \exists n_{\alpha+1} \mid n < \omega \rangle \) be the witnesses to the weak mixing property of \( \langle \hat{\varphi}_{\alpha+1,\alpha}, \pi_{\alpha+1,\alpha} \rangle \), as defined in the proof of Subclaim 3.6.3.

Recalling Definition 2.3.2(1), for all nonzero \( \alpha \leq \mu^+ \) and \( n < \omega \), we need to identify a candidate for a dense subposet \( \check{P}_{\alpha n} = (P_{\alpha n}, \leq_\alpha) \) of \( P_{\alpha n} \). We do this next.
Definition 3.8. Let $n < \omega$. Set $\hat{P}_n := 1(\hat{Q}_n)$.

Then, for each $\alpha \in [2, \mu^+]$, define $\hat{P}_{\alpha n}$ by recursion:

$$\hat{P}_{\alpha n} := \begin{cases} \{ p \in P_\alpha \mid \pi_{\alpha,\beta}(p) \in \hat{P}_{\beta n} \& \text{mtp}_{\beta+1}(p) = 0 \}, & \text{if } \alpha = \beta + 1; \\ \{ p \in P_\alpha \mid \pi_{\alpha,1}(p) \in \hat{P}_n \& \forall \gamma \in B_p \text{ mtp}_\gamma(\pi_{\alpha,\gamma}(p)) = 0 \}, & \text{otherwise}. \end{cases}$$

Lemma 3.9. Let $n < \omega$ and $1 \leq \beta < \alpha \leq \mu^+$. Then:

1. $\pi_{\alpha,\beta}^*\hat{P}_{\alpha n} \subseteq \hat{P}_{\beta n}$;
2. For every $p \in \hat{P}_{\beta n}$, $p * \emptyset_\alpha \in \hat{P}_{\alpha n}$.

Proof. By straight-forward induction, relying on Clause (4) of Definition 2.23.

We are now left with addressing Clause (iv) of Goal 3.2. Prior to that we will provide a sufficient condition securing that for each $\alpha \in \text{acc}(\mu^+ + 1)$, the pair $(P_\alpha, \ell_\alpha)$ has property $D$. For this, we establish a version of the Weak Mixing Property (see Definition 2.25) for limit stages.

Lemma 3.10. Let $\alpha \in \text{acc}(\mu^+ + 1)$. For all $\alpha \in P_\alpha$, $n < \omega$, $\bar{r}$, and $p' \leq^0 \pi_{\alpha,1}(a)$ and for every function $g : W_n(\pi_{\alpha,1}(a)) \rightarrow P_\alpha \downarrow a$, if all of the following hold:

1. $(B_{g(\bar{r})} \mid \xi < \chi)$ is $\subseteq$-weakly increasing. Put $B := \bigcup_{\xi < \chi} B_{g(\bar{r})}$, and for each $\gamma \in B$, let $\iota_\gamma := \min\{ \xi < \chi \mid \gamma \in B_{g(\bar{r})} \}$;
2. $\bar{r} = (r_\xi \mid \xi < \chi)$ is a good enumeration of $W_n(\pi_{\alpha,1}(a))$;
3. $\langle \pi_{\alpha,1}(g(\bar{r})) \mid \xi < \chi \rangle$ is diagonalizable with respect to $\bar{r}$ as witnessed by $p'$;
4. For all $\gamma \in B$, $\xi \in (\iota_\gamma, \chi)$, and $i \in \{ \text{dom}(\text{tp}_\gamma(\pi_{\alpha,\gamma}(a))), \sup_{\eta < \xi} \text{dom}(\text{tp}_\gamma(\pi_{\alpha,\gamma}(g(\bar{r}_\eta)))) \}$, $\text{tp}_\gamma(\pi_{\alpha,\gamma}(g(\bar{r}_\xi)))(i) \leq \text{mtp}_\gamma(\pi_{\alpha,\gamma}(a))$;
5. For all $\gamma \in B$, $\sup_{\eta \leq \xi < \chi} \text{mtp}_\gamma(\pi_{\alpha,\gamma}(g(\bar{r}_\xi))) < \omega$, then there exists $b \in P_\alpha$ such that:

(a) $\pi_{\alpha,1}(b) = p'$;
(b) For all $\gamma \in B_\alpha$, $\pi_{\alpha,\gamma}(b) \subseteq \pi_{\alpha,\gamma}(a)$;
(c) For all $q' \in W_n(p')$, $\pi_{\alpha,1}(b)(q') \leq^0 g(w(\pi_{\alpha,1}(a), q'))$.

Proof. Let $a \in P_\alpha$, $n < \omega$, $\bar{r}$, $p'$ and $g : W_n(a \upharpoonright 1) \rightarrow P_\alpha \downarrow a$ be as above. Let $\langle \gamma_\tau \mid \tau < \theta \rangle$ be the increasing enumeration of $B = \bigcup_{\xi < \chi} B_{\bar{r}_\xi}$. From Goal 3.2(i) and $\chi < \mu$, we infer that $\theta < \mu$. For each $\tau < \theta$:

- as $\gamma_\tau$ is a successor ordinal, we let $\beta_\tau$ denote its predecessor;
Proof. By (II) above, we may let $\text{dom}(r^\gamma) := \langle r^\gamma_\xi : \xi < \chi \rangle$ is a good enumeration of $W_\alpha(a \upharpoonright \beta_\gamma)$. 

By Fact 2.19, $r^\gamma := \langle r^\gamma_\xi : \xi < \chi \rangle$ is a good enumeration of $W_\alpha(a \upharpoonright \beta_\gamma)$; 

- derive a map $g_\tau : W_\alpha(a \upharpoonright \beta_\gamma) \rightarrow P_{\gamma_\tau} \downarrow (a \upharpoonright \gamma_\tau)$ via 

$$g_\tau(r^\gamma_\xi) := g(r^\gamma_\xi) \upharpoonright \gamma_\tau.$$ 

Claim 3.10.1. Suppose there is a sequence $\langle (b_\tau, p^\tau) \mid \tau < \theta \rangle \in \prod_{\tau < \theta} (P_{\gamma_\tau} \times P_{\beta_\gamma})$ satisfying that for all $\tau < \theta$:

(I) $b_0 \upharpoonright 1 = p^0 \upharpoonright 1 = p'$. 

(II) $b_\tau \upharpoonright \tau' = b_{\tau'}$ for all $\tau' < \tau$. 

(III) $b_\tau$ witnesses the conclusion of Definition 2.25 with respect to the tuple $(a \upharpoonright \gamma_\tau, r^\gamma, p^\gamma, g_\tau, \iota_{\gamma_\tau})$. In particular, $p^\gamma \leq^0_{\beta_\gamma} (a \upharpoonright \gamma_\tau)$ diagonalizes $(g_\tau(r^\gamma_\xi) \upharpoonright \gamma_\tau \mid \xi < \chi)$.

Then there is $b \in P_\alpha$ as in the conclusion of the Lemma.

Proof. By (II) above, we may let $b^* := \bigcup_{\tau < \theta} b_\tau$, so that $b^* \in P_\delta$ for $\delta := \text{dom}(b^*)$. For each $\tau < \theta$, Clause (III) yields 

$$b^* \upharpoonright \gamma_\tau = b_\tau \upharpoonright \gamma_\tau \upharpoonright \alpha \upharpoonright \gamma_\tau,$$

and hence $b^* \leq_\delta^0 (a \upharpoonright \delta)$. So we may let $b := \check{\eta}_{\alpha, \delta}(b^*)$, and infer from (I) that $b \upharpoonright 1 = p'$. Also, we have that $b \upharpoonright \gamma \upharpoonright a \upharpoonright \gamma_\tau$ for each $\gamma \in B_\alpha$. This shows that Clauses (a) and (b) of the lemma hold.

We are now left with verifying Clause (c). Let $q' \in W_\alpha(p')$; we want to show that $\check{\eta}_{\alpha, 1}(b)(q') \leq_\gamma^0 g(w(a \upharpoonright 1, q'))$. Note that by Lemma 3.5(2), $B_\alpha \subseteq B_{\beta_\gamma} = B_\delta$, so that $b = b^* \upharpoonright \emptyset_\alpha$. Hence, $\{ \gamma_\tau \mid \tau < \theta \}$ is cofinal in $B_\alpha$, and so it suffices to prove that, for each $\tau < \theta$, 

$$\check{\eta}_{\alpha, 1}(b)(q') \upharpoonright \gamma_\tau \leq_{\gamma_\tau}^0 g(w(a \upharpoonright 1, q')) \upharpoonright \gamma_\tau.$$ 

For each $\tau < \theta$, combining Clause (II) with Lemma 3.5(1) we have 

$$\check{\eta}_{\alpha, 1}(b)(q') \upharpoonright \gamma_\tau = \check{\eta}_{\gamma_\tau, 1}(b \upharpoonright \gamma_\tau)(q') = \check{\eta}_{\gamma_\tau, 1}(b_\tau)(q'),$$

hence it suffices to check that 

(*): 

$$\check{\eta}_{\gamma_\tau, 1}(b_\tau)(q') \leq_{\gamma_\tau}^0 g(w(a \upharpoonright 1, q')) \upharpoonright \gamma_\tau.$$

By (*) from Page 16 it is not hard to check that 

(**): 

$$\check{\eta}_{\gamma_\tau, 1}(b_\tau)(q') = \check{\eta}_{\gamma_\tau, \beta_\gamma}(b_\gamma)(\check{\eta}_{\beta_\gamma, 1}(b_\tau \upharpoonright \beta_\gamma)(q')).$$

Since $b_\tau \upharpoonright 1 = p'$ and $q' \in W_\alpha(p')$, Lemma 3.6 yields that $r := \check{\eta}_{\beta_\gamma, 1}(b_\tau \upharpoonright \beta_\gamma)(q')$ is in $W_\alpha(b_\tau \upharpoonright \beta_\gamma)$. Combining equation (**) with (III), we infer that 

$$\check{\eta}_{\gamma_\tau, 1}(b_\tau)(q') = \check{\eta}_{\gamma_\tau, \beta_\gamma}(b_\gamma)(r) \leq_{\gamma_\tau}^0 g_\tau(w(a \upharpoonright \beta_\gamma, r)) = g_\tau(w(a \upharpoonright 1, q')) \upharpoonright \gamma_\tau,$$

where the rightmost equality follows from the definition of $g_\tau$ and the fact that $r \upharpoonright 1 = q'$. This verifies equation (*) and yields the claim. 

Let us now argue by induction that such $\langle (b_\tau, p^\tau) \mid \tau < \theta \rangle$ exists.

Claim 3.10.2. There is a pair $(b_0, p^0)$ for which Clauses (I)–(III) hold.
Proof. Clause (II) is trivial at this stage. Setting \( p^0 := \bar{n}_{\beta_0,1} (a \upharpoonright \beta_0)(p') \) takes care of the second part of Clause (I), and we shall come back to the first part towards the end. Now, let us examine the tuple \( (a, p', g, \tau_0, \gamma_0) \) against the clauses of Definition 2.25 with respect to the forking projection \( (\bar{n}_{\gamma_0, \beta_0}, g_{\gamma_0, \beta_0}) \): Clause (1) is obvious and Clauses (3), (4) and (5) follow combining the corresponding clauses in the lemma with the definition of \( g_0 \).

Regarding Clause (2), we claim that \( p^0 \) diagonalizes \( \langle g_0(r_\xi^0) \upharpoonright \beta_0 \mid \xi < \chi \rangle \).

To that effect we will check (\( \alpha \)) and (\( \beta \)) of Proposition 2.22, when this is regarded with respect to the forking projection \( (\bar{n}_{\beta_0,1}, \pi_{\beta_0,1}) \), and the parameters \( a \upharpoonright \beta_0, p^0, \langle g_0(r_\xi^0) \upharpoonright \beta_0 \mid \xi < \chi \rangle, p' \) and \( p^0 \), respectively.

(\( \alpha \)) Note that \( g_0(r_\xi^0) \upharpoonright 1 = g(r_\xi) \upharpoonright 1 \), for each \( \xi < \chi \). Therefore, Clause (1) implies that \( p' \) diagonalizes \( \langle g_0(r_\xi^0) \upharpoonright 1 \mid \xi < \chi \rangle \).

(\( \beta \)) Note that by Clause (1) of the lemma, \( p' \leq_\beta a \upharpoonright 1 \), hence \( p^0 \leq_\beta a = \beta_0 \).

Let \( q' \in W_\beta (p') \). Again by Clause (1), \( q' \leq_1 \langle g(r_\xi) \upharpoonright 1, \xi \rangle \), where \( \xi \) is the unique index such that \( r_\xi = w(a \upharpoonright 1, q') \).

Finally, combining Lemma 3.5(5) and Lemma 3.6 we have

\[
\bar{n}_{\beta_0,1}(p^0)(q') \leq_\beta \bar{n}_{\beta_0,1}(a \upharpoonright \beta_0)(g(r_\xi) \upharpoonright 1) = g(r_\xi) \upharpoonright 1 \upharpoonright \beta_0 = g_0(r_\xi^0) \upharpoonright \beta_0,
\]

where the above equalities follow from \( \beta_0 < \min(\bigcup_{\xi < \chi} B_{g(r_\xi)}) \).

Altogether, \( g_0 \) witnesses Clauses (1)–(5). Thus, appealing to Lemma 3.6, we obtain \( b \in P_{\gamma_0} \) such that \( b \upharpoonright \beta_0 = p^0 \) and \( b \supseteq_{\gamma_0} a \upharpoonright \gamma_0 \) that witnesses the conclusion of Definition 2.25. Clearly, \( b_0 := b \) and \( p^0 \) are as wanted. \( \square \)

Suppose now \( \tau < \theta \), and that \( \langle (b_{\tau'}, p_{\tau'}) \mid \tau' < \tau \rangle \) has been constructed maintaining (I)–(III). Set \( b^* := \bigcup_{\tau' < \tau} b_{\tau'} \) and \( \delta := \text{dom}(b^*) \). Note that \( \delta \leq \beta_{\tau} \), as \( \gamma_{\tau} \in \text{nacc} (\mu^+) \). Also, using (I) and (II) of the induction, \( b^* \in P_{\delta} \) and \( \pi_{\beta_0,1}(b^*) = p' \).

Claim 3.10.3. There is a pair \( (b_{\tau}, p_{\tau}) \) satisfying Clause (III).

Proof. As in the previous claim, it suffices to show that

\[
p_{\tau} := \bar{n}_{\beta_{\tau}, \delta}(a \upharpoonright \beta_{\tau})(b^*)
\]

diagonalizes \( \langle g_{\tau}(r_\xi^\tau) \upharpoonright \beta_{\tau} \mid \xi < \chi \rangle \).

Once again, we want to appeal to Proposition 2.22, but this time regarded with respect to \( (\bar{n}_{\beta_{\tau}, \pi_{\beta_{\tau},1}}, a \upharpoonright \beta_{\tau}, \bar{\tau}', g_{\tau}(r_\xi^\tau) \upharpoonright \beta_{\tau} \upharpoonright \xi < \chi), p_{\tau} \) and \( p^\tau \).

(\( \alpha \)) The verification is exactly the same as in Claim 3.10.2.

(\( \beta \)) By (II) and (III) of the induction hypothesis, \( b^* \leq_\beta a \upharpoonright \delta \) and \( b^* \upharpoonright 1 = p' \).

Hence, \( p_{\tau} \in P_{\beta_{\tau}}, p_{\tau} \leq_\beta a \upharpoonright \beta_{\tau} \) and \( p_{\tau} \upharpoonright 1 = p' \).

Let \( q' \in W_{\beta_{\tau}} (p_{\tau}) \). Our aim is to show that

\[
\bar{n}_{\beta_{\tau},1}(p_{\tau})(q') \leq_\beta g_{\tau}(r_\xi^\tau) \upharpoonright \beta_{\tau},
\]

for the unique index \( \xi \) such that \( r_\xi = w(a \upharpoonright 1, q') \).

By virtue of Lemma 3.5(5), \( B_{\bar{n}_{\beta_{\tau},1}(p_{\tau})}(q') = B_{p_{\tau}} = B_{p'} \). Hence, it will be enough to check that \( \bar{n}_{\beta_{\tau},1}(p_{\tau})(q') \upharpoonright \delta \leq_\delta g_{\tau}(r_\xi^\tau) \upharpoontright \delta. \)
Proof. By Claim 3.10.3 and (II) of the induction hypothesis with Clauses (1) and (3) of Lemma 3.5 we have
\[ \langle \beta_{\tau'}, 1 \rangle (p_{\tau'})(q') | \gamma_{\tau'} = \langle \gamma_{\tau'}, 1 \rangle (b_{\tau'})(q') = \langle \gamma_{\tau'}, \beta_{\tau'} \rangle (s_{\tau'}), \]
where \( s_{\tau'} := \langle \beta_{\tau'}, 1 \rangle (b_{\tau'} \restriction \beta_{\tau'})(q'). \)

Thus, by (III) of our induction hypothesis,
\[ \langle \beta_{\tau'}, 1 \rangle (p_{\tau'})(q') | \gamma_{\tau'} = \langle \gamma_{\tau'}, \beta_{\tau'} \rangle (b_{\tau'})(s_{\tau'}) \leq_{\gamma_{\tau'}}^{0} g_{\tau'}(r_{\xi}^{\tau'}), \]
where \( \xi \) is the unique index such that \( r_{\xi}^{\tau'} = w(a \restriction \beta_{\tau'}, s_{\tau'}). \)

Since \( g_{\tau'}(r_{\xi}^{\tau'}) | \gamma_{\tau'} = g_{\tau'}(r_{\xi}^{\tau'}), \) the above expression actually yields
\[ \langle \beta_{\tau'}, 1 \rangle (p_{\tau'})(q') | \gamma_{\tau'} \leq_{\gamma_{\tau'}}^{0} g_{\tau'}(r_{\xi}^{\tau'}) | \gamma_{\tau'}. \]
Altogether,
\[ \langle \beta_{\tau'}, 1 \rangle (p_{\tau'})(q') | \delta \leq_{\delta} g_{\tau'}(r_{\xi}^{\tau'}) | \delta. \]

Finally, note that
\[ r_{\xi} = r_{\xi}' | 1 = w(a \restriction \beta_{\tau'}, s_{\tau'}) | 1 = w(a \restriction 1, q'), \]
where the last equality follows from Lemma 3.6 and \( s_{\tau'} \restriction 1 = q'. \)

The above shows that \((a \restriction \gamma_{\tau'}, p_{\tau'}, g_{\tau'}, \iota_{\gamma_{\tau'}})\) fulfills the assumptions of Definition 2.25 with respect the pair \((\langle \gamma_{\tau'}, \beta_{\tau'}, \iota_{\gamma_{\tau'}} \rangle)\). Appealing to Lemma 3.6 and Definition 3.7 we obtain \( b_{\tau} \subseteq_{n} a \restriction \gamma_{\tau'} \) with \( b_{\tau} \restriction \beta_{\tau'} = p_{\tau'} \). Thus, \((b_{\tau}, p_{\tau'})\) witnesses (III). □

Let \((b_{\tau}, p_{\tau'})\) be given by Claim 3.10.3. Note that, in particular, \( p_{\tau'} \restriction \delta = b^{*} \).

**Claim 3.10.4.** \( b_{\tau} \) witnesses (I) and (II).

**Proof.** By Claim 3.10.3 and (II) of the induction hypothesis, for each \( \tau' < \tau, \)
\[ b_{\tau} \restriction \gamma_{\tau'} = (p_{\tau} \restriction \delta) \restriction \gamma_{\tau'} = b^{*} \restriction \gamma_{\tau'} = b_{\tau'}. \]
Similarly, by (I) of the induction hypothesis, \( b_{\tau} \restriction 1 = b_{\tau'} \restriction 1 = p'. \)
Altogether, \( b_{\tau} \) witnesses Clauses (I) and (II). □

The above completes the induction and yields the lemma. □

The following technical lemma yields a sufficient condition for the pair \((\mathbb{P}_{\alpha}, \ell_{\alpha})\) to have property \( D \).

**Lemma 3.11.** Let \( \alpha \in \text{acc}(\omega^{+} + 1), a \in P_{\alpha}, n < \omega \) and \( \vec{s} = \langle s_{\xi} | \xi < \chi \rangle \) be a good enumeration of \( W_{n}(a) \). Set \( l := \ell_{\alpha}(a) \). If \((\mathbb{P}_{\alpha})_{l+n} \) forms a dense subposet of \((\mathbb{P}_{\alpha})_{l+n} \), then \( I \) has a winning strategy for the game \( G_{\vec{s}}(a, \vec{s}, (\mathbb{P}_{\alpha})_{l+n}) \) such that by using this strategy, for any outcome \( \langle (a_{\xi}, b_{\xi}) | \xi < \chi \rangle \) of the game, there will be \( b \in P_{\alpha} \) such that:
- \( b \) diagonalizes \( \langle b_{\xi} | \xi < \chi \rangle \);
- for all \( \gamma \in B_{\alpha}, \pi_{\alpha, \gamma}(b) \subseteq_{\gamma}^{n} \pi_{\alpha, \gamma}(a) \).

\[ ^{14} \text{For this latter equality, see equation (**) above. } \]
Proof. Set \( p := \pi_{\alpha,1}(a) \) and \( r_\xi := \pi_{\alpha,1}(s_\xi) \) for each \( \xi < \chi \). By Clauses (4) and (5) of Definition 2.13, \( \mathcal{F} = \langle r_\xi \mid \xi < \chi \rangle \) is a good enumeration of \( W_n(p) \).

We now describe our strategy for \( \mathbf{I} \). Suppose that \( \xi < \chi \) and that \( \langle (a_\eta, b_\eta) \mid \eta < \xi \rangle \) is an initial play of the game \( \mathcal{G}_{\alpha}(a, s, (s_\eta)_{i+n}) \); we need to define \( a_\xi \).

- If \( \xi = 0 \), then let \( p_0 \) the 0th-move of \( \mathbf{I} \) in the game \( \mathcal{G}_{p_1}(p, \overline{r}) \), which is available by virtue of Building Block I. Let \( t_0 \) be with \( t_0 \leq p_0, r_\xi \).

If \( B_\alpha \) is empty, then let \( a_0 := \mathcal{G}_{\alpha,1}(a)(p_0) \) and \( z_0 := \mathcal{G}_{\alpha,1}(a)(t_0) \). Note that \( z_0 \leq a_0, s_0 \), so that \( a_0 \) is a legitimate move for \( \mathbf{I} \).

Suppose now that \( B_\alpha \) is nonempty, and let \( \langle \gamma_\tau \mid \tau \leq \theta \rangle \) be the increasing enumeration of closure of \( B_\alpha \). For every \( \tau \in \text{nacc}(\theta + 1) \), \( \gamma_\tau \) is a successor ordinal, so we let \( \beta_\tau \) denote its predecessor. By recursion on \( \tau < \theta \), we shall define a coherent sequence \( \langle (a_\tau^0, z_\tau^0) \mid \tau \leq \theta \rangle \in \prod_{\tau \leq \theta} (P_{\gamma_\tau} \times P_{\beta_\tau}) \); then we shall let \( a_0 := \mathcal{G}_{\alpha,\gamma_\theta}(a)(a_\theta^0) \) and \( z_0 := \mathcal{G}_{\alpha,\gamma_\theta}(a)(z_\theta^0) \).

The idea is to craft the \( a_\tau^0 \)'s so that for all \( \gamma \in B_\alpha \), \( a_\gamma \mid \gamma \) satisfies (1)–(4) of Lemma 3.10. Also, \( z_\tau^0 \) will be an auxiliary condition witnessing that \( z_\tau^0 \leq \gamma \), \( s_0 \leq a_0, \gamma_\tau \). This will ensure that \( a_0 \) is a legitimate move for \( \mathbf{I} \).

\( \blacktriangleright \triangleright \) Set \( a_0^0 := \text{dom}(\text{tp}_{\gamma_0}(a \mid \gamma_0)) + \omega + 1 \), and then let

\[
\begin{align*}
a_0^0 &:= \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(p_0) \cap \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0) \\
z_0^0 &:= \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0) \cap \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0)
\end{align*}
\]

where the \( \cap \) operation is provided by Definition 2.23(5) with respect to the \( \text{type} \text{tp}_{\gamma_0} \) over \( (\mathcal{G}_{\gamma_0,\beta_0}, \pi_{\gamma_0,\beta_0}) \).

Since \( p_0 \preceq a \preceq 1 \), \( a_0^0 \preceq P_{\gamma_0} \) and also \( a_0^0 \leq z_0^0 \preceq a \mid \gamma_0 \). Similarly, \( z_0^0 \in P_{\gamma_0} \).

\textbf{Claim 3.11.1.} \( z_0^0 \leq a_0^0, s_0 \leq \gamma_0 \).

\textit{Proof.} Combining Clause (5)(a) of Definition 2.23 with Lemma 3.6,

\( z_0^0 \leq a_0^0 \preceq \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0) \preceq \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0) = s_0 \preceq \gamma_0 \).

On the other hand, \( \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0) \preceq \mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(p_0) \) and

\[
\text{dom}(\text{tp}_{\gamma_0}(\mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(t_0))) = \text{dom}(\text{tp}_{\gamma_0}(\mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(p_0))
\]

where this last equality follows from Clause (3) of Definition 2.23.

Combining this with Definition 2.23(6) we get \( z_0^0 \leq a_0^0, s_0 \), as desired. \( \square \)

\textbf{Claim 3.11.2.} For all \( i \in \text{dom}(\text{tp}_{\gamma_0}(a \mid \gamma_0), \text{dom}(\text{tp}_{\gamma_0}(a_0^0))) \),

\[
\text{tp}_{\gamma}(a_0^0)(i) \leq \text{mt}_{\gamma}(a \mid \gamma_0).
\]

\textit{Proof.} Let \( i \) be as above. By Definition 2.23(3), \( \text{dom}(\text{tp}_{\gamma_0}(\mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(p_0))) = \text{dom}(\text{tp}_{\gamma_0}(a \mid \gamma_0)) \). So, combining Clauses (2) and (5) of Definition 2.23

\[
\text{tp}_{\gamma_0}(a_0^0)(i) \leq \text{mt}_{\gamma_0}(\mathcal{G}_{\gamma_0,1}(a \mid \gamma_0)(p_0)) \leq \text{mt}_{\gamma_0}(a \mid \gamma_0).
\]

\( \square \)

\footnote{Recall that this is part of the rules of \( \mathcal{G}_{p_1}(p, \overline{r}) \) (see Definition 2.10).
\footnote{Note that \( \mathcal{G}_{\gamma_0,\beta_0}(a \mid \gamma_0)(p_0) = \mathcal{G}_{\gamma_0,\beta_0}(a \mid \gamma_0)(\mathcal{G}_{\gamma_0,1}(a \mid \beta_0)(p_0)) \) (See (*)) at Page 16.}
For every $\tau < \theta$ such that both $a_0^\tau$ and $z_0^\tau$ have already been defined, set $g_0^{\tau+1} := \text{dom}(\text{tp}_{\gamma^{\tau+1}}(a \upharpoonright \gamma^{\tau+1})) + \omega + 1$, and then let

$$a_0^{\tau+1} := n_{\gamma^{\tau+1},\gamma}(a \upharpoonright \gamma^{\tau+1})(a_0^\tau)^{\times}g_0^{\tau+1},$$

$$z_0^{\tau+1} := n_{\gamma^{\tau+1},\gamma}(a \upharpoonright \gamma^{\tau+1})(z_0^\tau)^{\times}g_0^{\tau+1},$$

where the $\times$ operation is with respect to the type $\text{tp}_{\gamma^{\tau+1}}$.

Claim 3.11.3. For all $\tau' \leq \tau$, $a_0^{\tau+1} \upharpoonright \gamma_{\tau'} = a_0^{\tau'}$ and $z_0^{\tau+1} \upharpoonright \gamma_{\tau'} = z_0^{\tau'}$.

Proof. Let $\tau' \leq \tau$. By Clause (5)(a) of Definition 2.23,

$$a_0^{\tau+1} \upharpoonright \gamma_{\tau+1} = n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau),$$

hence Lemma 3.5(5) yields $a_0^{\tau+1} \upharpoonright \gamma_{\tau'} = a_0^{\tau'}$. Using the induction hypothesis, we get $a_0^{\tau+1} \upharpoonright \gamma_{\tau'} = a_0^{\tau'}$. The argument for $z_0^{\tau+1}$ is the same. \qed

Claim 3.11.4. $z_0^{\tau+1} \leq \gamma_{\tau+1}, a_0^{\tau+1}, s_0 \upharpoonright \gamma_{\tau+1}$.

Proof. By the induction hypothesis, $z_0^0 \leq_{\gamma_{\tau}} a_0^0, s_0 \upharpoonright \gamma_{\tau}$.

Thus, Clause (5) of Definition 2.23 and Lemma 3.6 combined yield

$$z_0^{\tau+1} \leq_{\gamma_{\tau+1}} n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(z_0^0) \leq_{\gamma_{\tau+1}} n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau).$$

Similarly, Lemma 3.6 yields

$$n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(z_0^0) \leq_{\gamma_{\tau+1}} n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau).$$

Also, by Clause (3) of Definition 2.23 and the remark made at Footnote 17

$$\text{dom}(\text{tp}_{\gamma^{\tau+1}}(n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(z_0^0))) = \text{dom}(\text{tp}_{\gamma^{\tau+1}}(n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau))).$$

Therefore, Definition 2.23(6) yields $z_0^{\tau+1} \leq_{\gamma_{\tau+1}} a_0^{\tau+1}$, as desired. \qed

Finally, the following can be proved exactly as in Claim 3.11.2.

Claim 3.11.5. For all $i \in [\text{dom}(\text{tp}_{\gamma^{\tau+1}}(a \upharpoonright \gamma_{\tau+1})), \text{dom}(\text{tp}_{\gamma^{\tau+1}}(a_0^{\tau+1})))$,

$$\text{tp}_{\gamma^{\tau+1}}(a_0^{\tau+1})(i) \leq \text{mtp}_{\gamma^{\tau+1}}(a \upharpoonright \gamma_{\tau+1}).$$

For every $\tau \in \text{acc}(\theta + 1)$, let $a_0^{\tau'} := \bigcup_{\tau' < \tau} a_0^{\tau'}$ and $z_0^{\tau'} := \bigcup_{\tau' < \tau} z_0^{\tau'}$. By the induction hypothesis, $\langle (a_0^{\tau'}, z_0^{\tau'}) \mid \tau' \leq \tau \rangle$ is clearly coherent. Additionally, arguing as in Claim 3.11.4 we have $z_0^\tau \leq_{\gamma_{\tau+1}} a_0^\tau, s_0 \upharpoonright \gamma_{\tau}$. At the end of the recursion, we define $a_0$ and $z_0$ as mentioned before. Note that by our construction $z_0$ witnesses that $a_0$ is a legitimate move for $I$ so, in response, $I$ plays a condition $b_0$ in $(\mathcal{B}_\alpha)$, extending $a_0$ and satisfying $b_0 \leq_0 s_0$. Finally, note that $B_\alpha \subseteq B_{a_0} \subseteq B_0^{18}$ and for every $\gamma \in B_\alpha$,

$$\text{dom}(\text{tp}_{\gamma}(a \upharpoonright \gamma)) + 1 < \text{dom}(\text{tp}_{\gamma}(a_0 \upharpoonright \gamma)).$$

17 Note that $n_{\gamma_{\tau+1},\gamma}(a \upharpoonright \gamma_{\tau+1})(a_0^\tau) = n_{\gamma_{\tau+1},\beta^{\tau+1}}(a \upharpoonright \gamma_{\tau+1})(n_{\beta^{\tau+1},\gamma}(a \upharpoonright \beta^{\tau+1})(a_0^\tau))$.

18 The inclusion $B_{a_0} \subseteq B_0$ is obvious. For the other we use Clause (4) of Definition 2.23, noting that for all $\gamma \in B_{a_0}$, $\text{dom}(\text{tp}_{\gamma}(a_0 \upharpoonright \gamma)) \neq 0$. 
Also, for all \(i \in \{\text{dom}(\mathbf{tp}_\gamma(a \upharpoonright \gamma)), \text{dom}(\mathbf{tp}_\gamma(a_0 \upharpoonright \gamma))\},\)

\[
\text{tp}_\gamma(a_0 \upharpoonright \gamma)(i) \leq \text{mtp}_\gamma(a \upharpoonright \gamma).
\]

- Suppose that \(0 < \eta < \chi\). Recall that \(((a_\eta, b_\eta) : \eta < \xi)\) is an initial play of the game and that we want to define \(a_\xi\). To that effect, let \(p_\xi\) the \(\xi\)th-move of \(I\) in the game \(\mathcal{O}_{x}(p, T)\), provided the previous ones are \(((a_\eta : \eta \leq \xi), \eta < \xi)\). Let \(t_\xi\) be such that \(t_\xi \leq_1 p_\xi, s_\xi\) and set \(B_\xi := \bigcup_{\eta < \xi} B_{\eta_\xi}\).

If \(B_\xi\) is empty then again set \(a_\xi := \hat{a}_{\eta_1}(a)(p_\xi)\) and \(z_\xi := \hat{a}_{\eta_1}(a)(t_\xi)\) and argue as in the case \(\eta < \xi\). Otherwise, \(B_\xi\) is nonempty and we let \(\langle \gamma_\tau, \tau \leq \theta \rangle\) be the increasing enumeration of the closure of \(B_\xi\). By recursion on \(\tau \leq \theta\), we define a coherent sequence \(\langle (a_\xi, z_\xi) : \tau \leq \theta \rangle \in \prod_{\tau \leq \theta} (P_\tau \times P_\tau)\), and then we shall let \(a_\xi := \hat{a}_{\alpha, \gamma_a}(a)(a_\xi)\) and \(z_\xi := \hat{a}_{\alpha, \gamma_a}(a)(z_\xi)\). The construction and the subsequent verifications are the same as in the case \(\eta < \xi\): the only difference is that now, for each \(\tau \in \text{nacc}(\theta + 1)\), we set \(\bar{a}_\xi := \langle \sup_{\eta < \xi} \text{dom}(\mathbf{tp}_\gamma(b_\eta \upharpoonright \gamma_\tau)) \rangle + \omega + 1\).

Thereby, as in the case \(\eta < \xi\), we get a condition \(a_\xi\) which is a legitimate move for \(I\). In response, \(II\) plays a condition \(b_\xi\) in \((\bar{P}_\alpha)_{\tau + \eta}\) extending \(a_\xi\) and satisfying \(b_\xi \leq_0 s_\xi.\) Finally, note that \(B_\xi \subseteq B_{\alpha_\xi} \subseteq B_{\beta_\xi}\) and for all \(\gamma \in B_\xi\),

\[
\left(\text{dom}(\mathbf{tp}_\gamma(b_\eta \upharpoonright \gamma)) \right) + 1 < \text{dom}(\mathbf{tp}_\gamma(a_\xi \upharpoonright \gamma)).
\]

Also, for all \(i \in \{\text{dom}(\mathbf{tp}_\gamma(a \upharpoonright \gamma)), \text{sup}_{\eta < \xi} \text{dom}(\mathbf{tp}_\gamma(b_\eta \upharpoonright \gamma))\},\)

\[
\text{tp}_\gamma(a_\xi \upharpoonright \gamma)(i) \leq \text{mtp}_\gamma(a \upharpoonright \gamma).
\]

At the end we obtain a sequence \(\langle (a_\xi, b_\xi) : \xi < \chi \rangle\) which is a play of the game \(\mathcal{O}_{x}(a, s, (\bar{P}_\alpha)_{\tau + \eta})\). By our construction, for each \(\xi < \chi\), \(a_\xi \upharpoonright 1 = p_\xi,\) hence \(b_\xi \upharpoonright 1 \upharpoonright \xi < \chi\) is diagonalizable with respect to \(\bar{r}\). Let \(p' \leq_0 \pi_{a_\xi}(a)\) be a witness for this latter and set \(B_\xi := \bigcup_{\eta < \xi} B_{\eta_\xi}\) for all \(\xi < \chi\).

Our next task is to show that \(b_\xi \upharpoonright \xi < \chi\) is diagonalizable and that the witness \(b\) for it fulfils the requirements of the lemma.

**Claim 3.11.6.** The tuple \((a, r, p', g, B_\chi)\) meets the requirements of Lemma 3.10, where \(g: W_{\alpha}(\pi_{a_\xi}(a)) \to \bar{P}_\alpha \downarrow a\) is defined via \(g(r_\xi) := b_\xi\).

**Proof.** Let us go over the clauses of Lemma 3.10: Clause (0) holds by the construction of \(\langle B_{b_\xi} : \xi < \chi\rangle\). Clause (1) is obvious and Clause (2) follows from the discussion of the previous paragraph. So, let us address the rest.

For each \(\gamma \in B_\chi\), denote \(\omega_\gamma := \min\{\xi < \chi \mid \gamma \in B_{b_\xi}\}\).

(3): Let \(\gamma \in B_\chi\) and \(\xi < \chi:\)

- If \(\xi < \omega_\gamma\), then \(\gamma \notin B_{b_\xi}\) and so \(b_\xi \upharpoonright \gamma = [b_\xi \upharpoonright \gamma]^P\), where \(\gamma = \beta + 1\).
  - Thus, Lemma 3.6 and Definition 2.23(4) yield \(\text{dom}(\mathbf{tp}_\gamma(b_\xi \upharpoonright \gamma)) = 0\).
- If \(\xi = \omega_\gamma\), then \(\gamma \in B_{b_\xi}\) and so \(b_\xi \upharpoonright \gamma \neq [b_\xi \upharpoonright \gamma]^P\), where \(\gamma = \beta + 1\).
  - Again, Lemma 3.6 and Definition 2.23(4) yield \(\text{dom}(\mathbf{tp}_\gamma(b_\xi \upharpoonright \gamma)) \geq 1\).

---

19For details about the verification of \(\langle \upharpoonright \upharpoonright \rangle\), see Claim 3.11.2. Note that \(\sup_{\eta < \xi} \text{dom}(\mathbf{tp}_\gamma(b_\eta \upharpoonright \gamma)) < \text{dom}(\mathbf{tp}_\gamma(a_\xi \upharpoonright \gamma))\).
If $\xi > \iota_\gamma$, then $\gamma \in B_{b_\gamma} \subseteq B_\xi$. Combining (††) above with $b_\xi \leq a_\xi$ and Clause (2) of Definition 2.23 we get
\[
(sup \{dom(tp_\gamma(b_\eta \upharpoonright \gamma)) : \eta < \xi\}) + 1 < dom(tp_\gamma(b_\xi \upharpoonright \gamma)).
\]

(4): Let $\gamma \in B_\chi, \iota_\gamma < \xi < \chi$ and $i$ be as in Clause (4) of Lemma 3.10. By definition, $\gamma \in B_{b_\gamma} \subseteq B_\xi$, hence (†††) yields $tp_\gamma(a_\xi \upharpoonright \gamma)(i) \leq mtp_\gamma(a \upharpoonright \gamma)$. Combining this with Definition 2.23(3) and $b_\xi \leq a_\xi$ we arrive at
\[
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
Claim 3.13.1. Let \( n < \omega \). For every \( \gamma \in [2, \alpha] \) and every directed set \( D \) of conditions in \( (\dot{P}_\gamma)_n \) of size \( < \kappa_n \), there is \( q \in (\dot{P}_\gamma)_n \) such that \( q \) is a lower bound of \( D \) and \( B_q = \bigcup_{p \in D} B_p \).

Proof. We argue by induction on \( \gamma \in [2, \alpha] \). The base case \( \gamma = 2 \) can be proved similarly to the successor case below. So, we assume by induction that the statement holds for all \( \beta < \gamma \) and prove it for \( \gamma \).

Fix an arbitrary directed family \( D \subseteq (\dot{P}_\gamma)_n \) of size \( \kappa_n \).

- Suppose \( \gamma = \beta + 1 \). Then \( \{ \pi_{\gamma, \beta}(p) \mid p \in D \} \) is a directed subset of \( (\dot{P}_\beta)_n \) of size \( < \kappa_n \), so that the inductive assumption yields a lower bound \( p' \in (\dot{P}_\beta)_n \) such that \( B_{p'} := \bigcup_{p \in D} B_{\pi_{\gamma, \beta}(p)} \). Set \( \bar{D} := \{ \phi_{\gamma, \beta}(p') \mid p \in D \} \), and note that \( |\bar{D}| \leq |D| < \kappa_n \). By Lemma 3.6, \( (\phi_{\gamma, \beta}, \pi_{\gamma, \beta}) \) is a forking projection from \( (\dot{P}_\beta, \ell_\beta) \) to \( (\dot{P}_\gamma, \ell_\gamma) \). So, by Definition 2.13(7) together with Remark 2.24, \( \bar{D} \) is a directed subset of \( (\dot{P}_\gamma)^{\pi_{\gamma, \beta}} \).

Recalling that \( (\dot{P}_\gamma)^{\pi_{\gamma, \beta}} \) is isomorphic to the \( \kappa_n \)-directed-closed poset \( \dot{\kappa}_n^\pi \) given by Building Block II, we pick a lower bound \( q \in (\dot{P}_\gamma)^{\pi_{\gamma, \beta}} \) for \( D \) such that \( \pi_{\gamma, \beta}(q) = p' \). It is clear that \( q \) is the desired lower bound.

- Suppose \( \gamma \) is limit. Let \( C := \text{cl}(\bigcup_{p \in D} B_p) \cup \{ 1, \gamma \} \). We shall define a \( \subseteq \)-increasing sequence \( \langle p_\beta \mid \beta \in C \rangle \in \prod_{\beta \in C}(\dot{P}_\beta)_n \) such that, for all \( \beta \in C \), \( p_\beta \) is a lower bound for \( \{ \pi_{\alpha, \beta}(p) \mid p \in D \} \) with \( B_{p_\beta} = \bigcup_{p \in D} B_{\pi_{\alpha, \beta}(p)} \). Note that for each \( \beta \in C \), Lemma 3.9 yields \( \{ \pi_{\alpha, \beta}(p) \mid p \in D \} \subseteq (\dot{P}_\beta)_n \). We define the sequence \( \langle p_\beta \mid \beta \in C \rangle \) by recursion on \( \beta \in C \):

  - For \( \beta = 1 \), \( \{ \pi_{\alpha, 1}(p) \mid p \in D \} \) is a directed subset of \( (\dot{P}_1)_n \) of size \( < \kappa_n \). By Building Block I, \( (P_1, \ell_1, c_1) \) is \( \Sigma \)-Prikry, and hence we may find a lower bound \( p_1 \in (\dot{P}_1)_n \) for the set under consideration.
  - Suppose \( \beta > 1 \) is a non-accumulation point of \( C \cap \alpha \). Set \( \beta := \tau + 1 \) and \( \varepsilon := \sup(C \cap \beta) \). Clearly, \( \varepsilon \leq \tau \), so that Lemma 3.5(5) yields
    \[
    \phi_{\tau, \varepsilon}(\pi_{\alpha, \tau}(p))(p_\varepsilon) = p_\varepsilon \ast \emptyset_\tau,
    \]
    for each \( p \in D \). Set \( q := p_\varepsilon \ast \emptyset \) and note that the induction hypothesis yields \( B_q = \bigcup_{p \in D} B_{\pi_{\alpha, \varepsilon}(p)} \). Set
    \[
    \bar{D} := \{ \phi_{\beta, \tau}(\pi_{\alpha, \beta}(p))(q) \mid p \in D \}.
    \]
    Let \( p \in D \). Then \( \pi_{\alpha, \beta}(p) \in (\dot{P}_\beta)_n \) and by Lemma 3.9, \( q \in (\dot{P}_\tau)_n \). Also, by Lemma 3.6, \( t_{p_\beta} \) is a type over \( (\phi_{\beta, \tau}, \pi_{\beta, \tau}) \), hence Remark 2.24 yields \( \phi_{\beta, \tau}(\pi_{\alpha, \beta}(p))(q) \in (\dot{P}_\beta)_n \). Altogether, \( \bar{D} \subseteq (\dot{P}_\beta)_n \).

Arguing as in the successor case above we find \( p_\beta \in (\dot{P}_\beta)_n \) a lower bound for \( \bar{D} \) such that \( B_{p_\beta} = B_q \cup \{ \beta \} \). So, \( B_{p_\beta} = \bigcup_{p \in D} B_{\pi_{\alpha, \beta}(p)} \).
Claim 3.13.2. If \( \gamma < \mu \) and \( \beta \in \text{acc}(C) \), then for all \( r \in D \), we have \( |B_{\beta}^r| < \mu \), so that \( p_\beta \in P_\beta \). Also, by the induction hypothesis, \( B_{p_\beta} = \bigcup_{p \in D} B_{\pi_{\alpha,\beta}}(p) \).

For all \( p \in D \) and all \( \alpha \in C \cap \beta \), we have \( \pi_{\alpha,\beta}(p_\beta) = p_\alpha \leq \pi_{\alpha,\beta}(p) \), hence \( p_\beta \) is a bound for \( \{ \pi_{\alpha,\beta}(p) \mid p \in D \} \) in \( (P_\beta)_n \).

We claim that \( p_\beta \in (P_\beta)_n \): Let \( \tau \in B_{p_\beta} \) and \( \alpha \in C \cap \beta \) be such that \( \tau \in B_{p_\alpha} \). By the induction hypothesis \( p_\alpha \in (P_\alpha)_n \), hence Lemma 3.9 yields \( \pi_{\beta,\tau}(p_\beta) = \pi_{\alpha,\tau}(p_\alpha) \in (P_\tau)_n \). Finally, by similar reasons, \( \pi_{\beta,1}(p_\beta) = \pi_{\alpha,1}(p_\alpha) \in (P_1)_n \). Altogether, \( p_\beta \in (P_\beta)_n \).

Suppose \( \beta = \gamma \), but \( \beta \notin \text{acc}(C) \). In this case, let \( \bar{\gamma} := \sup(C \cap \gamma) \), and then set \( p_\gamma := p_\bar{\gamma} * \theta_\gamma \). As the interval \( (\bar{\gamma}, \gamma] \) is disjoint from \( \bigcup_{p \in D} B_{p_\gamma} \) for every \( p \in D \),

\[
p_\gamma = p_\bar{\gamma} * \theta_\alpha \leq \pi_{\alpha,\bar{\gamma}}(p) * \theta_\gamma = p.
\]

Also, by the induction hypothesis, \( B_{p_\bar{\gamma}} = B_{p_\gamma} = \bigcup_{p \in D} B_{p_\gamma} \) and \( p_\gamma \in (P_n)_n \). Finally, Lemma 3.9 yields \( p_\gamma \in (P_\gamma)_n \), as wanted.

Clearly, \( p_\gamma \) is a lower bound for \( D \) in \( (P_\gamma)_n \) with the desired feature. \( \square \)

The next claim takes care of Clause (3)

**Claim 3.13.2.** Suppose \( p, p' \in P_\alpha \) with \( c_\alpha(p) = c_\alpha(p') \). Then, \( (P_\alpha)_0^p \cap (P_\alpha)_0^{p'} \) is nonempty.

**Proof.** If \( \alpha < \mu^+ \), then since \( (\phi_{\alpha,1}, \pi_{\alpha,1}) \) is a forking projection from \( (P_\alpha, \ell_\alpha, c_\alpha) \) to \( (P_1, \ell_1, c_1) \), we get from Clause (8) of Definition 2.13 that \( c_1(p \upharpoonright 1) = c_1(p' \upharpoonright 1) \), and then by Clause (3) of Definition 2.3, we may pick \( r \in (P_1)_0^{c_1} \cap (P_0)_1^{c_1} \). In effect, Clause (8) of Definition 2.13 entails \( \pi_{\alpha,1}(p)(r) = \pi_{\alpha,1}(p')(r) \). Finally, Fact 2.19(2) implies that \( \pi_{\alpha,1}(p)(r) \) is in \( (P_0)_0^p \) and that \( \pi_{\alpha,1}(p')(r) \) is in \( (P_0)_0^{p'} \). In particular, \( (P_0)_0^p \cap (P_0)_0^{p'} \) is nonempty.

From now on, assume \( \alpha = \mu^+ \). In particular, for all nonzero \( \beta < \gamma < \mu^+ \), \( (\mathbb{P}_\gamma, \ell_\gamma, c_\gamma) \) is a \( \Sigma \)-Prikry triple admitting a forking projection to \( (\mathbb{P}_\beta, \ell_\beta, c_\beta) \) as witnessed by \( (\pi_{\gamma,\beta}, \pi_{\gamma,\beta}) \). To avoid trivialities, assume also that \( |\{1_{\mu^+}, p, p'\}| = 3 \). For each \( q \in \{p, p'\} \), let \( C_q := \text{cl}(B_q) \) and define a function \( e_q : C_q \to H_\mu \) via

\[
e_q(\gamma) := (\phi_\gamma[C_q \cap \gamma], c_\gamma(q \upharpoonright \gamma)).
\]

Write \( i \) for the common value of \( c_{\mu^+}(p) \) and \( c_{\mu^+}(p') \). It follows that, for every \( \gamma \in C_p \cap C_{p'} \), \( e_q(\gamma) = e_{p'}(\gamma) = e_{p'}(\gamma) \), so that \( \phi_\gamma[C_p \cap \gamma] = \phi_\gamma[C_{p'} \cap \gamma] \) and hence \( C_p \cap \gamma = C_{p'} \cap \gamma \). Consequently, \( R := C_p \cap C_{p'} \) is an initial segment of \( C_p \) and an initial segment of \( C_{p'} \).
Let $\zeta := \max(C_p \cup C_{p'})$, so that $p = (p \upharpoonright \zeta) * \emptyset_{\mu^+}$ and $p' = (p' \upharpoonright \zeta) * \emptyset_{\mu^+}$.

Set $\gamma_0 := \max(\{0\} \cup R)$. By the above analysis, $C_p \cap (\gamma_0, \zeta]$ and $C_{p'} \cap (\gamma_0, \zeta]$ are two disjoint closed sets.

If $\gamma = \zeta$, then $e_p(\zeta) = e_{p'}(\zeta)$, so that $c_\zeta(p \upharpoonright \zeta) = c_\zeta(p' \upharpoonright \zeta)$, and hence $(P_{\gamma_0})_0^{\zeta} \cap (P_{\gamma_0})_0^{\zeta}$ is nonempty. Pick $r$ in that intersection. Then $r * \emptyset_{\mu^+}$ is an element of $(P_{\mu^+})_0^{\zeta} \cap (P_{\mu^+})_0^{\zeta}$.

Next, suppose that $\gamma < \zeta$. Consequently, there exists a finite increasing sequence $(\gamma_j : j \leq k)$ of ordinals from $C_p \cup C_{p'}$ such that $\gamma_{k+1} = \zeta$ and, for all $j \leq k$:

(i) if $\gamma_j \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_p$;

(ii) if $\gamma_j \notin C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (C_p \cup C_{p'}) \subseteq C_{p'}$.

We now define a sequence $\langle r_j \mid j \leq k+1 \rangle$ in $\prod_{j=0}^{k+1} ((P_{\gamma_j})_0^{\gamma_j} \cap (P_{\gamma_j})_0^{\gamma_j})$ as follows.

- For $j = 0$, if $\gamma_0 \in C_p \cap C_{p'}$, then $e_p(\gamma_0) = e_{p'}(\gamma_0)$, so that $c_{\gamma_0}(p \upharpoonright \gamma_0) = c_{\gamma_0}(p' \upharpoonright \gamma_0)$, and we may indeed pick $r_0 \in (P_{\gamma_0})_0^{\gamma_0} \cap (P_{\gamma_0})_0^{\gamma_0}$.
- If $\gamma_0 \notin C_p \cap C_{p'}$, then $\gamma_0 = 0$, and we simply let $r_0 := \emptyset$.

Suppose that $j < k + 1$, where $r_j$ has already been defined. Let $q := \langle \gamma_j \rangle \upharpoonright (\gamma_j, \gamma_{j+1}]$ and $q' := \langle \gamma_j \rangle \upharpoonright (\gamma_j, \gamma_{j+1}]$. By Lemma 3.5(2), $B_q = (B_{\gamma_j} \cap \gamma_{j+1}) \cup B_{r_j}$ and $B_{q'} = (B_{\gamma_j} \cap \gamma_{j+1}) \cup B_{r_j}$.

In particular, if $\gamma_{j+1} \in C_p$, then $(\gamma_j, \gamma_{j+1}] \cap (B_{\gamma_j} \cup B_{q'}) \subseteq B_{q'}$, so that $q' \leq r_j * r_{j+1}$ and $q \leq r_{j+1} * q'$ by Clauses (5) and (6) of Lemma 3.5, respectively. Likewise, if $\gamma_{j+1} \notin C_p$, then $q = r_j \upharpoonright r_{j+1}$, so that $q' \leq r_{j+1} * q$. Thus, $\{q, q'\} \cap (P_{\gamma_j})_0^{\gamma_j} \cap (P_{\gamma_j})_0^{\gamma_j} = \emptyset$, and we may let $r_{j+1}$ be an element of that set.

Evidently, $r_{k+1} * \emptyset_{\mu^+}$ is an element of $(P_{\mu^+})_0^{\zeta} \cap (P_{\mu^+})_0^{\zeta}$.

(4) Let $p \in P_\alpha$, $n, m < \omega$ and $q \in (P_\alpha^\omega)_{n+m}$ be arbitrary. Recalling that $(\langle \alpha \rangle_{n+1})_{\langle \alpha \rangle_0}$ is a forking projection from $(P_\alpha, \alpha)$ to $(P_1, \ell_1)$, we infer from Clause (4) of Definition 2.13 that $\langle \alpha \rangle_{n+1}(p)(m(p \upharpoonright 1, q \upharpoonright 1))$ is the greatest element of $\{r \leq_n p \mid q \leq_m r\}$.

(5) Recalling that $(P_1, \ell_1, c_1)$ is $\Sigma$-Prikry, and that $(\langle \alpha \rangle_{n+1}, \langle \alpha \rangle_0$ is a forking projection from $(P_\alpha, \alpha)$ to $(P_1, \ell_1)$, we infer from Fact 2.19(1) that, for every $p \in P_\alpha$, $|W(p)| = |W(p \upharpoonright 1)| < \mu$.

(6) Let $p', p \in P_\alpha$, with $p' \leq_m p$. Let $q \in W(p')$ be arbitrary. For all $\gamma < \alpha$, the pair $(\langle \alpha \rangle_{n+1}, \langle \alpha \rangle_0)$ is a forking projection from $(P_\alpha, \alpha)$ to $(P_\gamma, \ell_\gamma)$, so that by the special case $m = 0$ of Clause (4) of Definition 2.13,

$$w(p, q) = \langle \alpha \rangle_{\gamma}(p)(w(p \upharpoonright 1, q \upharpoonright 1))$$

Now, for all $q' \leq_m q$, the induction hypothesis implies that, for all $\gamma < \alpha, w(p \upharpoonright 1, q' \upharpoonright 1) \leq \gamma w(p \upharpoonright 1, q \upharpoonright 1)$. Together with Clause (5) of Definition 2.13, it follows that, for all $\gamma < \alpha$,

$$w(p, q') \upharpoonright \gamma = w(p \upharpoonright 1, q' \upharpoonright 1) \leq \gamma w(p \upharpoonright 1, q \upharpoonright 1) = w(p, q) \upharpoonright \gamma.$$
So, by the definition of $\leq_{\alpha}$, $w(p, q') \leq_{\alpha} w(p, q)$, as desired.

(7) Since $(\mathfrak{n}_{\alpha, 1}, \pi_{\alpha, 1})$ is a forking projection from $(\mathbb{P}_\alpha, \ell_\alpha)$ to $(\mathbb{P}_1, \ell_1)$, $(\mathbb{P}_1, \ell_1, c_1)$ is $\Sigma$-Prikry. By Corollary 3.12, $(\mathbb{P}_\alpha, \ell_\alpha)$ has property $D$.

It thus follows from Lemma 2.21 that $(\mathbb{P}_\alpha, \ell_\alpha)$ has the CPP.

To complete our proof we shall need the following claim.

Claim 3.13.3. For each $\alpha$ with $1 \leq \alpha \leq \mu^+$, $\mathbb{P}_\alpha \models P_\alpha \models \mu = \kappa^+$.

Proof. Suppose that $\mathbb{P}_\alpha \models P_\alpha \models \mu \neq \kappa^+$. As $\mathbb{P}_\alpha$ projects to $\mathbb{P}_1$, this means that there exists $p \in P_\alpha$ such that $p \models P_\alpha [\mu] \leq |\kappa|$. Since $P_1$ is isomorphic to the poset $Q$ of Building Block I, and since $1_Q \not\models \kappa$ is singular”\footnote{This is the sole part of the whole proof to make use of the fact that the poset given by Building Block I forces $\kappa$ to be singular.}, $\mathbb{P}_1 \models P_1 \models \kappa$ is singular”. As $\mathbb{P}_\alpha$ projects to $\mathbb{P}_1$, in fact $p \models P_\alpha \models \text{cf}(\mu) < \kappa$. Thus, Lemma 2.7(2) yields a condition $p' \leq_{\alpha} p$ with $|W(p')| \geq \mu$, contradicting Clause (5) above.

This completes the proof of Lemma 3.13. \qed

4. An application

In this section, we present the first application of our iteration scheme. We will be constructing a model of finite simultaneous reflection at a successor of a singular strong limit cardinal $\kappa$ in the presence of $\neg \text{SCH}_\kappa$.

Definition 4.1. For cardinals $\theta < \mu = \text{cf}(\mu)$ and stationary subsets $S, \Gamma$ of $\mu$, $\text{Refl}(\theta, S, \Gamma)$ stands for the following assertion. For every collection $S$ of stationary subsets of $\mu$, with $|S| < \theta$ and sup($\{\text{cf}(\alpha) \mid \alpha \in \bigcup S\}$) $< \mu$, there exists $\delta \in \Gamma \cap E^\mu_{\omega}$ such that, for every $S \in \mathcal{S}$, $S \cap \delta$ is stationary in $\delta$.

We write $\text{Refl}(\theta, S, \mu)$ for $\text{Refl}(\theta, S, \Gamma)$.

A proof of the following folklore fact may be found in [PRS20, §4].

Fact 4.2. If $\kappa$ is a singular strong limit cardinal admitting a stationary subset $S \subseteq \kappa^+$ for which $\text{Refl}(\text{cf}(\kappa)^+, S)$ holds, then $2^\kappa = \kappa^+$.

In particular, if $\kappa$ is a singular strong limit cardinal of countable cofinality for which $\text{SCH}_\kappa$ fails, and $\text{Refl}(\theta, \kappa^+)$ holds, then $\theta \leq \omega$. We shall soon show that $\theta := \omega$ is indeed feasible.

The following general statement about simultaneous reflection will be useful in our verification later on.

Proposition 4.3. Suppose that $\mu$ is non-Mahlo cardinal, and $\theta \leq \text{cf}(\mu)$. For stationary subsets $T, \Gamma, R$ of $\mu$, $\text{Refl}(\theta, T, \Gamma) + \text{Refl}(\theta, \Gamma, R)$ entails $\text{Refl}(\theta, T \cup \Gamma, R)$.

Proof. Given a collection $S$ of stationary subsets of $T \cup \Gamma$, with $|S| < \theta$ and sup($\{\text{cf}(\alpha) \mid \alpha \in \bigcup S\}$) $< \mu$, we shall first attach to any set $S \in \mathcal{S}$, a stationary subset $S'$ of $\Gamma$, as follows.
If $S \cap \Gamma$ is stationary, then let $S' := S \cap \Gamma$.

If $S \cap \Gamma$ is nonstationary, then for every (sufficiently thin) club $C \subseteq \mu$, $S \cap C$ is a stationary subset of $T$, and so by Refl($<\kappa, T, \Gamma$), there exists $\alpha \in \Gamma \cap E^\mu_{\omega \cdot \omega}$ such that $(S \cap C) \cap \alpha$ is stationary in $\alpha$, and in particular, $\alpha \in C$. So, the set $\{\alpha \in \Gamma \mid S \cap \alpha$ is stationary$\}$ is stationary, and, as $\mu$ is non-Mahlo, we may pick $S'$ which is a stationary subset of it and all of its points consists of the same cofinality.

Next, as $|S| < \text{cf}(\mu)$, we have $\sup(\{\text{cf}(\alpha) \mid \alpha \in S', S \in S\}) < \mu$, and so, from Refl($<\theta, \Gamma, R$), we find some $\alpha \in R$ such that $S' \cap \alpha$ is stationary for all $S \in S$.

**Claim 4.3.1.** Let $S \in S$. Then $S \cap \alpha$ is stationary in $\alpha$.

**Proof.** If $S' = S$, then $S \cap \alpha = S' \cap \alpha$ is stationary in $\alpha$, and we are done. Next, assume $S' \neq S$, and let $c$ be an arbitrary club in $\alpha$. As $S' \cap \alpha$ is stationary in $\alpha$, we may pick $\delta \in \text{acc}(c) \cap S'$. As $\delta \in S' \subseteq E^\mu_{\omega \cdot \omega}$, $c \cap \delta$ is a club in $\delta$, and as $\delta \in S'$, $S \cap \delta$ is stationary, so $S \cap c \cap \delta \neq \emptyset$. In particular, $S \cap c \neq \emptyset$.

This completes the proof.

4.1. **About Building Block II.** In this subsection, we describe Building Block II that we will be feeding to the iteration scheme of the preceding section. We were originally planning to use the functor given by [PRS20, §6], but unfortunately we found a gap in the proof of the mixing property [PRS20, Lemma 6.16]. To mitigate this gap, we shall relax Clause (4) of [PRS20, Definition 6.2] and prove that the outcome is a functor satisfying the weak mixing property (Lemma 4.15 below). The upshot of this subsection is encapsulated by Corollary 4.17. We commence by describing our setup.

**Setup 4.** Suppose that $(\mathbb{P}, \ell, c)$ is a given $\Sigma$-Prikry notion of forcing and that $(\mathbb{P}, \ell)$ has property $D$. Denote $\mathbb{P} = (P, \leq)$ and $\Sigma = \langle \kappa_n \mid n < \omega \rangle$. Also, define $\kappa$ and $\mu$ as in Definition 2.3, and assume that $\mathbb{P}_\ell \forces " \kappa \text{ is singular}"$ and that $\mu^{<\mu} = \mu$. Recall that for each $n < \omega$, we denote by $\mathbb{P}_n$ a dense $\kappa_n$-directed-closed subposet of $\mathbb{P}_n$. Our universe of sets is denoted by $V$, and we assume that, for all $n < \omega$, $V^{\mathbb{P}_n} \models \text{Refl}(1, E^\mu_{\omega \cdot \omega}, E^\mu_{\kappa_n})$. Write $\Gamma := \{\alpha < \mu \mid \omega < \text{cf}^V(\alpha) < \kappa\}$. Suppose $r^* \in P$ forces that $\tilde{T}$ is a $\mathbb{P}$-name for a stationary subset $T$ of $(E^\mu_{\omega \cdot \omega})^V$ that does not reflect in $\Gamma$ and that $R$ is the binary relation:

$$R := \{(\alpha, q) \in \mu \times P \mid q \leq r^* \ & \ \forall r \leq q[\ell(r) \in I \rightarrow r \forces_{\mathbb{P}_\ell(\alpha)} \tilde{\alpha} \in \tilde{C}_{\ell(\alpha)}]\}.$$  

A moment’s reflection makes it clear that, for all $(\alpha, q) \in R$, $q \forces \tilde{\alpha} \notin \tilde{T}$. Also, if $(\alpha, q) \in R$ and $q' \leq q$ then $(\alpha, q') \in R$, as well.

**Definition 4.4** (relaxed form of [PRS20, Definition 6.2]). Suppose $p \in P$. A labeled $p$-tree is a function $S : \mathcal{W}(p) \rightarrow [\mu]^{<\mu}$ such that for all $q \in \mathcal{W}(p)$:

\[\mathcal{C}_{\ell(q)} \text{ is a club in } \hat{\mu}.\]  

For more details, see [PRS20, §5 and §6].
(1) \( S(q) \) is a closed bounded subset of \( \mu \);
(2) \( S(q') \supseteq S(q) \) whenever \( q' \leq q \);
(3) \( q \vDash S(q) \cap T = \emptyset \);
(4) there is a natural number \( m \) such that for any pair \( q' \leq q \) of elements of \( W(p) \), if \( S(q') \neq \emptyset \) and \( \ell(q) \geq \ell(p) + m \), then \( \max(S(q')) \in R \).
The least such \( m \) is denoted by \( m(S) \).

Remark 4.5. Note that for \( m \) given by Clause (4), if \( q \in W_{\geq m}(p) \) is incompatible with \( r^* \), then \( S(q') = \emptyset \) for all \( q' \geq q \) in \( W(p) \).

Definition 4.6 ([PRS20, Definition 6.3]). For \( p \in P \), we say that \( \vec{S} = \langle S_i \mid i \leq \alpha \rangle \) is a \( p \)-strategy iff all the following hold:

1. \( \alpha < \mu \);
2. \( S_i \) is a labeled \( p \)-tree for all \( i \leq \alpha \);
3. for every \( i < \alpha \) and \( q \in W(p) \), \( S_i(q) \subseteq S_{i+1}(q) \);
4. for every \( i < \alpha \) and a pair \( q' \leq q \) in \( W(p) \), \( (S_{i+1}(q) \setminus S_i(q)) \subseteq (S_{i+1}(q') \setminus S_i(q')) \);
5. for every limit \( i \leq \alpha \) and \( q \in W(p) \), \( S_i(q) \) is the ordinal closure of \( \bigcup_{j<i} S_j(q) \). In particular, \( S_0(q) = \emptyset \) for all \( q \in W(p) \).

Now, we are ready to describe our functor.

Definition 4.7 ([PRS20, Definition 6.4]). Let \( \mathbb{A}(\mathbb{P}, \bar{T}) \) be the notion of forcing \( \mathbb{A} := (A, \subseteq) \), where:

1. \( (p, \vec{S}) \in A \) iff \( p \in P \), and \( \vec{S} \) is either the empty sequence, or a \( p \)-strategy;
2. \( (p', \vec{S}') \leq (p, \vec{S}) \) iff:
   - (a) \( p' \leq p \);
   - (b) \( \dom(\vec{S}') \geq \dom(\vec{S}) \);
   - (c) \( S'_i(q) = S_i(w(p, q)) \) for all \( i \in \dom(\vec{S}) \) and \( q \in W(p') \).

For all \( p \in P \), denote \( [p]^\mathbb{A} := (p, \emptyset) \).

Definition 4.8 ([PRS20, Definitions 6.10 and 6.11]).

- Define \( c_\mathbb{A} : A \to H_\mathbb{A} \) by letting, for all \( (p, \vec{S}) \in A \),
  \[ c_\mathbb{A}(p, \vec{S}) := (c(p), \{ (i, c(q), S_i(q)) \mid i \in \dom(\vec{S}), q \in W(p) \} \).
- Define \( \pi : A \to P \) by stipulating \( \pi(p, \vec{S}) := p \) and \( \ell_\mathbb{A} := \ell \circ \pi \).
- Given \( a = (p, \vec{S}) \) in \( A \), define \( \vec{\pi}(a) : \mathbb{P} \downarrow p \to A \) by letting for each \( p' \leq p \), \( \vec{\pi}(a)(p') := (p', \vec{S}') \), where \( \vec{S}' \) is the sequence \( \langle S'_i : W(p') \to [\mu]^{<\mu} \mid i < \dom(\vec{S}) \rangle \) satisfying:

\[ S'_i(q) := S_i(w(p, q)) \text{ for all } i \in \dom(\vec{S}') \text{ and } q \in W(p'). \]

Even after relaxing Clause (4) of [PRS20, Definition 6.2] to that of Definition 4.4, the following remains valid, with essentially the same proofs.

Fact 4.9 ([PRS20, Corollary 4.13, Lemma 6.6, Theorem 6.8]).
Definition 4.12. and \( q \) from the outset. In particular, we will have \( \dot{\mu} \) over \( (\mathbb{P}, \ell, c) \) and by Definition 4.8.

Definition 4.11. For two conditions \( p, p' \leq \pi(a) \); we shall show that \( \mathcal{H}(a)(p') \in A \) and \( \mathcal{H}(a)(p') \leq a \).

Write \( a \) as \( (p, S) \). If \( S = \emptyset \), then \( \mathcal{H}(a)(p') = [p']^A \) and we are done.

Next, suppose that \( \text{dom}(S) = \alpha + 1 \). Let \( (p', \tilde{S}) := \mathcal{H}(a)(p') \). Let \( i \leq \alpha \) and we shall verify that \( S'_i \) is a \( p' \)-labeled tree with \( m(S'_i) \leq m(S_i) \). We go over the clauses of Definition 4.4. To this end, let \( q' \leq q \) be arbitrary pair of elements of \( W(p') \).

(2) By Definition 2.3(6), we have \( w(p, q') \leq w(p, q) \), so that \( S'_i(q') = S_i(w(p, q')) \supseteq S_i(w(p, q)) = S'_i(q) \).

(3) As \( q \leq w(p, q), w(p, q) \Vdash \bigcap_{\mathcal{P}} S_i(w(p, q)) \cap T = \emptyset \), so that, since \( S'_i(q) = S_i(w(p, q)) \), we clearly have \( q \Vdash S'_i(q) \cap T = \emptyset \).

(4) To avoid trivialities, Suppose that \( S'_i(q') \neq \emptyset \) and \( \ell(q) \geq m(S_i) \). Write \( \gamma := \max(S'_i(q')) \). As \( \ell(w(p, q)) = \ell(q) \geq m(S_i) \) and \( \gamma = \max(S_i(w(p, q'))), \) we infer that \( (\gamma, w(p, q)) \in R \). In addition, \( q \leq w(p, q), \) so by the definition of \( R \) it follows that \( (\gamma, q) \in R \). Recalling that \( m(S'_i(q)) = \gamma \), we are done.

To prove that \( (p', \tilde{S}) \) is a condition in \( A \) it now remains to argue that \( \tilde{S} \) fulfills the requirements described in Clauses (3) and (5) of Definition 4.6 but this already follows from the definition of \( \tilde{S} \) and the fact that \( \tilde{S} \) is a \( p \)-strategy. Finally \( \mathcal{H}(a)(p') = (p', \tilde{S}) \leq (p, S) = a \) by the very choice of \( p' \) and by Definition 4.8. \( \square \)

We now introduce a sequence of orderings \( \langle \sqsubseteq^n | n < \omega \rangle \) of \( A \) and a type \( \text{tp} \) over \( (\mathcal{H}, \pi) \) that will witness together the weak mixing property of \( (\mathcal{H}, \pi) \).

**Definition 4.11.** For two conditions \( a = (p, S_i | i < \alpha) \) and \( b = (p', \{T_i | i < \alpha'\}) \), and \( n < \omega \), we let \( b \sqsubseteq^n a \) if \( b \sqsubseteq^0 a \) and, if \( \alpha \neq 0 \), then for all \( i < \alpha' \) and \( q \in W(p') \) with \( \ell(q) < \ell(p') + n, T_i(q) = S_{\min(i, \alpha)}(w(p, q)) \).

**Definition 4.12.** Define a map \( \text{tp} : A \to ^{<\omega}_\omega \), as follows.

Given \( a = (p, S) \) in \( A \), write \( \tilde{S} \) as \( \langle S_i | i < \beta \rangle \), and then let

\[ \text{tp}(a) := \langle m(S_i) | i < \beta \rangle. \]

We shall soon verify that \( \text{tp} \) is a type, but will use the mtp notation of Definition 2.23 from the outset. In particular, we will have \( \dot{A} = (A, \sqsubseteq), \) with \( A := \{ a \in A | \pi(a) \in P_{\ell(\pi(a))} \text{ & mtp}(a) = 0 \} \). Note that the supercollection \( \{ a \in A | \text{mtp}(a) = 0 \} \) coincides with the set \( A \) from [PRS20, Definition 6.4].

\[ \text{Here, Claim 4.14.1 below plays the role of [PRS20, Lemma 6.7].} \]
In particular, the proof of [PRS20, Lemma 6.15] goes through, yielding the following fact.

**Fact 4.13.** For all \( n < \omega \), \( \hat{\kappa}_n^\pi \) is \( \mu \)-directed-closed. \( \square \)

**Lemma 4.14.** The map \( \tp \) is a type over \((\hat{\kappa}, \pi)\).

**Proof.** We go over the clauses of Definition 2.23:

1. This follows from the mere definition of \( \tp \).
2. Write \( b = (p', \vec{S}') \) and \( a = (p, \vec{S}) \). By Definitions 4.7 and 4.12, \( \dom(\tp(b)) = \dom(\vec{S}') \geq \dom(\vec{S}) = \dom(\tp(a)) \). Fix \( i \in \dom(\tp(a)) \) and let us show that \( \tp(b)(i) \leq \tp(a)(i) \), i.e., that \( m(S'_i) \leq m(S_i) \).

   Let \( q' \leq q \) be a pair of elements in \( W(p') \) with \( S'_i(q') \neq \emptyset \) and \( \ell(q) \geq m(S_i) \).

   By Definition 4.7(2c), \( S'_i(q') = S_i(w(p, q')) \), hence \( w(p, q') \leq w(p, q) \) is a pair of elements in \( W(p) \) with \( S_i(w(p, q')) \neq \emptyset \). Set \( \gamma := \max(S_i(w(p, q'))) \). By Definition 4.4(4), \( (\gamma, w(p, q)) \in R \) hence the definition of \( R \) yields \( (\gamma, q) \in R \).

   Noting that \( \gamma = \max(S'_i(q')) \) it finally follows that \( m(S'_i) \leq m(S_i) \).

3. This follows from Definition 4.8(*).

4. Let \( a \in A \).
   - If \( a = [\pi(a)]^\vec{A} \) then \( a = (p, \emptyset) \), and so \( \tp([\pi(a)]^\vec{A}) \) is the empty sequence. Conversely, if \( \tp(a) \) is the empty sequence then Definition 4.12 implies that \( a \) takes the form \((\pi(a), \emptyset)\), hence \( a = [\pi(a)]^\vec{A} \).
   - Write \( a \) as \( (p, \langle S_i \mid i < \dom(\tp(a)) \rangle) \) and let \( \alpha \in \mu \setminus \dom(\tp(a)) \). There are two cases to consider:
     - If \( \dom(\tp(a)) = 0 \), then let \( a^\ominus := (p, \langle T_i \mid i \leq \alpha \rangle) \), where \( T_i : W(p) \to \{\emptyset\} \) is constant for every \( i \leq \alpha \).
     - Otherwise, say \( \dom(\tp(a)) = \beta + 1 \), let \( a^\ominus := (p, \langle T_i \mid i \leq \alpha \rangle) \), where \( T_i := S_{\min(i, \alpha)} \) for every \( i \leq \alpha \).

   It is routine to check that \( a^\ominus \) is as desired.

5. Write \( b = (p', \vec{S}') \) and \( a = (p, \vec{S}) \). If \( \dom(\tp(b)) = \dom(\tp(a)) = 0 \), then \( b^\ominus \trianglelefteq a^\ominus \) follows simply from \( p' \leq p \). Otherwise, by the above clause \( b^\ominus = (p', \vec{T}') \), where \( \vec{T}' := \langle T'_i \mid i \leq \alpha \rangle \) and \( T'_i := S'_{\min(i, \alpha)} \).

   Similarly, \( a^\ominus = (p, \vec{T}) \), where \( \vec{T} := \langle T_i \mid i \leq \alpha \rangle \) and \( T_i := S_{\min(i, \alpha)} \).

   Using that \( b \subseteq a \), Definition 4.7 yields \( b^\ominus \trianglelefteq a^\ominus \), as desired.

6. Let \( a = (p, \vec{S}) \in A \). To avoid trivialities, let us assume that \( \vec{S} \neq \emptyset \).

   Suppose \( p \) is incompatible with \( r^* \). Then, by Remark 4.5, for all \( i < \dom(\tp(a)) \) and all \( q \in W(p), S_i(q) = \emptyset \). Therefore, \( mtp(a) = 0 \). Using Definition 2.3(2) find \( p' \leq p \in P \) and set \( b := \hat{\kappa}(a)(p') \). Combining Clauses (2) and (3) above with the fact that \( mtp(a) = 0 \) it easily follows that \( mtp(b) = 0 \). Also, \( \pi(b) = p' \in \hat{P}_{\ell(p)} \).

   Thus, \( b \in \hat{A}_{\ell(p)} \downarrow a \), as wanted.

   Suppose \( p \leq r^* \). The following claim will give us the desired condition.

**Claim 4.14.1.** For every \( \epsilon < \mu \), there exist \( \alpha > \epsilon \) and \( b = (q, \vec{T}) \trianglelefteq^0 a \) such that \( b \in \hat{\kappa} \), \( \dom(\vec{T}) = \alpha + 1 \), and for all \( r \in W(q) \), \( \max(T_a(r)) = \alpha \).

**Proof.** Let \( \epsilon < \mu \) be arbitrary. Since \( (\mathbb{P}, \ell, c) \) is \( \Sigma \)-Prikry, we infer from Definition 2.3(5) that \( |W(p)| < \mu \). Thus, by possibly extending \( \epsilon \), we may
assume that $S_i(q) \subseteq \epsilon$, for all $q \in W(p)$ and $i \in \text{dom}(\text{tp}(a))$. By Clause (5), we may also assume that $\text{dom}(\text{tp}(a))$ is a successor ordinal, say, it is $\delta + 1$.

As $p \leq r^\pi$, by the very same proof of [PRS20, Claim 5.6.2(1)] and using Clause (2) of Definition 2.3, we may fix $(\alpha, q) \in R$ with $\alpha > \beta + \epsilon$, $q \leq \omega^0 p$ and $q \in \check{P}(\ell(p))$. Define $T' = \langle T_i : W(q) \to [\mu]^{< \mu} | i \leq \alpha \rangle$ by letting for all $r \in W(q)$ and $i \in \text{dom}(T')$:

$$T_i(r) := \begin{cases} S_i(w(p, r)), & \text{if } i \leq \delta; \\ S_\delta(w(p, r)) \cup \{\alpha\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $T_i$ is a labeled $q$-tree for each $i \leq \alpha$. By Definitions 4.6, 4.7 and 4.8, we also have that $b = (q, T)$ is a condition in $\check{A}$ with $b \leq \omega^0 a$ and $\pi(b) \in \check{P}(\ell(p))$. As $(\alpha, q) \in R$, also $\text{mtp}(b) = 0$. Thus $b$ condition in $\check{A}$ and so $\alpha$ and $b$ are as desired. 

This completes the proof.

Lemma 4.15 (Weak Mixing Property). For all $a \in A$, $n < \omega$, $\vec{r}$, and $p' \leq \omega^0 \pi(a)$, and for every function $g : W_n(\pi(a)) \to A \downarrow a$, if there exists an ordinal $\iota$ such that all of the following hold:

1. $\vec{r} = (r_\xi | \xi < \chi)$ is a good enumeration of $W_n(\pi(a))$;
2. $(\pi(g(r_\xi)) | \xi < \chi)$ is diagonalizable with respect to $\vec{r}$, as witnessed by $p'$;
3. for every $\xi < \chi$:
   - if $\xi < \iota$, then $\text{dom}(\text{tp}(g(r_\xi))) = 0$;
   - if $\xi = \iota$, then $\text{dom}(\text{tp}(g(r_\xi))) \geq 1$;
   - if $\xi > \iota$, then $(\sup_{\eta < \xi} \text{dom}(\text{tp}(g(r_\eta)))) + 1 < \text{dom}(\text{tp}(g(r_\xi)))$;
4. for all $\xi \in (\iota, \chi)$ and $i \in \{\text{dom}(\text{tp}(a)), \sup_{\eta < \xi} \text{dom}(\text{tp}(g(r_\eta)))\}$,
   $$\text{tp}(g(r_\xi))(i) \leq \text{mtp}(a),$$
5. $\sup_{\xi < \chi} \text{mtp}(g(r_\xi)) < \omega$,

then there exists $b \subseteq^n a$ with $\pi(b) = p'$ and $\text{mtp}(b) \leq n + \sup_{\xi < \chi} \text{mtp}(g(r_\xi))$, such that for all $q' \in W_n(p')$,

$$\vdash (b)(q') \leq^0 g(w(\pi(a), q')).$$

Proof. Let $a := (p, \vec{S})$. Using Clause (2), for each $\xi < \chi$, set $(p_\xi, \vec{S}_\xi) := g(r_\xi)$.

Claim 4.15.1. If $\iota \geq \chi$ then there is $b \in A$ as desired.

Proof. If $\iota \geq \chi$ then Clause (3) yields $\text{dom}(\text{tp}(g(r_\xi))) = 0$, for all $\xi < \chi$. Hence, Clause (4) of Definition 2.23 yields $g(r_\xi) = [\pi(g(r_\xi))]^A_{< \chi}$ for all $\xi < \chi$.

Set $b := [p']^A_{< \chi}$. Clearly, $\pi(b) = p'$ and $b \leq \omega^0 a$, hence also $b \subseteq^n a$.

Let $q' \in W_n(p')$. By Clause (1) above, $q' \leq^0 \pi(g(r_\xi))$, where $\xi$ is the unique index such that $r_\xi = w(\pi(a), q')$. Also, by Definition 2.13(6),

$$\vdash (b)(q') = [q']^A_{< \chi} \leq^0 [\pi(g(r_\xi))]^A_{< \chi} = g(r_\xi),$$

as desired. 

\[\square\]
Hereafter suppose that $\iota < \chi$. For each $\xi$ with $\iota \leq \xi < \chi$, combining Clause (3) and Definition 4.12 we have $\dom(\vec{S}_i) = \alpha_\xi + 1$, for some $\alpha_\xi < \mu$. Moreover, Clause (3) implies that for each $\xi$ with $\iota < \xi < \chi$, $\sup_{i \leq \eta < \xi} \alpha_\eta < \alpha_\xi$. Similarly, the same clause yields $\vec{S}_i = \emptyset$ for all $\xi < \iota$.

Let $(s_\tau | \tau < \theta)$ be a good enumeration $W_n(p')$. By Fact 2.19, $\theta < \mu$. For each $\tau < \theta$, set $r_\xi := w(p, s_\tau)$. Note that Clause (1) above implies that for each $\tau < \theta$,

$$s_\tau \leq^{\alpha_\iota} \pi(g(w(p, s_\tau))) = \pi(g(r_\xi)) = p_\xi.$$

Set $\alpha' := \sup_{i \leq \iota < \chi} \alpha_\xi$ and $\alpha := \dom(tp(a))$. Note that, by regularity of $\mu$, $\alpha \leq \alpha' < \mu$. Our goal is to define a sequence $\vec{T} := (T_i : W(p') \to [\mu]^{<\mu} | i \leq \alpha')$ for which $b := (p', \vec{T})$ satisfies the conclusion of the lemma.

As $(s_\tau | \tau < \theta)$ is a good enumeration of the $n^{\text{th}}$-level of the $p'$-tree $W(p')$, Fact 2.6 entails that, for each $q \in W(p')$, there is a unique ordinal $\tau_q < \theta$, such that $q$ is comparable with $s_{\tau_q}$. It thus follows from Fact 2.6(3) that, for all $q \in W(p')$, $\ell(q) - \ell(p') \geq n$ iff $q \in W(s_{\tau_q})$. Moreover, for each $q \in W_{\geq n}(p')$, $q \leq s_{\tau_q} \leq p_{\xi_{\tau_q}}$, hence $w(p_{\xi_{\tau_q}}, q)$ is well-defined.

Now, for all $i \leq \alpha'$ and $q \in W(p')$, let:

$$T_i(q) := \begin{cases} S_{\xi_{\tau_q} \in \iota, \alpha_{\iota}} w(p_{\xi_{\tau_q}}, q), & \text{if } q \in W(s_{\tau_q}) \& i \leq \xi_{\tau_q}; \\
S_{\min_{i, \alpha}} w(p, q), & \text{if } q \notin W(s_{\tau_q}) \& \vec{S} \neq \emptyset; \\
\emptyset, & \text{otherwise.} \end{cases}$$

Claim 4.15.2. Let $i \leq \alpha'$. Then $T_i$ is a labeled $p'$-tree.

Proof. Fix $q \in W(p')$ and let us go over the Clauses of Definition 4.4. The verification of (1), (2) and (3) are the same as in [PRS20, Claim 6.16.1], so we just provide details for the new Clause (4).

For each $i < \alpha'$, set

$$\xi(i) := \min\{\xi \in [i, \chi) | i \leq \alpha_\xi\}.$$ 

Subclaim 4.15.2.1. If $i \leq \alpha'$, then

$$m(T_i) \leq n + \max\{mtp(a), \sup_{i < \xi < \xi(i)} mtp(g(r_\eta)), tp(g(r_{\xi(i)}))(i)\}.$$ 

Proof. Let $q' \leq q$ be in $W(p')$ with $q \in W_k(p')$, where

$$k \geq n + \max\{mtp(a), \sup_{i < \xi < \xi(i)} mtp(g(r_\eta)), tp(g(r_{\xi(i)}))(i)\}.$$ 

Suppose that $T_i(q') \neq \emptyset$. Denote $\tau := \tau_{q'}$ and $\delta := \max(T_i(q'))$. Since $\ell(q) \geq \ell(p') + n$, note that $q, q' \in W(s_\tau)$. Also, $i \leq \xi_\tau$, as otherwise $T_i(q') = \emptyset$. Therefore, we fall into the first option of the casuistic getting

$$T_i(q') = S_{\min_{i, \alpha_{\iota}}} w(p_{\xi_{\tau}}, q').$$

Assume that $\xi_\tau < \xi(i)$. Then, $\alpha_{\xi_\tau} < i$ and so

$$T_i(q') = S_{\alpha_{\xi_\tau}, \alpha_{\iota}} w(p_{\xi_\tau}, q').$$
We have that \( w(p_\xi, q') \leq w(p_\xi, q) \) is a pair in \( W_{k-n}(p_\xi) \) and that the set \( S^{S_{\xi, \tau}}_\alpha(w(p_\xi, q')) \) is non-empty. Also, \( k - n \geq \text{mtp}(g(r_\xi)) = m(S^{S_{\xi, \tau}}_\alpha) \). So, by Clause (4) for \( S^{S_{\xi, \tau}}_\alpha \), we have that \( (\delta, w(p_\xi, q)) \in R \), and thus \( (\delta, q) \in R \).

\[ T_i(q') = S^{S_{\xi, \tau}}_\alpha(w(p_\xi, q')). \]

If \( \text{dom}(\text{tp}(a)) \leq i \leq \sup_{\xi \leq \chi} \alpha_\xi \), by Clause (4) above,

\[ \text{tp}(g(r_\xi))((i) \leq \text{mtp}(a). \]

Otherwise, if \( \sup_{\xi \leq \chi} \alpha_\xi < i \leq \alpha_{\xi(i)} \), again by Clause (4) above

\[ \text{tp}(g(r_\xi))((i) \leq \max\{\text{mtp}(a), \text{tp}(g(r_{\xi(i)}))(i)\}. \]

In either case, \( w(p_\xi, q) \in W_{k-n}(p_\xi) \) and \( k - n \geq \text{tp}(g(r_\xi))((i) = m(S^{S_{\xi, \tau}}) \).

So by Clause (4) of \( S^{S_{\xi, \tau}}_\alpha \) we get that \( (\delta, w(p_\xi, q)) \in R \), hence \( (\delta, q) \in R \).

**Subclaim 4.15.2.2.** \( m(T_{\alpha'}) \leq n + \sup_{\xi \leq \chi} \text{mtp}(g(r_\xi)). \)

**Proof.** Let \( q' \leq q \) be in \( W(p') \) with \( q \in W_k(p') \) and \( k \geq n + \sup_{\xi \leq \chi} \text{mtp}(g(r_\xi)) \) and suppose that \( T_{\alpha'}(q') \neq \emptyset \). Denote \( \tau := \tau_{q'} \) and \( \delta := \max(T_{\alpha'}(q')). \)

Since \( k \geq n, q, q' \in W(s_\tau) \). Also, \( i \leq \xi_{\tau} \), as otherwise \( T_{\alpha'}(q') = \emptyset \). Hence, \( T_{\alpha'}(q') = S^{S_{\xi, \tau}}_\alpha(w(p_\xi, q')).\) Then \( w(p_\xi, q') \leq w(p_\xi, q) \) is a pair in \( W_{k-n}(p_\xi) \) with \( k - n \geq \text{mtp}(g(r_\xi)) = m(S^{S_{\xi, \tau}}_\alpha) \). So, by Definition 4.4(4) regarded with respect to \( S^{S_{\xi, \tau}}_\alpha \), it follows that \( (\delta, w(p_\xi, q)) \in R \). Thus, \( (\delta, q) \in R \), as wanted.

The combination of the above subclaims yield Clause (4) for \( T_i \).

**Claim 4.15.3.** The sequence \( \vec{T} = \langle T_i : W(p') \to [\mu]^{<\mu} | i \leq \alpha' \rangle \) is a \( p' \)-strategy.

**Proof.** We need to go over the clauses of Definition 4.6. However, Clause (1) is trivial, Clause (2) is established in the preceding claim, and Clauses (3) and (5) follow from the corresponding features of \( S \) and the \( S^{S_{\xi, \tau}} \)'s. Finally, Clause (4) can be proved exactly as in [PRS20, Claim 6.16.2].

Thus, we have established that \( b := (p', \vec{T}) \) is a legitimate condition in \( A \), such that \( \text{mtp}(b) \leq n + \sup_{\xi \leq \chi} \text{mtp}(g(r_\xi)) \).

The next claims take care of the second bullet concerning \( b \).

**Claim 4.15.4.** Let \( \tau < \theta \). For each \( q \in W_\alpha(s_\tau) \), \( w(p', q) = w(s_\tau, q) = q \).

**Proof.** The first equality can be proved exactly as in [PRS20, Claim 6.16.4]. For the second, notice that \( q \) and \( w(s_\tau, q) \) are conditions in \( W(s_\tau) \) with the same length. Hence, Fact 2.6(2) yields \( q = w(s_\tau, q) \), as wanted.

**Claim 4.15.5.** \( \pi(b) = p' \) and \( b \leq^0 a \).

**Proof.** The proof of this can be found in [PRS20, Claim 6.16.3].
Claim 4.15.6. For each $\tau < \theta$, $\Diamond(b)(s_\tau) \leq^0 g(r_\xi)$.\footnote{Recall that $\langle s_\tau \mid \tau < \theta \rangle$ was an enumeration of $W_n(p')$.}

Proof. Let $\tau < \theta$ and $T^\tau_\xi$ be denote the $s_\tau$-strategy such that $\Diamond(b)(s_\tau) = (s_\tau, T^\tau_\xi)$. By Corollary 4.10 we have that $\pi(\Diamond(b)(s_\tau)) = s_\tau \leq^0 p_\xi$.

Thus, if $\xi < \iota$ then $\Diamond(b)(s_\tau) \leq^0 [g(r_\xi)]^\mu = g(r_\xi)$, and we are done.

So let us assume that $\iota \leq \xi$ and let $i \leq \alpha_\xi$ and $q \in W(s_\tau)$. By Definition 4.8(*), $T^\tau_i(q) = T_i(w(p',q))$ and by the preceding claim, $w(p',q) = w(s_\tau,q) = q$, hence $T^\tau_i(q) = T_i(q)$. Also $r_{\xi_\eta} = w(p,s_\eta) = w(p,s_\tau) = r_\xi$, hence

$$T^\tau_i(q) = S^\xi_{\min\{i,\alpha_\xi\}}(w(p_\xi,q)) = S^\xi_{\alpha_\xi}(w(p_\xi,q)).$$

Altogether, $\Diamond(b)(s_\tau) \leq^0 g(r_\xi)$, as wanted. \hfill \Box

The above claims yield the proof of the lemma. \hfill \Box

Combining Lemmas 4.10 and 4.15 we arrive at:

Corollary 4.16. ($\Diamond, \pi$) is a forking projection from $(A, \ell_A, c_A)$ to $(P, \ell, c_A)$ having the weak mixing property. \hfill \Box

Now we take advantage of this latter corollary to establish that $(A, \ell_A, c_A)$ is $\Sigma$-Prikry and that $(A, \ell_A)$ has property $D$. On this respect, note that the latter statement follows combining Corollary 4.16, Lemma 2.28 and property $D$ of $(P, \ell)$ (Setup 4). For the former let us go over the clauses of Definition 2.3: Clauses (1),(3),(4),(5) and (6) follow from lemmas 4.5, 4.7, 4.8 and 4.9 of [PRS20], respectively. Clause (7) follows combining property $D$ of $(P, \ell)$ with Corollary 4.16 and Corollary 2.29. Also, by [PRS20, Corollary 4.13], $1_A \Vdash _A \mu = \kappa^+$. Finally, note that Clause (2) follows from Lemma 2.30 together with Corollary 4.16 and Fact 4.13.

Altogether, we arrive at the main result of this section:

Corollary 4.17. Suppose:

\begin{enumerate}
  \item $(P, \ell, c)$ is a $\Sigma$-Prikry notion of forcing such that the pair $(P, \ell)$ has property $D$;
  \item $P \Vdash P \mu = \kappa^+$;
  \item $P = (P, \leq)$ is a subset of $H_{\mu^+}$;
  \item $P_\tau \in P$ forces that $z$ is a $P_\tau$-name for a stationary subset of $(E^\mu_{\nu^+})^V$ that does not reflect in $\{\alpha < \mu \mid \omega < cf^V(\alpha) < \kappa\}$.
\end{enumerate}

Then, there exists a $\Sigma$-Prikry triple $(A, \ell_A, c_A)$ such that $(A, \ell_A)$ has property $D$ and for which the following are true:

\begin{enumerate}
  \item $(A, \ell_A, c_A)$ admits a forking projection $(\Diamond, \pi)$ to $(P, \ell, c)$ that has the weak mixing property;
  \item for each $n < \omega$, $\mathbb{H}_n^\tau$ is $\mu$-directed-closed;
  \item $1_A \Vdash A \mu = \kappa^+$;
  \item $A = (A, \subseteq)$ is a subset of $H_{\mu^+}$;
  \item $[P_\tau]^A$ forces that $z$ is nonstationary.
\end{enumerate}
Proof. Item (1) and the assertion that \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) is \(\Sigma\)-Prikry and that \((\mathbb{A}, \ell_\mathbb{A})\) has property \(D\) follow from our previous arguments. Item (2) follows from Fact 4.13 and items (3), (4) and (5) are consequence of Fact 4.9. \(\Box\)

4.2. Connecting the dots. For the rest of this section, we make the following assumptions:

- \(\Sigma = \langle \kappa_n \mid n < \omega \rangle\) is an increasing sequence of Laver-indestructible supercompact cardinals;
- \(\kappa := \sup_{n<\omega} \kappa_n\), \(\mu := \kappa^+\) and \(\lambda := \kappa^{++}\);
- \(2\kappa = \kappa^+\) and \(2\mu = \mu^+\);
- \(\Gamma := \{ \alpha < \mu \mid \omega < \text{cf}^V(\alpha) < \kappa \}\).

Under these assumptions, [PRS20, Corollary 5.11] reads as follows:

**Fact 4.18.** If \((\mathbb{P}, \ell, c)\) is a \(\Sigma\)-Prikry notion of forcing such that \(1_{\mathbb{P}} \Vdash \mathbb{P} \leq \mu = \kappa^+\), then \(V^\mathbb{P} \models \text{Refl}(<\omega, \Gamma)\).

We now want to appeal to the iteration scheme of the previous section. For this, we need to introduce our three building blocks of choice.

**Building Block I.** Let \(Q\) be the Extender Based Prikry Forcing (EBPF) for blowing up \(2^\kappa\) to \(\kappa^{++}\). By results in [Pov20, Ch.10, §2.5], the EBPF can be regarded as a \(\Sigma\)-Prikry triple \((Q, \ell, c)\) and \((Q, \ell)\) has property \(D\). Also, \(Q\) is a subset of \(H_\mu^+\) and \(1_{\mathbb{Q}} \Vdash Q \leq \mu = \kappa^+\). In addition, \(\kappa\) is singular, so that \(1_{\mathbb{Q}} \Vdash \kappa\) is singular”.

**Building Block II.** For every \(\Sigma\)-Prikry triple \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) having property \(D\) such that \(P = (\mathbb{P}, \leq)\) is a subset of \(H_\mu^+\) and \(1_{\mathbb{P}} \Vdash \mu = \kappa^+\), every \(r^* \in P\), and every \(\mathbb{P}\)-name \(z \in H_\mu^+\), we are given a corresponding \(\Sigma\)-Prikry triple \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) having property \(D\) such that:

(a) \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) admits a forking projection \((\mathfrak{h}, \pi)\) to \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) that has the weak mixing property;
(b) for each \(n < \omega\), \(\mathbb{A}_n\) is \(\kappa_n\)-directed-closed;\(^{24}\)
(c) \(1_{\mathbb{A}} \Vdash \mu = \kappa^+\);
(d) \(\mathbb{A} = (A, \leq)\) is a subset of \(H_\mu^+\);
(e) each element of \(A\) is a pair \((x, y)\) with \(\pi(x, y) = x\);
(f) if \(r^* \in P\) forces that \(z\) is a \(\mathbb{P}\)-name for a stationary subset of \((E^\mu_\mathbb{P})^V\) that does not reflect in \(\Gamma\), then \([r^*]_{\mathbb{A}} \Vdash \text{“}z\text{ is nonstationary}”\).

**Remark 4.19.** The above block is obtained as follows.

- If \(r^* \in P\) forces that \(z\) is a \(\mathbb{P}\)-name for a stationary subset of \((E^\mu_\mathbb{P})^V\) that does not reflect in \(\Gamma\), then we invoke Corollary 4.17.
- Otherwise, let \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) and \((\mathfrak{h}, \pi)\) be as in Example 2.15. By virtue of Lemma 2.27, \((\mathfrak{h}, \pi)\) has the weak mixing property. Hence, Lemma 2.29 and Lemma 2.21 combined imply that \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) is \(\Sigma\)-Prikry and that \((\mathbb{A}, \ell_\mathbb{A})\)

\(^{24}\)Recall Footnote 11 in page 14.
Lemma 4.22. For all \( \alpha \in [2, \mu^+] \), \( n < \omega \), and \( \epsilon < \mu \),
\[
D_{\alpha,n}^\epsilon := \{ b \in (\mathbb{P}_\alpha)_n \mid \forall \beta + 1 \in B_{\alpha} \ [\epsilon < \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(b), \pi_{\alpha,\beta}(b))] \}
\]
is dense in \((\mathbb{P}_\alpha)_n\).

In particular, for all \( \alpha \in [2, \mu^+] \) and \( n < \omega \), \((\mathbb{P}_\alpha)_n\) is dense in \((\mathbb{P}_\alpha)_{\omega}\).

Proof. We proceed by induction on \( \alpha \geq 2 \). The base case \( \alpha = 2 \) can be derived from Claim 4.14.1.\(^{25}\) So, let us suppose that we are given \( \alpha \in (2, \mu^+] \) such that for all \( \beta \in [2, \alpha) \), \( n < \omega \) and \( \epsilon < \mu \), \( D_{\alpha,n}^\epsilon \) is dense in \((\mathbb{P}_\beta)_n\).

\(^{25}\) For more details, see the discussion of the successor step below.
Case 1: Suppose that $\alpha = \beta + 1$ is a successor ordinal. Let $\alpha \in P_\alpha$ and $\epsilon < \mu$. Fix $\vec{S}$ such that $a = \pi_{\alpha,\beta}(a) \prec \vec{S}$. Appealing to Claim 4.14.1, we find $a' \leq_{\beta} (a \upharpoonright \beta)$ and $\epsilon > \max\{\epsilon, \sigma(a)\}$ with $(a', \epsilon) \in R_\beta$. By the inductive hypothesis, we may now pick $b' \in D^\beta_{\alpha,\epsilon}(a)$ extending $a'$. Note that $(b', \epsilon) \in R_\beta$, as well.

Set $b := b' \prec (S'_i \mid i \leq \text{dom}(\vec{S}) + 1)$, where for each $q \in W(a'' \upharpoonright \beta)$, $S'_i(q)$ is defined as follows:

$$S'_i(q) := \begin{cases} S_i(w(a \upharpoonright \beta, q)), & \text{if } i \leq \text{dom}(\vec{S}); \\ S_i(w(a \upharpoonright \beta, q)) \cup \{\epsilon\}, & \text{otherwise.} \end{cases}$$

By our choice of $\epsilon$, it follows that $b \in P_\alpha$ and that $b \leq_{\alpha} a$. Moreover, $\text{mtp}_{\beta+1}(b) = 0$. Thus, since $b' \in P_\beta$, it follows that $b \in P_\alpha$. Finally, since $B_0 = B_\eta \cup \{\alpha\}$, our choice of $b'$ implies that $b$ is in $D^\epsilon_{\alpha,\epsilon}(a)$.

Case 2: Suppose that $\text{cf}(\alpha) > \kappa$. Let $a \in P_\alpha$ and $\epsilon < \mu$. Then $B_\alpha$ is bounded in $\alpha$. Fix $\gamma < \alpha$ such that $a = \pi_{\alpha,\gamma}(a) \ast \emptyset_\alpha$. By the inductive hypothesis, we find $a' \in D^\epsilon_{\gamma,\epsilon}(a)$ extending $a \upharpoonright \gamma$. Set $b := a' \ast \emptyset_\alpha$, so that $B_0 = B_\alpha$. Then $b \in D^\epsilon_{\alpha,\gamma}(a)$ extends $a$, as desired.

Case 3: Suppose that $1 < \text{cf}(\alpha) \leq \kappa$. As $\kappa$ is the limit of the strictly increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ (see Building Block I), we may let $n < \omega$ be the least such that $\text{cf}(\alpha) < \kappa_n$.

Claim 4.22.1. For all $l \geq n$ and $\epsilon < \mu$, $D^\epsilon_{\alpha,l}$ is dense in $(\mathbb{P}_\alpha)_l$.

Proof. Let $a \in P_\alpha$ and $\epsilon < \mu$ such that $l := \ell_\alpha(a)$ is greater or equal to $n$. By the proof of Case 2, we may assume that $B_\alpha$ is unbounded in $\alpha$. Let $(\langle \gamma_\tau \mid \tau < \text{cf}(\alpha) \rangle)$ be the increasing enumeration of some cofinal subsets of $B_\alpha$ of size $\text{cf}(\alpha)$. For every $\tau < \text{cf}(\alpha)$, $\gamma_\tau$ is a successor ordinal, so we let $\beta_\tau$ denote its predecessor. We shall construct a sequence $(b_\tau \mid \tau < \text{cf}(\alpha))$ such that, for each $\tau < \text{cf}(\alpha)$, the following hold:

1. $b_\tau \in \mathbb{P}_\tau$, and $b_\tau \leq_{\beta_\tau} a \upharpoonright \gamma_\tau$;
2. if $\tau$ is successor, then for all $\beta + 1 \in B_{\beta_\tau}$, $\epsilon < \sigma^{\beta+1}(b_\tau, \pi_{\gamma_\tau,\beta_\tau}(b_\tau))$;
3. $\pi_{\gamma_\tau,\eta}(b_\tau) \leq_{\eta} b_\eta$ for all $\eta \leq \tau$.

Let $\vec{S}$ be such that $\pi_{\alpha,\beta_0+1}(a) = \pi_{\alpha,\beta_0}(a) \prec \vec{S}$. By Claim 4.14.1, we may find $b_0^* \leq_{\beta_0} \pi_{\alpha,\beta_0}(a)$ and $\epsilon > \max\{\epsilon, \sigma(a)\}$ such that $(b_0^*, \epsilon) \in R_{\beta_0}$. Also, by the inductive assumption, we may assume that $b_0^* \in P_{\beta_0}$. Now set

$$b_0 := b_0^* \prec (S'_i \mid i \leq \text{dom}(\vec{S}) + 1)$$

where the latter is defined as in Case 1. Once again, $\vec{S}'$ is a $b_0^*$-strategy, and so $b_0 \leq_{\beta_0+1} \pi_{\alpha,\beta_0+1}(a)$. Also, thanks to our choice of $\epsilon$, $\text{mtp}_{\beta_0+1}(b_0) = 0$, hence $b_0 \in P_{\beta_0+1}$. Note also that $\sigma_{\beta_0+1}(b_0, b_0^*) = \epsilon > \epsilon$, so that (2) holds as well. Finally, (3) is obvious.
Next, let us assume that \( \tau < \text{cf}(\alpha) \) and we have already successfully defined \( \langle b_\eta \mid \eta < \tau \rangle \).

- If \( \tau = \eta + 1 \), then set \( c := \rho_{\gamma_\eta}(\pi_{\alpha,\gamma_\eta}(a))(b_\eta) \). By Clause (1), \( c \in \mathcal{P}_{\gamma_\eta} \).

Hence we may appeal to Case 1 above and find \( b_\tau \in \mathcal{P}_{\gamma_\eta} \) with \( b_\tau \leq_\gamma c \), witnessing (2). Clearly, \( b_\tau \) satisfies (1)–(3).

- Otherwise, Lemma 3.9 and (3) of the inductive assumption imply that \( \langle b_\eta \mid \eta < \tau \rangle \) is a \( \leq_{\gamma_\eta} \)-decreasing sequence in \( (\mathcal{P}_{\gamma_\eta})_\ell \). Thus, by Lemma 3.13, it admits a lower bound \( b_\tau^* \in (\mathcal{P}_{\gamma_\eta})_\ell \).

Thereby, a sequence \( \langle b_\tau \mid \tau < \text{cf}(\alpha) \rangle \) witnessing (1)–(3) has been constructed. Repeating the argument of the limit case above we find a lower bound \( b \) in \( (\mathcal{P}_{\alpha})_l \) for this sequence. By virtue of Claim 3.13.1 we may assume that \( B_b = \bigcup_{\tau < \text{cf}(\alpha)} B_{b_\tau} \). Clearly, \( b \leq_\alpha a \). We claim that \( b \in D_{a,l}^\ell \). To see this, fix \( \beta + 1 \in B_b \). Let \( \tau \in \text{nacc}(\text{cf}(\alpha)) \) be such that \( \beta + 1 \in B_{b_\tau} \). Since \( \pi_{\alpha,\gamma_\tau}(b) \leq_\gamma b_\tau \), Fact 4.21 and Clause (2) above imply \( \epsilon < \sigma^{\beta + 1}(b_\tau, \pi_{\gamma_\tau,\beta_\tau}(b_\tau)) \leq \sigma^{\beta + 1}(\pi_{\alpha,\gamma_\tau}(b), \pi_{\alpha,\beta_\tau}(b)) \).

For each \( l < \omega \), let us say that \( \upharpoonright_l \) holds if for each \( \epsilon < \mu \), \( D_{a,l}^\epsilon \) is dense in \( (\mathcal{P}_{\alpha})_l \). By Claim 4.22.1, \( \upharpoonright_l \) holds for all \( l \geq n \). In particular, if \( n = 0 \), then we are done with the proof of the lemma. So, let us suppose that \( n \geq 1 \), and let us show how to bring this down to \( \upharpoonright_0 \).

**Claim 4.22.2.** Let \( l < \omega \) be such that \( \upharpoonright_{l + 1} \) holds. Then \( \upharpoonright_l \) holds, as well.

**Proof.** Let \( a \in \mathcal{P}_\alpha \) and \( \epsilon < \mu \) such that \( \ell_\alpha(a) = l \). We need to find a condition \( b \in D_{a,l}^\epsilon \) extending \( a \).

Set \( a_0 := a \) and \( \epsilon_0 := \epsilon \). Letting \( \hat{s} \) be a good enumeration of \( W_1(a_0) \) we can play \( \mathcal{D}_{\mathcal{P}_\alpha}^\epsilon(a_0, \hat{s}, D_{a,l+1}^\epsilon) \) and produce a sequence \( \langle b_\xi \mid \xi < \chi \rangle \) corresponding to the moves of \( \Pi \).

Since \( \upharpoonright_{l+1} \) holds, \( D_{a,l+1}^\epsilon \) (hence, \( (\mathcal{P}_\alpha)_{l+1} \)) is a dense subset of \( (\mathcal{P}_\alpha)_{l+1} \). So, we can appeal to Lemma 3.11, finding a condition \( a_1 \in \mathcal{P}_\alpha \) which diagonalizes \( \langle b_\xi \mid \xi < \chi \rangle \) and such that, for all \( \beta + 1 \in B_{a_0} \),

\[
\pi_{\alpha,\beta+1}(a_1) \subseteq_{\beta+1} \pi_{\alpha,\beta+1}(a_0). 
\]

Recalling Definition 4.11, this means the following.

1. For each \( \beta + 1 \in B_{a_0} \), and \( i \leq \max(\text{dom}(\hat{S}_{\beta,a_1})) \),

\[
S^\beta,a_1_i(\pi_{\alpha,\beta}(a_1)) = S^\beta,a_0_i(\pi_{\alpha,\beta}(a_0)),
\]

where for each \( i \in \{0, 1\} \) and \( \beta + 1 \in B_{a_0} \), we set

\[
\pi_{\alpha,\beta+1}(a_1) := \pi_{\alpha,\beta}(a_1)^{\hat{S}_{\beta,a_1}}, \quad \hat{S}_{\beta,a_1} := \langle S^\beta,a_1_j \mid j \leq \alpha_i \rangle.
\]

By 0-extending \( a_1 \), we may suppose that \( a_1 \vDash (\mathcal{P}_\alpha)_l \), \( \epsilon_1 \in \bigcap_{\beta + 1 \in B_{a_0}} C^\beta_{l_1} \), where \( \epsilon_1 \) is some ordinal such that \( \sup_{\xi < \chi} \sigma(b_\xi) < \epsilon_1 < \mu \). Therefore:

\[\text{Since } \pi_{\alpha,\beta}(a_1) \leq_\beta \pi_{\alpha,\beta}(a_0) \text{ note that } w(\pi_{\alpha,\beta}(a_0), \pi_{\alpha,\beta}(a_1)) = \pi_{\alpha,\beta}(a_0).\]
(II) For $\beta + 1 \in B_{a_0}$, $\pi_{\alpha,\beta}(a_1) \Vdash (p_{\beta})^{c_0} \epsilon_1 \in C^\beta_{\beta}$. Also, $\sigma(a_0) < \epsilon_1$.\footnote{Note that the last assertion follows from $b_\xi \leq a_0$ and our choice of $\epsilon_1$.}

The whole point of diagonalizing conditions in $D^0_{\alpha,l+1}$ is (III) and (IV):

**Subclaim 4.22.2.1.**

(III) For each $\beta + 1 \in B_{a_0}$ and $q \in W_1(\pi_{\alpha,\beta}(a_1))$, 

$$\epsilon_0 < \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(a_1), q).$$

**Proof.** Let $\beta + 1$ and $q$ be as in the statement. Defining $c := \cap_{\alpha,\beta}(a_1)(q)$ we have that $c \in W_1(a_1)$, and so there is $\xi < \chi$ such that $c \leq a_0$. Since $b_\xi \leq a_0$, then $B_{a_0} \subseteq B_{b_\xi}$, and so $\beta + 1 \in B_{b_\xi}$. Hence,

$$\epsilon_0 < \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), \pi_{\alpha,\beta}(b_\xi)) \leq \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(c), q),$$

where the second inequality follows from $\pi_{\alpha,\beta+1}(c) \leq_{\beta+1} \pi_{\alpha,\beta+1}(b_\xi)$ and $\pi_{\alpha,\beta}(c) = q$. Also, by Fact 4.21,

$$\sigma^{\beta+1}(\pi_{\alpha,\beta+1}(c), q) = \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(a_1), q),$$

which yields the desired inequality. \qed

**Subclaim 4.22.2.2.**

(IV) For all $c \in W_{\geq 1}(a_1)$ and $\beta + 1 \in B_{a_1}$,

$$\pi_{\alpha,\beta}(c) \Vdash (p_{\beta})^{c_0} \hat{C}^\beta_{\lambda_0(c)} \cap (\hat{c}_0, \hat{c}_1) \neq \emptyset.$$ 

**Proof.** Fix $c \in W_{\geq 1}(a_1)$ and $\beta + 1 \in B_{a_1}$. Appealing to Fact 2.6, let $\hat{c}$ be the unique condition in $W_1(a_1)$ such that $c \leq \hat{c}$.

Since $a_1$ diagonalizes $\langle b_\xi \mid \xi < \chi \rangle$, there is $\xi < \chi$ such that $\hat{c} \leq a_0$. Recall that $b_\xi \in P_{a_1}$, hence Lemma 3.9 yields $\pi_{\alpha,\beta+1}(b_\xi) \in P_{\beta+1}$. In particular, we have that $\text{mtp}_{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi)) = 0$ or, equivalently,

$$(\pi_{\alpha,\beta}(b_\xi), \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), q)) \in R_{\beta}, \text{ for all } q \in W(\pi_{\alpha,\beta}(b_\xi)).$$\footnote{See Definition 4.4(4) and Definition 4.12.}

Since $\pi_{\alpha,\beta}(c) \leq \pi_{\alpha,\beta}(\hat{c}) \leq \pi_{\alpha,\beta}(b_\xi)$,

$$\pi_{\alpha,\beta}(c) \Vdash (p_{\beta})^{c_0} \sigma^{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), \pi_{\alpha,\beta}(b_\xi)) \in \hat{C}^\beta_{\lambda_0(c)}.$$ 

Finally, as $\sigma^{\beta+1}(\pi_{\alpha,\beta+1}(b_\xi), \pi_{\alpha,\beta}(b_\xi)) \in (\epsilon_0, \epsilon_1)$, the claim follows. \footnote{Recall that $\epsilon_1 > \sup_{\xi < \chi} \sigma(b_\xi)$.}

Repeating the above argument $\omega$-many times we obtain a $\leq_0$-decreasing sequence $\langle a_n \mid n < \omega \rangle$ and an increasing sequence of ordinals $\langle \epsilon_n \mid n < \omega \rangle$ such that, for each $n < \omega$, $\langle a_n, a_{n+1}, \epsilon_n, \epsilon_{n+1} \rangle$ witnesses together (I), (II), (III) and (IV). Note that any $\leq_0$-lower bound $b \in P_{a_1}$ for this sequence would give a condition in $D^0_{\alpha,l}$ extending $a$.

**Subclaim 4.22.2.3.** There is $b \in P_{a_1}$ such that $b \leq_0 a_n$ for all $n < \omega$.

In particular, there is $b \in D^0_{\alpha,l}$ such that $b \leq_0 a$.\footnote{Recall that $\epsilon_1 > \sup_{\xi < \chi} \sigma(b_\xi)$.}
We construct this sequence by recursion on $\tau < \theta$. The condition $b$ will be defined as the limit of a sequence $\langle b_\tau \mid \tau < \theta \rangle$ such that

1. $b_\tau \in (P_{\beta_{\tau + 1}}) / \tau$ and $B_{b_\tau} = \{ \beta_0 + 1 \mid \varphi \leq \tau \}$;
2. $b_\tau \leq 0_{\beta_{\tau + 1}} \pi_{\alpha,\beta_{\tau + 1}}(a_n)$ for all $n < \omega$;
3. $\pi_{\beta_{\tau + 1},\beta_{\tau + 1}}(b_\tau) = b_\varphi$ for all $\varphi \leq \tau$.

We construct this sequence by recursion on $\tau < \theta$.

- Suppose $\tau = 0$. Since $\langle \pi_{\alpha,1}(a_n) \mid n < \omega \rangle$ is a decreasing sequence in $(P_1)_l = (\tilde{P}_1)_l$, we can appeal to Clause (2) of Definition 2.3 and let $p \in (P_1)_l$ be a lower bound for it.

For each $n < \omega$, set

$$
c_n := \pi_{\gamma_0,\beta_0}(\pi_{\alpha,\gamma_0}(a_n))([p]^{P_{\beta_0}}).
$$

For each $n < \omega$, $[p]^{P_{\beta_0}} \leq 0_{\beta_0} \pi_{\alpha,\beta_0}(a_n)$, hence $c_n \in (P_{\beta_0})_{\beta_0}^\infty$. Actually, $\vec{c} := \langle c_n \mid n < \omega \rangle$ is $\leq_{\pi_{\gamma_0,\beta_0}}$-decreasing, hence, as in the proof of [PRS20, Lemma 6.15], $\langle \vec{c} \rangle$ is order-isomorphic to $(\omega, \emptyset)$.

Let $n_0 < \omega$ denote the least such that $\gamma_0 \in B_{a_n}$ for each $n \geq n_0$. For each $n \geq n_0$, set $c_n := [p]^{P_{\beta_0}} \cap S^n$. By Clause (5) of Definition 2.23

$$\alpha_n = \max(\text{dom}(S^{3\beta_0,a_n})) = \max(\text{dom}(S^n)).$$

Set $\alpha := \sup_{n < \omega} \alpha_n$. Note that for every $n \geq n_0$ and $i \leq \alpha_n$ we have

$$S^n_i([p]^{P_{\beta_0}}) = S^{3\beta_0,a_n}_i(w(\pi_{\alpha,\beta_0}(a_n), [p]^{P_{\beta_0}})) = S^{3\beta_0,a_n}_i(\pi_{\alpha,\beta_0}(a_n)),
$$

where the right-most equality follows from $[p]^{P_{\beta_0}} \leq_{\beta_0} 0 \pi_{\alpha,\beta_0}(a_n)$.

Combining this with Clause (1) we have

$$S^n_i([p]^{P_{\beta_0}}) = S^{1\beta_0,a_{\alpha_n}}_{\min(i,\alpha_n)}(\pi_{\alpha,\beta_0}(a_n)).
$$

Let us now define a $\tilde{P}_{\gamma_0}$-lower bound for $\langle c_n \mid n < \omega \rangle$.

If $[p]^{P_{\beta_0}}$ is incompatible with $r_{\beta_0}^\ast$, then, for each $n < \omega$, Remark 4.5 yields

$$S^n = \langle S^n_i : W([p]^{P_{\beta_0}}) \rightarrow \{ \emptyset \} \mid i \leq \alpha_n \rangle.
$$

Setting $b_0 := [p]^{P_{\beta_0}} \cap S$, where $S := \langle S_i \mid i \leq \alpha \rangle$ is the sequence defined according to the following casuistic:

- For $i < \alpha$, $S_i(q)$ is defined as the unique element of

$$\{ S^n_i(q) \mid n \geq n_0, \alpha_n \geq i \}.$$

By (*), note that $S_i([p]^{P_{\beta_0}}) = S^{3\beta_0,a_{\alpha_n}}_{\min(i,\alpha_n)}(\pi_{\alpha,\beta_0}(a_{\alpha_n}))$.

- At stage $\alpha$, we distinguish two cases:
  - If $S_i(q) = \emptyset$ for all $i < \alpha$, then we continue and let $S_\alpha(q) := \emptyset$.

30For the verification of $[p]^{\gamma_0} \in \tilde{P}_{\beta_0}$, see the proof of Fact 4.22.2.3.2 below.
Otherwise, let
\[
S_\alpha(q) := \begin{cases} 
\bigcup_{i \leq \alpha} S_i(q) \cup \{\epsilon\}, & \text{if } \ell_\alpha(q) \geq 1, \\
S^{\delta_\alpha,a_{n_0}}_{\alpha,a_{n_0}}(\pi_{\alpha,\beta_0}(a_{n_0})) \cup \{\epsilon\}, & \text{otherwise.}
\end{cases}
\]

**Subsubclaim 4.22.2.3.1.** For each \( q \in W([p]^{\beta_0}), \max(S_\alpha(q)) = \epsilon. \)

**Proof.** In case \( q = [p]^{\beta_0} \) we appeal to Clause (II), noting that \( \sigma(a_{n_0}) < \epsilon_{n_0+1} < \epsilon. \) Otherwise, it is enough to show that
\[
\epsilon = \sup\{\max(S^a_{\alpha,\beta_0}(q)) \mid S^a_{\alpha,\beta_0}(q) \neq \emptyset, n_0 \leq n < \omega\},
\]
which follows combining Clauses (II) and (III). \( \square \)

**Subsubclaim 4.22.2.3.2.** \( b_0 \) witnesses (1)–(3).

**Proof.** A moment’s reflection makes it clear that we only need to verify (1).

Considering Fact 4.22.2.3.1 and that \( S^a_{\alpha,\beta_0} \) were \([p]^{\beta_0}\)-strategies, it is clear that we only need to verify that \( S_\alpha \) is a labeled \([p]^{\beta_0}\)-tree with \( m(S_\alpha) = 0. \)

For this it is enough to check Clauses (3) and (4) of Definition 4.4:

(3) To avoid trivialities, assume that \( S_\alpha(q) \neq \emptyset. \) We already know that
\[
q \models^{\beta_0} S^a_{\alpha,\beta_0}(q) \cap \hat{T} = \emptyset, \text{ for each } n \geq n_0.
\]
Thus, to establish the clause it suffices to show that \( (\epsilon, [p]^{\beta_0}) \in R_{\beta_0}. \)
First note that \([p]^{\beta_0} \leq_{\beta_0} r_{\beta_0}. \) Also, for each \( n < \omega, [p]^{\beta_0} \leq_{\beta_0} \pi_{\alpha,\beta_0}(a_n), \) hence Clause (II) yields \([p]^{\beta_0} \models^{(p_{\beta_0})_\ell} \epsilon_n \in C^\beta_0, \) for all \( n \geq n_0. \)
Thus,
\[
[p]^{\beta_0} \models^{(p_{\beta_0})_\ell} \epsilon \in C^\beta_0.
\]
Now, let \( r \leq_{\beta_0} [p]^{\beta_0}. \) If \( r \leq_{\beta_0} [p]^{\beta_0} \) then the above yields
\[
r \models^{(p_{\beta_0})_\ell} \epsilon \in C^\beta_0.
\]

Otherwise, there is some \( c \in W_{\geq 1}(a_{n_0}) \) such that \( r \leq_{\beta_0} \pi_{\alpha,\beta_0}(c) \) and so Clause (IV) yields \( \pi_{\alpha,\beta_0}(c) \models^{(p_{\beta_0})_\ell} \epsilon \in C^\beta_0_{\ell_{\beta_0}(r)}. \)
Therefore,
\[
\pi_{\alpha,\beta_0}(r) \models^{(p_{\beta_0})_\ell_{\beta_0}(r)} \epsilon \in C^\beta_0_{\ell_{\beta_0}(r)}, \text{ as wanted.}
\]

Altogether, we have shown that \((\epsilon, [p]^{\beta_0}) \in R_{\beta_0}. \)

(4) The clause follows combining Fact 4.22.2.3.1 with the previous argument. Actually, it follows that \( m(S_\alpha) = 0. \)

Thus, \( b_0 \in P_{\gamma_0} \) and \( \operatorname{mtp}_{\gamma_0}(b_0) = m(S_\alpha) = 0. \)

Also, since \( p \in \hat{P}_1 \) and \([p]^{\beta_0} = p \uparrow \emptyset_{\beta_0}, \) Lemma 3.9 yields \([p]^{\beta_0} \in \hat{P}_{\beta_0}. \)
Altogether, \( b_0 \in \hat{P}_{\gamma_0} \) and \( B_{\beta_0} = \{\beta_0 + 1\}. \) \( \square \)

This completes the construction when \( \tau = 0. \)

Assume that we have already defined \( \langle b_\tau \mid \tau < \eta \rangle \) and let us show how to construct \( b_\eta. \) For each \( n < \omega, \) set
\[
c_n := \hat{m}_{\gamma_n,\beta_0}(\pi_{\alpha,\gamma_n}(a_n))(b^*_\eta).
\]
where, \( b_\eta^* := (\bigcup_{\tau<\eta} b_\tau) \ast \emptyset_{\beta_\eta} \).

**Subsubclaim 4.22.2.3.3.** \( b_\eta^* \in \dot{P}_{\beta_\eta} \) and \( b_\eta^* \leq^0_{\alpha,\beta_\eta} (a_n) \), for each \( n < \omega \).

**Proof.** Note that once we have established the first assertion the second will follow automatically from (2) and (3) of our induction hypothesis.

- Assume \( \eta = \tau + 1 \). Then Clauses (1) and (3) of our induction hypothesis yield \( \bigcup_{\zeta<\eta} b_\zeta = b_\tau \in \dot{P}_{\beta_\tau} \). Using Lemma 3.9 we have that \( b_\eta^* \in \dot{P}_{\beta_\eta} \).

- Otherwise, \( \eta \) is limit. Set \( \varrho := \sup_{\tau<\eta} \beta_\tau \) and note that \( \varrho \leq \beta_\eta \). By Clause (3), \( \bigcup_{\tau<\eta} b_\tau \in P_\varrho \). Also, combining Clauses (1) and (3) it is not hard to check that for all \( \beta + 1 \in B_{\bigcup_{\tau<\eta} b_\tau} \), \( \text{mtp}_{\beta+1}(\pi_\varrho,\beta+1(\bigcup_{\tau<\eta} b_\tau)) = 0 \). Thus, \( \bigcup_{\tau<\eta} b_\tau \in \dot{P}_\varrho \). Once again, using Lemma 3.9, we have that \( b_\eta^* \in \dot{P}_{\beta_\eta} \).

\( \Box \)

An outright consequence of the above is that \( c_n \in P_{\gamma_\eta} \), for each \( n < \omega \).

Once again, \( \vec{c} := \langle c_n \mid n<\omega \rangle \) is \( \leq_{\pi_{\gamma_\eta}^{\alpha_\eta}} \) decreasing. As before, let \( n_\eta \) be the least integer such that \( \gamma_\eta \in B_{a_n} \) for all \( n \geq n_\eta \).

Set \( b_\eta := b_\eta^* \cap S \), where \( S \) is defined exactly as in (*) and (†) above, but using \( S_{\beta_\eta, a_\eta} \) instead of \( S_{\beta_0, a_0} \).

**Subsubclaim 4.22.2.3.4.** \( b_\eta \) witnesses (1)-(3).

**Proof.** For (1) and (2) one proceeds exactly as in Subsubclaims 4.22.2.3.1 and 4.22.2.3.2, complementing the argument with Subsubclaim 4.22.2.3.3 above. Finally, Clause (3) follows from our induction hypothesis.

\( \Box \)

The above completes the inductive construction of a sequence \( \langle b_\tau \mid \tau < \theta \rangle \) witnessing (1)-(3) above. Set \( b := (\bigcup_{\tau<\theta} b_\tau) \ast \emptyset_\alpha \).

**Subclaim 4.22.2.3.5.** \( b \in \dot{P}_\alpha \) and \( b \leq^0_\alpha a_n \) for all \( n < \omega \).

**Proof.** The latter assertion is an outright consequence of (1) and (2). As for the former, let \( \beta + 1 \in B_b \). By Clause (1), \( B_b = \{ \beta_\tau \mid \tau < \theta \} \), hence \( \beta + 1 \in B_{b_\alpha} \) for some \( \tau < \theta \). Combining (1) and (3) we have

\[
\text{mtp}_{\beta+1}(\pi_{\alpha,\beta+1}(b)) = \text{mtp}_{\beta+1}(\pi_{\beta_\tau+1,\beta+1}(b_\tau)) = 0.
\]

Thus, \( b \in \dot{P}_\alpha \), as wanted.

\( \Box \)

The above fact completes the proof of the subclaim.

Subclaim 4.22.2.3 yields a condition \( b \in D_{\alpha,1}^\epsilon \) such that \( b \leq^0_\alpha a \). Thereby, for every \( \epsilon < \mu \) the set \( D_{\alpha,1}^\epsilon \) is dense in \( (\mathcal{P}_\alpha)_\ell \), and thus \( \dagger_1 \) holds.

Appealing to Claim 4.22.2 iteratively we have that \( \dagger_0 \) holds. Namely, the moreover part of the lemma is satisfied. This completes the proof.

Thanks to Lemma 4.22 we can now appeal to the iteration scheme of Section 3 with respect to the building blocks of this section and obtain, in return, a \( \Sigma \)-Prikry triple \( (\mathcal{P}_\mu, \ell_\mu, c_\mu) \).

**Theorem 4.23.** In \( V^{\mathcal{P}_\mu} \) all of the following hold true:
(1) Any cardinal in \( V \) remains a cardinal and retains its cofinality;
(2) \( \kappa \) is a singular strong limit of countable cofinality;
(3) \( 2^\kappa = \kappa^+ \);
(4) \( \text{Refl}(\omega, \kappa^+) \).

Proof. (1) By Fact 2.7(1), no cardinal \( \leq \kappa \) changes its cofinality; by Fact 2.7(3), \( \kappa^+ \) is not collapsed, and by Definition 2.3(3), no cardinal \( > \kappa^+ \) changes its cofinality.

(2) In \( V \), \( \kappa \) is a singular strong limit of countable cofinality, and so by Fact 2.7(1), this remains valid in \( V^{\kappa^+} \).

(3) In \( V \), we have that \( 2^\kappa = \kappa^+ \). In addition, by Remark 3.3(1), \( P_\mu^+ \) is isomorphic to a subset of \( H_{\mu^+} \), so that, from \( |H_{\mu^+}| = \kappa^+ \), we infer that \( V^{\kappa^+} \models 2^\kappa \leq \kappa^{++} \). Finally, as \( P_\mu^+ \) projects to \( P_1 \) which is isomorphic to \( Q \), we get that \( V^{\kappa^+} \models 2^\kappa \geq \kappa^{++} \). Altogether, \( V^{\kappa^+} \models 2^\kappa = \kappa^{++} \).

(4) As \( \kappa^+ = \mu \) and \( \kappa \) is singular, \( \text{Refl}(\omega, \kappa^+) \) is equivalent to \( \text{Refl}(\omega, E_\mu^{<\kappa}) \).

By Fact 4.18, we already know that \( V^{\kappa^+} \models \text{Refl}(\omega, \Gamma) \). So, by Proposition 4.3, it suffices to verify that \( \text{Refl}(\omega, (E_\mu^{<\kappa})^V, \Gamma) \) holds in \( V^{\kappa^+} \).

Let \( G \) be \( P_{\mu^+} \)-generic over \( V \) and hereafter work within \( V[G] \). Towards a contradiction, suppose that there exists a subset \( T \) of \( (E_\mu^{<\kappa})^V \) that does not reflect in \( \Gamma \). Fix \( r^* \in G \) and a \( P_{\mu^+} \)-name \( \tau \) such that \( \tau_G \) is equal to such a \( T \) and such that \( r^* \) forces \( \tau \) to be a stationary subset of \( (E_\mu^{<\kappa})^V \) that does not reflect in \( \Gamma \). Furthermore, we may require that \( \tau \) be a nice name, i.e., each element of \( \tau \) is a pair \( (\xi, p) \) where \( (\xi, p) \in (E_\mu^{<\kappa})^V \times P_{\mu^+} \), and, for all \( \xi \in (E_\mu^{<\kappa})^V \), the set \( \{ p \mid (\xi, p) \in \tau \} \) is an antichain.

As \( P_{\mu^+} \) satisfies Clause (3) of Definition 2.3, \( P_{\mu^+} \) has the \( \mu^+-cc \). Consequently, there exists a large enough \( \beta < \mu^+ \) such that

\[
B_{r^*} \cup \bigcup \{ B_p \mid (\xi, p) \in \tau \} \subseteq \beta.
\]

Let \( r := r^* \restriction \beta \) and set

\[
\sigma := \{ (\xi, p \restriction \beta) \mid (\xi, p) \in \tau \}.
\]

From the choice of Building Block III, we may find a large enough \( \alpha < \mu^+ \) with \( \alpha > \beta \) such that \( \psi(\alpha) = (\beta, r, \sigma) \). As \( \beta < \alpha \), \( r \in P_\beta \) and \( \sigma \) is a \( P_\beta \)-name, the definition of our iteration at step \( \alpha + 1 \) involves appealing to Building Block II with \( (P_\alpha, \ell_\alpha, c_\alpha) \), \( r^* := r \star \emptyset_\alpha \) and \( z := \iota_\beta^\alpha(\sigma) \). For any ordinal \( \eta < \mu^+ \), denote \( G_\eta := \pi_{\mu^+-\eta}[G] \). By the choice of \( \beta \), and as \( \alpha > \beta \), we have

\[
\tau = \{ (\xi, p \star \emptyset_{\mu^+}) \mid (\xi, p) \in \sigma \} = \{ (\xi, p \star \emptyset_{\mu^+}) \mid (\xi, p) \in z \},
\]

so that, in \( V[G] \),

\[
T = \tau_G = \sigma_G \circ z_{G_\alpha}.
\]

In addition, \( r^* = r^* \star \emptyset_{\mu^+} \).

Finally, as \( r^* \) forces \( \tau \) is a stationary subset of \( (E_\mu^{<\kappa})^V \) that does not reflect in \( \Gamma \), \( r^* \) forces that \( z \) is a stationary subset of \( (E_\mu^{<\kappa})^V \) that does not reflect
in $\Gamma$. So, since $\pi_{\mu+\alpha+1}(r^*) = r^* \in [r^*]^\alpha$ is in $G_{\alpha+1}$, Clause (f) of Building Block II entails that, in $V[G_{\alpha+1}]$, there exists a club in $\mu$ which is disjoint from $T$. In particular, $T$ is nonstationary in $V[G]$, contradicting its very choice. □

Thus, we arrive at the following strengthening of the theorem announced by Sharon in [Sha05].

**Corollary 4.24.** Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, converging to a cardinal $\kappa$. Then there exists a forcing extension where the following properties hold:

1. $\kappa$ is a singular strong limit cardinal of countable cofinality;
2. $2^\kappa = \kappa^{++}$, hence $\text{SCH}_\kappa$ fails;
3. $\text{Refl}(\langle \omega, \kappa^+ \rangle)$ holds.

**Proof.** Let $L$ be the inverse limit of the iteration $\langle L_n; \dot{\mathcal{Q}}_n \mid n < \omega \rangle$, where $L_0$ is the trivial forcing and for positive integer $n$, if $L_n \models \text{"$\kappa_{n-1}$ is supercompact"}$, then $L_n \models \text{"$\dot{\mathcal{Q}}_n$ is a Laver preparation for $\kappa_n$ above $\kappa_{n-1}$"}$. After forcing with $L$, each $\kappa_n$ remains supercompact and, moreover, becomes indestructible under $\kappa_n$-directed-closed forcing. Also, the cardinals and cofinalities of interest are preserved.

Working in $V^L$, set $\mu := \kappa^+$, $\lambda := \kappa^{++}$ and $C := \text{Add}(\lambda, 1)$. Finally, work in $W := V^{L[\dot{C}]}$. Since $\kappa$ is singular strong limit of cofinality $\omega < \kappa_0$ and $\kappa_0$ is supercompact, $2^\kappa = \kappa^+$. Also, thanks to the forcing $C$, $2^\mu = \mu^+$. Altogether, in $W$, all the following hold:

- $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-Indestructible supercompact cardinals;
- $\kappa := \sup_{n < \omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$.

Now, appeal to Theorem 4.23. □

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31 Recall that by Fact 4.2, the extent of reflection obtained is optimal.
References


