SIGMA-PRIKRY FORCING II:
ITERATION SCHEME

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Abstract. In Part I of this series [PRS20], we introduced a class of notions of forcing which we call Σ-Prikry, and showed that many of the known Prikry-type notions of forcing that centers around singular cardinals of countable cofinality are Σ-Prikry. We proved that given a Σ-Prikry poset $P$ and a $P$-name for a non-reflecting stationary set $T$, there exists a corresponding Σ-Prikry poset that projects to $P$ and kills the stationarity of $T$. In this paper, we develop a general scheme for iterating Σ-Prikry posets, as well as verify that the Extender Based Prikry Forcing is Σ-Prikry. As an application, we blow up the power of a countable limit of Laver-indestructible supercompact cardinals, and then iteratively kill all non-reflecting stationary subsets of its successor. This yields a model in which the singular cardinal hypothesis fails and simultaneous reflection of finite families of stationary sets holds.

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1. Introduction

In the introduction to Part I of this series [PRS20], we described the need for iteration schemes and the challenges involved in devising such schemes, especially at the level of successor of singular cardinals. The main tool available to obtain consistency results at the level of singular cardinals and their successors is the method of forcing with large cardinals and, in particular, *Prikry-type forcings*. By Prikry-type forcings one usually means to a poset $\mathbb{P} = (P, \leq)$ having the following property.

**Prikry Property.** There exists an ordering $\leq^*$ on $P$ coarser than $\leq$ (typically, of a better closure degree) satisfying that for every sentence $\varphi$ in the forcing language and every $p \in P$ there exists $q \in P$ with $q \leq^* p$ deciding $\varphi$.

In this paper, we develop an iteration scheme Prikry-type posets, specifically, for the class of $\Sigma$-Prikry forcings that we introduced in [PRS20] (see Definition 2.3 below). Of course, viable iteration schemes for Prikry-type posets already exists, namely, the Magidor iteration and the Gitik iteration (see [Git10, S6]). In both these cases the ordering $\leq^*$ witnessing the Prikry Property of the iteration can be roughly described as the finite-support iteration of the $\leq^*$-orderings of its components. As the expectation from the final $\leq^*$ is to have an eventually-high closure degree, the two schemes are typically useful in the context where one carries an iteration $\langle \mathbb{P}_\alpha; \dot{\mathbb{Q}}_\alpha | \alpha < \rho \rangle$ with each $\dot{\mathbb{Q}}_\alpha$ being a $\mathbb{P}_\alpha$-name for either a trivial forcing, or a Prikry-type forcing concentrating on the combinatorics of the inaccessible cardinal $\alpha$.

This should be compared with the iteration to control the power function $\alpha \mapsto 2^\alpha$ below some cardinal $\rho$.

In contrast, in this paper, we are interested in carrying out an iteration of length $\kappa^{++}$, where $\kappa$ is a singular cardinal (or, more generally, forced by the first step of the iteration to become one), and all components of the iteration are Prikry-type forcings that concentrate on the combinatorics of $\kappa$ or its successor. For this, we will need to allow a support of arbitrarily large size below $\kappa$. To be able to lift the Prikry property through an infinite-support iteration, members of the $\Sigma$-Prikry class are thus required to possess the following stronger property, which is inspired by the concepts coming from the study of topological Ramsey spaces [Tod10].

**Complete Prikry Property.** There is a partition of the ordering $\leq$ into countably many relations $\{\leq_n \mid n < \omega\}$ such that, if we denote $\text{cone}_n(q) := \{r \mid r \leq_n q\}$, then, for every 0-open $U \subseteq P$ (i.e., $q \in U \implies \text{cone}_0(q) \subseteq U$), every $p \in P$ and every $n < \omega$, there exists $q \leq_0 p$ such that $\text{cone}_n(q)$ is either a subset of $U$ or disjoint from $U$.

Another parameter that requires attention when devising an iteration scheme is the chain condition of the components to be used. In view of the goal of solving a problem concerning the combinatorics of $\kappa$ or its successor through an iteration of length $\kappa^{++}$, there is a need to know that all counterexamples to our problem will show up at some intermediate stage of the
iteration, so that we at least have the chance to kill them all. The standard way to secure the latter is to require that the whole iteration $P_{\kappa^{++}}$ would have the $\kappa^{++}$-chain condition ($\kappa^{++}$-cc). As the $\kappa$-support iteration of $\kappa^{++}$-cc posets need not have the $\kappa^{++}$-cc (see [Ros18] for an explicit counterexample), members of the $\Sigma$-Prikry class are required to satisfy the following strong form of the $\kappa^{++}$-cc:

**Linked$_0$ Property.** There exists a map $c : P \to \kappa^+$ satisfying that for all $p, q \in P$, if $c(p) = c(q)$, then $p$ and $q$ are compatible, and, furthermore, $\text{cone}_0(p) \cap \text{cone}_0(q)$ is nonempty.

In particular, our verification of the chain condition of $P_{\kappa^{++}}$ will not go through the $\Delta$-system lemma; rather, we will take advantage of a basic fact concerning the density of box products of topological spaces.

Now that we have a way to ensure that all counterexamples show up at intermediate stages, we fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++} \rangle$, and shall want that, for any $\alpha < \kappa^{++}$, $P_{\alpha+1}$ will amount to forcing over the model $V^P_{\alpha}$ to solve a problem suggested by $z_\alpha$. The standard approach to achieve this is to set $P_{\alpha+1} := P_\alpha \ast Q_\alpha$, where $Q_\alpha$ is a $P_\alpha$-name for a poset that takes care of $z_\alpha$. However, the disadvantage of this approach is that if $P_1$ is a notion of forcing that blows up $2^\kappa$, then any typical poset $Q_1$ in $V^P_1$ which is designed to add a subset of $\kappa^+$ via bounded approximations will fail to have the $\kappa^{++}$-cc. To work around this, in our scheme, we set $P_{\alpha+1} := A(P_\alpha, z_\alpha)$, where $A(\cdot, \cdot)$ is a functor that, to each $\Sigma$-Prikry poset $P$ and a problem $z$, produces a $\Sigma$-Prikry poset $A(P, z)$ that projects onto $P$ and solves the problem $z$. A key feature of this functor is that the projection from $A(P, z)$ to $P$ splits, that is, in addition to a projection map $\pi$ from $A(P, z)$ onto $P$, there is a map $\triangleleft$ that goes in the other direction, and the two maps commute in a very strong sense. The exact details may be found in our definition of forking projection (see Definition 2.7 below).

A special case of the main result of this paper may be roughly stated as follows.

**Main Theorem.** Suppose that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of regular uncountable cardinals, converging to a cardinal $\kappa$. For simplicity, let us say that a notion of forcing $P$ is nice if $P \subseteq H_{\kappa^{++}}$ and $P$ does not collapse $\kappa^+$. Now, suppose that:

- $Q$ is a nice $\Sigma$-Prikry notion of forcing;
- $A(\cdot, \cdot)$ is a functor that produces for every nice $\Sigma$-Prikry notion of forcing $P$ and every $z \in H_{\kappa^{++}}$, a corresponding nice $\Sigma$-Prikry notion of forcing $A(P, z)$ that admits a forking projection to $P$;
- $2^{2^\kappa} = \kappa^{++}$, so that we may fix a bookkeeping list $\langle z_\alpha \mid \alpha < \kappa^{++} \rangle$.

Then there exists a sequence $\langle P_\alpha \mid \alpha \leq \kappa^{++} \rangle$ of nice $\Sigma$-Prikry forcings such that $P_1$ is isomorphic to $Q$, $P_{\alpha+1}$ is isomorphic to $A(P_\alpha, z_\alpha)$, and, for every pair $\alpha \leq \beta \leq \kappa^{++}$, $P_\beta$ projects onto $P_\alpha$. 
1.1. **Organization of this paper.** We assume no familiarity with [PRS20]. In Section 2, we recall the definitions of the Σ-Prikry class, and forking projections. We also prove a useful lemma concerning the canonical form of forking projections.

In Section 3, we verify that the Extender Based Prikry Forcing (EBPF) due to Gitik and Magidor [GM94, §3] fits into the Σ-Prikry framework.

In Section 4, we present our abstract iteration scheme for Σ-Prikry posets, and prove the Main Theorem of this paper.

In Section 5, we present the very first application of our scheme. We carry out an iteration of length $\kappa^{++}$, where the first step of the iteration is the EBPF for making $2^\kappa = \kappa^{++}$, and all the later steps are obtained by invoking the functor $A(P, z)$ from [PRS20, §6] for killing a nonreflecting stationary subset $z$. This functor is essentially due to Sharon [Sha05, §2], and as a corollary, we obtain a correct proof of the main result of [Sha05, §3]:

**Corollary.** If $\kappa$ is the limit of a countable increasing sequence of supercompact cardinals, then there exists a cofinality-preserving forcing extension in which $\kappa$ remains a strong limit, every finite collection of stationary subsets of $\kappa^+$ reflects simultaneously, and $2^\kappa = \kappa^{++}$.

1.2. **Notation and conventions.** Our forcing convention is that $p \leq q$ means that $p$ extends $q$. We write $\mathbb{P} \downarrow q$ for $\{ p \in \mathbb{P} \mid p \leq q \}$. Denote $E[\mu] := \{ \alpha < \mu \mid \text{cf} (\alpha) = \theta \}$. The sets $E[\mu]_{\leq \theta}$ and $E[\mu]_{> \theta}$ are defined in a similar fashion. For a stationary subset $S$ of a regular uncountable cardinal $\mu$, we write $\text{Tr}(S) := \{ \delta \in E[\mu]_{> \omega} \mid S \cap \delta \text{ is stationary in } \delta \}$. $H_\mu$ denotes the collection of all sets of hereditary cardinality less than $\mu$. For every set of ordinals $x$, we denote $\text{cl}(x) := \{ \sup (x \cap \gamma) \mid \gamma \in \text{Ord}, x \cap \gamma \neq \emptyset \}$, and $\text{acc}(x) := \{ \gamma \in x \mid \sup (x \cap \gamma) = \gamma > 0 \}$.

2. **Σ-Prikry forcing and forking projections**

In this section, we recall some definitions and facts from [PRS20, §2] and [PRS20, §4]. Our aim is to provide the non familiar reader with a brief summary of the notions of Σ-Prikry forcing and forking projections. We also pinpoint some of the connections between these two concepts.

**Definition 2.1.** We say that $(\mathbb{P}, \ell)$ is a graded poset iff $\mathbb{P} = (P, \leq)$ is a poset, $\ell : P \to \omega$ is a surjection, and, for all $p \in P$:

- For every $q \leq p$, $\ell(q) \geq \ell(p)$;
- There exists $q \leq p$ with $\ell(q) = \ell(p) + 1$.

**Convention 2.2.** For a graded poset as above, we denote $P_n := \{ p \in P \mid \ell (p) = n \}$, $P_n^p := \{ q \in P \mid q \leq p, \ell (q) = \ell (p) + n \}$, and sometime write $q \leq^n p$ (and say the $q$ is an $n$-step extension of $p$) rather than writing $q \in P^p_n$.

**Definition 2.3.** Suppose that $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $1$, and that $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$. Suppose
that $\mu$ is a cardinal such that $1 \vDash \beta = \kappa^+$. For functions $\ell : P \to \omega$ and $c : P \to \mu$, we say that $(P, \ell, c)$ is $\Sigma$-Prikry iff all of the following hold:

1. $(P, \ell)$ is a graded poset;
2. For all $n < \omega$, $P_n := (P_n \cup \{1\}, \leq)$ is $\kappa_n$-directed-closed;\footnote{That is, for every $D \in [P_n \cup \{1\}]^{< \kappa_n}$ with the property that for all $p, p' \in D$, there is $q \in D$ with $q \leq p, p'$, there exists $r \in P_n$ such that $r \leq p$ for all $p \in D$.}
3. For all $p, q \in P$, if $c(p) = c(q)$, then $P^p_0 \cap P^q_0$ is non-empty;
4. For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{ r \leq^n p \mid q \leq^m r \}$ contains a greatest element which we denote by $m(p, q)$;\footnote{By convention, a greatest element, if exists, is unique.} In the special case $m = 0$, we shall write $w(p, q)$ rather than $0(p, q)$;\footnote{Note that Clause (7) is the Introduction’s $\text{Linked}_0$ property. Often, we will want to avoid encodings and opt to define the function $c$ as a map from $P$ to some natural set $\mathcal{M}$ of size $\leq \mu$, instead of a map to the cardinal $\mu$ itself. In the special case that $\mu^{< \mu} = \mu$, we shall simply take $\mathcal{M}$ to be $H_\mu$.}
5. For all $p \in P$, the set $W(p) := \{ w(p, q) \mid q \leq p \}$ has size $< \mu$;
6. For all $p' \leq p$ in $P$, $q \mapsto w(p, q)$ forms an order-preserving map from $W(p')$ to $W(p)$;
7. Suppose that $U \subseteq P$ is a 0-open set, i.e., $r \in U$ iff $P^r_0 \subseteq U$. Then, for all $p \in P$ and $n < \omega$, there is $q \leq^0 p$, such that, either $P^q_n \cap U = \emptyset$ or $P^q_n \subseteq U$.

Remark 2.4.

1. Note that Clause (3) is the Introduction’s $\text{Linked}_0$ property. Often, we will want to avoid encodings and opt to define the function $c$ as a map from $P$ to some natural set $\mathcal{M}$ of size $\leq \mu$, instead of a map to the cardinal $\mu$ itself. In the special case that $\mu^{< \mu} = \mu$, we shall simply take $\mathcal{M}$ to be $H_\mu$.\footnote{Note that Clause (7) is the Introduction’s $\text{Linked}_0$ property. Often, we will want to avoid encodings and opt to define the function $c$ as a map from $P$ to some natural set $\mathcal{M}$ of size $\leq \mu$, instead of a map to the cardinal $\mu$ itself. In the special case that $\mu^{< \mu} = \mu$, we shall simply take $\mathcal{M}$ to be $H_\mu$.}
2. Note that Clause (7) is the Complete Prikry Property (CPP).

Definition 2.5. Let $p \in P$. For each $n < \omega$, we write $W_n(p) := \{ w(p, q) \mid q \in P^n_0 \}$. The object $W(p) := \bigcup_{n<\omega} W_n(p)$ is called the p-tree.\footnote{The nice features of the p-tree are listed in [PRS20, Lemma 2.8], but we shall not assume the reader is familiar with them.}

Fact 2.6 ([PRS20, Lemma 2.10]).

1. $P$ does not add bounded subsets of $\kappa$;
2. For every regular cardinal $\nu \geq \kappa$, if there exists $p \in P$ for which $p \vDash c f(\nu) < \kappa$, then there exists $p' \leq p$ with $|W(p')| \geq \nu$.\footnote{For future reference, we point out that this fact relies only on Clauses (1), (2), (4) and (7) of Definition 2.3. Furthermore, we do not need to know that $1$ decides a value for $\kappa^+$.}

In Section 3, we explore the Extender Based Prikry Forcing [GM94], verifying it is a $\Sigma$-Prikry poset.

Definition 2.7. Suppose that $(P, \ell_P, c_P)$ is a $\Sigma$-Prikry triple, $A = (A, \leq)$ is a notion of forcing, and $\ell_A$ and $c_A$ are functions with $\text{dom}(\ell_A) = \text{dom}(c_A) = A$.

A pair of functions $(\ell, \pi)$ is said to be a forking projection from $(A, \ell_A)$ to $(P, \ell_P)$ iff all of the following hold:

1. $\pi$ is a projection from $A$ onto $P$, and $\ell_A = \ell_P \circ \pi$;
(2) for all \( a \in A \), \( \hat{\kappa}(a) \) is an order-preserving function from \( (\mathbb{P} \downarrow \pi(a), \leq) \) to \( (A \downarrow a, \leq) \);

(3) for all \( p \in P \), \( \{ a \in A \mid \pi(a) = p \} \) admits a greatest element, which we denote by \( [p]^\mathbb{A} \);

(4) for all \( n, m < \omega \) and \( b \leq^{n+m} a \), \( m(a, b) \) exists and satisfies:

\[
m(a, b) = \hat{\kappa}(a)(m(\pi(a), \pi(b)))\;
\]

(5) for all \( a \in A \) and \( q \leq \pi(a) \), \( \pi(\hat{\kappa}(a)(q)) = q \);

(6) for all \( a \in A \) and \( q \leq \pi(a) \), \( a = [\pi(a)]^\mathbb{A} \) iff \( \hat{\kappa}(a)(q) = [q]^\mathbb{A} \);

(7) for all \( a \in A \), \( a' \leq^0 a \) and \( r \leq^0 \pi(a') \), \( \hat{\kappa}(a')(r) \leq \hat{\kappa}(a)(r) \).

The pair \((\hat{\kappa}, \pi)\) is said to be a forking projection from \((A, \ell_A, c_A)\) to \((\mathbb{P}, \ell_P, c_P)\) iff, in addition to all of the above, the following holds:

(8) for all \( a, a' \in A \), if \( c_A(a) = c_A(a') \), then \( c_P(\pi(a)) = c_P(\pi(a')) \) and, for all \( r \in P_0^{\pi(a)} \cap P_0^{\pi(a')} \), \( \hat{\kappa}(a)(r) = \hat{\kappa}(a')(r) \).

**Remark 2.8.** Intuitively speaking, \( \hat{\kappa}(a) \) is an operator that, for each \( p \in P \downarrow \pi(a) \), provides the \( \preceq \)-greatest condition \( b \preceq a \) such that \( \pi(b) = p \).

The notion of forking projection is crucial in the proof that the class of \( \Sigma \)-Prikry forcings is iterable, in the sense of Section 4. For instance, the next fact gives an idea about how we proceed when verifying clause (1) of Definition 2.3 for the different stages of the iteration:

**Fact 2.9 ([PRS20, Lemma 4.3]).** Suppose that \((\hat{\kappa}, \pi)\) is a forking projection from \((A, \ell_A, c_A)\) to \((\mathbb{P}, \ell_P)\), or, just a pair of maps satisfying Clauses (1), (2) and (4) of Definition 2.7. For each \( a \in A \), the following holds:

1. \( \hat{\kappa}(a) \upharpoonright W(\pi(a)) \) forms a bijection from \( W(\pi(a)) \) to \( W(a) \);

2. for all \( n < \omega \) and \( r \in P_n^{\pi(a)} \), \( \hat{\kappa}(a)(r) \in A_n^0 \).

In particular, \((A, \ell_A)\) is a graded poset.

Similarly, the following takes care of Clauses (2) and (3) of Definition 2.3, respectively:

**Fact 2.10 ([PRS20, Lemma 4.6]).** Suppose that \((\hat{\kappa}, \pi)\) is a forking projection from \((A, \ell_A)\) to \((\mathbb{P}, \ell_P)\), or, just a pair of maps satisfying Clauses (1), (2), (5) and (7) of Definition 2.7. Let \( n < \omega \). Suppose that for every directed family \( D \) of conditions in \( A_n \) with \( |D| < \kappa_n \), if the map \( d \mapsto \pi(d) \) is constant over \( D \), then \( D \) admits a lower bound in \( A_n \). Then \( A_n \) is \( \kappa_n \)-directed-closed.

**Fact 2.11 ([PRS20, Lemma 4.7]).** Suppose that \((\hat{\kappa}, \pi)\) is a forking projection from \((A, \ell_A, c_A)\) to \((\mathbb{P}, \ell_P, c_P)\), or, just a pair of maps satisfying Clauses (1), (2), (4), (7) and (8) of Definition 2.7. For all \( a, a' \in A \), if \( c_A(a) = c_A(a') \), then \( A_n^0 \cap A_n^0 \) is non-empty. In particular, if \( |\text{Im}(c_A)| \leq \mu \), then \( A_n \) is \( \mu^+ \)-2-linked.

Another central point in the proof of our main theorem is the verification of the CPP for the successive stages of an iteration of \( \Sigma \)-Prikry forcings. For such purpose we require that \((\hat{\kappa}, \pi)\) posses the following additional property:
Definition 2.12. A forking projection \((\mathfrak{h}, \pi)\) from \((A, \ell_A)\) to \((\mathbb{P}, \ell_\mathbb{P})\) is said to have the mixing property iff for all \(a \in A\), \(n < \omega\), \(q \in P^\pi(a)\), and a function \(g : W_n(q) \to \mathbb{A} \downarrow a\) such that \(\pi \circ g\) is the identity map,\(^6\) there exists \(b \in A_0^n\) with \(\pi(b) = q\) such that \(\mathfrak{h}(b)(r) \in A_0^n(r)\) for every \(r \in W_n(q)\).

Remark 2.13. One may think on the mixing property as giving a condition \(b \in A_0^n\) “diagonalizing” the set \(\{g(r) \mid r \in W_n(q)\}\). Specifically, it is arguably an abstraction of the following well-known fact about Prikry forcing: for each \(p = (s, A)\) and \(\{(s^\alpha,a), B_\alpha \mid \alpha \in A\}\) below \(p\), there is \(q \leq p\) such that, for each \(r \leq q\), if some \(\alpha \in A\) appears in the stem of \(r\) then \(r \leq (s^\alpha, B_\alpha)\).

The mixing property provides us the key to ensure that the CPP is not lost along an iteration of \(\Sigma\)-Prikry forcings.

Fact 2.14 ([PRS20, Lemma 4.12]). Suppose \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) is a \(\Sigma\)-Prikry triple. If \((\mathbb{A}, \ell_\mathbb{A})\) admits a forking projection to \((\mathbb{P}, \ell_\mathbb{P})\) that has the mixing property, then \((\mathbb{A}, \ell_\mathbb{A})\) has the CPP, that is, for every \(0\)-open \(U \subseteq A\), for all \(a \in A\) and \(n < \omega\), there is \(b \in A_0^n\) such that, either \(A_0^n \cap U = \emptyset\) or \(A_0^n \subseteq U\).

We refer the reader to [PRS20, §4] for more about the connections between forking projections and \(\Sigma\)-Prikry forcings. Finally, we close the section with some useful lemmas that will be invoked later, in Section 4.

Lemma 2.15. Suppose that \((\mathfrak{h}, \pi)\) is a forking projection from \((A, \ell_A)\) to \((\mathbb{P}, \ell_\mathbb{P})\). For every \(a \in A\), \(\mathfrak{h}(a)(\pi(a)) = a\).

Proof. By Definition 2.7(4), using \((n,m,b) := (0,0,a)\), we infer that \(\mathfrak{h}(a)(\pi(a)) = \mathfrak{h}(a)(w(\pi(a), \pi(a))) = w(a, a) = a\). \(\square\)

Lemma 2.16 (Canonical form). Suppose that \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) and \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) are both \(\Sigma\)-Prikry notions of forcing. Denote \(\mathbb{P} = (P, \leq)\) and \(\mathbb{A} = (A, \leq)\).

If \((\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})\) admits a forking projection to \((\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})\) as witnessed by a pair \((\mathfrak{h}, \pi)\), then we may assume that all of the following hold true:

1. each element of \(A\) is a pair \(x, y\) with \(\pi(x, y) = x\);
2. for all \(a \in A\), \([\pi(a)]^\mathbb{A} = (\pi(a), \emptyset)\);
3. for all \(p, q \in P\), if \(c_\mathbb{P}(p) = c_\mathbb{P}(q)\), then \(c_\mathbb{A}([p]^\mathbb{A}) = c_\mathbb{A}([q]^\mathbb{A})\).

Proof. By applying a bijection, we may assume that \(A = |A|\) with \(I_\mathbb{A} = \emptyset\). To clarify what we are about to do, we agree to say that “\(a\) is a lift” iff \(a = [\pi(a)]^\mathbb{A}\). Now, define \(f : A \to P \times A\) via:

\[
f(a) := \begin{cases} (\pi(a), \emptyset), & \text{if } a \text{ is a lift;} \\ (\pi(a), a), & \text{otherwise.} \end{cases}
\]

Claim 2.16.1. \(f\) is injective.

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\(^6\)Equivalently, a function \(g : W_n(q) \to A\) such that \(g(r) \leq a\) and \(\pi(g(r)) = r\) for every \(r \in W_n(q)\).
Proof. Suppose $a, a' \in A$ with $f(a) = f(a')$.

- If $a$ is not a lift and $a'$ is not a lift, then from $f(a) = f(a')$ we immediately get that $a = a'$.

- If $a$ is a lift and $a'$ is a lift, then from $f(a) = f(a')$, we infer that $\pi(a) = \pi(a')$, so that $a = [\pi(a)]^A = [\pi(a')]^A = a'$.

- If $a$ is not a lift, but $a'$ is a lift, then from $f(a) = f(a')$, we infer that $a = \emptyset = 1_A$, contradicting the fact that $1_A \neq 1_A$. This completes the proof. \hfill \Box

Let $B := \Im(f)$ and $B := \{(f(a), f(b)) \mid a \leq b\}$, so that $B := (B, \leq B)$ is isomorphic to $A$. Define $\ell_B := \ell_A \circ f^{-1}$ and $\pi_B := \pi \circ f^{-1}$. Also, define $\mathfrak{g}_B$ via $\mathfrak{g}_B(b)(p) := f(\mathfrak{g}(f^{-1}(b))(p))$. It is clear that $b \in B$ is a lift iff $f^{-1}(a)$ is a lift iff $b = (\mathfrak{g}_B(b), \emptyset)$.

Next, define $c_B : B \to \mu \times 2$ by letting for all $b \in B$:

$$c_B(b) := \begin{cases} (c_B(\pi_B(b)), 0), & \text{if } b \text{ is a lift;} \\ (c_A(f^{-1}(b)), 1), & \text{otherwise.} \end{cases}$$

Claim 2.16.2. Suppose $b_0, b_1 \in B$ with $c_B(b_0) = c_B(b_1)$. Then $c_B(\pi_B(b_0)) = c_B(\pi_B(b_1))$ and, for all $r \in P_0^{\pi_B(b_0)} \cap P_0^{\pi_B(b_1)}$, $\mathfrak{g}_B(b_0)(r) = \mathfrak{g}_B(b_1)(r)$.

Proof. We focus on verifying that for all $r \in P_0^{\pi_B(b_0)} \cap P_0^{\pi_B(b_1)}$, $\mathfrak{g}_B(b_0)(r) = \mathfrak{g}_B(b_1)(r)$. For each $i < 2$, denote $a_i := f^{-1}(b_i)$ and $p_i := \pi_B(b_i)$, so that $\pi(a_i) = p_i$. Suppose $r \in P_0^{\pi_B} \cap P_0^{\pi_B}$.

- If $b_0$ is a lift, then so are $b_1, b_0, a_1$. Therefore, for each $i < 2$, Definition 2.7(6) implies that $\mathfrak{g}_B(b_i)(r) = f(\mathfrak{g}(a_i)(r)) = f([r]^B) = [r]^B$. In effect, $\mathfrak{g}_B(b_0)(r) = \mathfrak{g}_B(b_1)(r)$, as desired.

- Otherwise, $c_A(a_0) = c_A(a_1)$. As $r \in P_0^{\pi(a_0)} \cap P_0^{\pi(a_1)}$, $\mathfrak{g}_B(b_0)(p) = f(\mathfrak{g}(a_0)(p)) = f(\mathfrak{g}(a_1)(p)) = \mathfrak{g}_B(b_1)(p)$. \hfill \Box

This completes the proof. \hfill \Box

3. Extender Based Prikry Forcing

In this section, we recall the definition of the Extender Based Prikry Forcing (EBPF) due to Gitik and Magidor [GM94, §3] (see also [Git96] and [Git10, §2]), and verify it fits into the $\Sigma$-Prikry framework. Unlike other expositions of this forcing, we shall not assume the GCH, as we want to be able to conduct various forcing preparations (such as Laver’s) that messes up the GCH. Specifically, our setup is as follows:

- $\Sigma = (\kappa_n \mid n < \omega)$ is a strictly increasing sequence of cardinals;
- $\kappa := \sup_{n < \omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := 2^\mu$;
- $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$;
- For each $n < \omega$, $\kappa_n$ is $(\lambda + 1)$-strong.

In particular, we are assuming that, for each $n < \omega$, there is a $(\kappa_n, \lambda + 1)$-extender $E_n$ whose associated embedding $j_n : V \to M_n$ is such that $M_n$ is a
transitive class, $\kappa_n M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda$. For each $n < \omega$, and each $\alpha < \lambda$, define

$$E_{n,\alpha} := \{ X \subseteq \kappa_n \mid \alpha \in j_n(X) \}.$$  

Note that $E_{n,\alpha}$ is a non-principal $\kappa_n$-complete ultrafilter over $\kappa_n$, provided that $\alpha \geq \kappa_n$. Moreover, in the particular case of $\alpha = \kappa_n$, $E_{n,\kappa_n}$ is also normal. For ordinals $\alpha < \kappa_n$ the measures $E_{n,\alpha}$ are principal so the only reason to consider them is for a more neat presentation.

For each $n < \omega$, we shall consider an ordering $\leq_{E_n}$ over $\lambda$, as follows:

**Definition 3.1.** For each $n < \omega$, set

$$\leq_{E_n} := \{ (\beta, \alpha) \in \lambda \times \lambda \mid \beta \leq \alpha, \land \exists f \in \kappa_n \kappa_n j_n(f)(\alpha) = \beta \}.$$  

It is routine to check that $\leq_{E_n}$ is reflexive, transitive and antisymmetric, hence $(\lambda, \leq_{E_n})$ is a partial order. The intuition behind the ordering $\leq_{E_n}$ is, provided $\beta \leq_{E_n} \alpha$, that one can represent the seed of $E_{n,\beta}$ by means of the seed of $E_{n,\alpha}$, and so the ultrapower $Ult(V, E_{n,\beta})$ can be encoded within $Ult(V, E_{n,\alpha})$. Formally speaking, and it is straightforward to check it, if $\beta \leq_{E_n} \alpha$ then $E_{n,\beta} \leq_{RK} E_{n,\alpha}$ as witnessed by any function $f : \kappa_n \rightarrow \kappa_n$ such that $j_n(f)(\alpha) = \beta$.

In case $\beta \leq_{E_n} \alpha$, we shall fix in advance a witnessing map $\pi_{\alpha,\beta} : \kappa_n \rightarrow \kappa_n$. In the special case where $\alpha = \beta$, by convention $\pi_{\alpha,\beta} = \text{id}$. Observe that $\leq_{E_n} \langle (\kappa_n \times \kappa_n) \rangle$ is exactly the $\in$-order over $\kappa_n$ so that when we refer to $\leq_{E_n}$ we will really be speaking about the restriction of this order to $\lambda \setminus \kappa_n$.

The following lemma lists some key features of the poset $(\lambda, \leq_{E_n})$:

**Lemma 3.2.** Let $n < \omega$.

1. For every $a \in [\lambda]^{<\kappa_n}$, there are $\lambda$-many $\alpha < \lambda$ above $\text{sup}(a)$ such that for every $\gamma, \beta \in a$:
   - $\gamma, \beta \leq_{E_n} \alpha$;
   - if $\gamma \leq_{E_n} \beta$, then $\{ \nu \in \kappa_n \mid \pi_{\alpha,\gamma}(\nu) = \pi_{\beta,\gamma}(\pi_{\alpha,\beta}(\nu)) \} \in E_{n,\alpha}$.

2. For all $\gamma < \beta$, $\gamma \leq_{E_n} \alpha$, and $\beta \leq_{E_n} \alpha$,
   $$\{ \nu \in \kappa_n \mid \pi_{\alpha,\gamma}(\nu) < \pi_{\alpha,\beta}(\nu) \} \in E_{n,\alpha}.$$  

3. For all $\alpha, \beta < \lambda$ with $\beta \leq_{E_n} \alpha$, $\pi_{\alpha,\beta} : \kappa_n \rightarrow \kappa_n$ is a projection map, such that for each $A \in E_{n,\alpha}$, $\pi_{\alpha,\beta}"A \in E_{n,\beta}$.

**Proof.** All of this is proved in [Git10, §2], under the unnecessary hypothesis of GCH. Instead, let us define $\Delta$ to be the set of all infinite cardinals $\delta \leq \kappa_n$ satisfying $\delta^{<\text{cf}(\delta)} = \delta$. Clearly, $\Delta$ is a closed set, and as $\kappa_n$ is a measurable cardinal, $\text{max}(\Delta) = \kappa_n$. It thus follows that we may recursively construct an enumeration $\langle a_\alpha \mid \alpha < \kappa_n \rangle$ of $[\kappa_n]^{<\kappa_n}$ such that, for every $\delta \in \Delta$:
   - $\{ a_\alpha \mid \alpha < \delta \} \subseteq [\delta]^{<\delta}$;
   - for each $a \in [\delta]^{<\text{cf}(\delta)}$, $\{ \alpha < \delta \mid a_\alpha = a \}$ has size $\delta$.

Write $\langle a_\alpha \mid \alpha < j_n(\kappa_n) \rangle := j_n(\langle a_\alpha \mid \alpha < \kappa_n \rangle)$.

---

7The notation $\leq_{RK}$ stands for the usual Rudin-Keisler ordering (cf. [Git10, p. 1366]).
Claim 3.2.1. \( \{ a_\alpha \mid \alpha < \lambda \} = [\lambda]^{< \lambda} \) and each element is enumerated cofinally often.

Proof. As \( V_\lambda \subseteq M_n \), \( (\lambda^{< \lambda})^M = \lambda^{< \lambda} = \lambda \). Also, \( j_n(\kappa_n) > \lambda \), so that \( \lambda \in j(\Delta) \) and:

- \( \{ a_\alpha \mid \alpha < \lambda \} \subseteq [\lambda]^{< \lambda} \);
- for each \( a \in [\lambda]^{< \lambda} \), \( \{ \alpha < \lambda \mid a_\alpha = a \} \) has size \( \lambda \). \( \square \)

The rest of the proof is now identical to that in [Git10, §2]. Specifically:

1. By Lemmas 2.1, 2.2 and 2.4 of [Git10].
2. This is Lemma 2.3 of [Git10].
3. This is obvious. \( \square \)

3.1. The EBPF triple. In this subsection we revisit the EBPF and show that it can be interpreted as a \( \Sigma \)-Prikry triple \((\mathbb{P}, \ell, c)\). We shall first need the following building blocks:

Definition 3.3. Let \( n < \omega \). Define \( Q_{n0}, Q_{n1}, \) and \( Q_n \) as follows:

\( (0)_n Q_{n0} := (Q_{n0}, \leq_{n0}) \), where elements of \( Q_{n0} \) are triples \( p = (a^p, A^p, f^p) \) meeting the following requirements:

(a) \( f^p \) is a function from some \( x \in [\lambda]^{< \kappa} \) to \( \kappa_n \);
(b) \( a^p \in [\lambda]^{< \kappa_n} \), and \( a^p \) contains a \( \leq_{E_n} \)-maximal element, which hereafter is denoted by \( mc(a^p) \);
(c) \( \text{dom}(f^p) \cap a^p = \emptyset \);
(d) \( A^p \in E_{n, mc(a^p)} \);
(e) if \( \beta < \alpha \) is a pair in \( a \), for all \( \nu \in A \), \( \pi_{mc(a^p), \beta}(\nu) < \pi_{mc(a^p), \alpha}(\nu) \);
(f) if \( \alpha, \beta, \gamma \in a \) with \( \gamma \leq_{E_n} \beta \leq_{E_n} \alpha \), then, for all \( \nu \in \pi_{mc(a^p), \alpha}A \), \( \pi_{\alpha, \gamma}(\nu) = \pi_{\beta, \gamma}(\pi_{\alpha, \beta}(\nu)) \).

The ordering \( \leq_{n0} \) is defined as follows: \( (a^p, A^p, f^p) \leq_{n0} (a^q, A^q, f^q) \) iff the following are satisfied:

(i) \( f^p \supseteq f^q \);
(ii) \( a^p \supseteq a^q \);
(iii) \( \pi_{mc(a^p), mc(a^q)}A^p \subseteq A^q \).

\( (1)_n Q_{n1} := (Q_{n1}, \leq_{n1}) \), where \( Q_{n1} := \bigcup \{ \{ x^\kappa \mid x \in [\lambda]^{< \kappa} \} \) and \( \leq_{n1} := \supseteq \).

\( (2)_n Q_n := (Q_{n0} \cup Q_{n1}, \leq_{n1}) \), where the ordering \( \leq_{n} \) is defined as follows:

- for each \( p, q \in Q_n, p \leq_{n} q \) if
- \( (a) \) either \( p, q \in Q_{n1} \) for some \( \alpha \), or \( p \leq_{n1} q \), and, for some \( \nu \in A^q \), \( p \leq_{n1} q^\gamma(\nu) \), where \( q^\gamma(\nu) := f^q \cup \{ (\beta, \pi_{mc(a^q), \beta}(\nu)) \mid \beta \in a^q \} \).

Remark 3.4. By Lemma 3.2, Clauses (b)–(f) of \( (0)_n \) may indeed hold simultaneously.

Definition 3.5 (EBPF). Extender Based Prikry Forcing is the poset \( \mathbb{P} := (P, \leq) \) defined by the following clauses:

- Conditions in \( P \) are sequences \( p = (p_n \mid n < \omega) \in \prod_{n<\omega} Q_n \).
- For all \( p \in P \):
– There is $n < \omega$ such that $p_n \in Q_{n0}$;
– For every $n < \omega$, if $p_n \in Q_{n0}$, then $p_{n+1} \in Q_{n0}$ and $a^{p_n} \subseteq a^{p_{n+1}}$.

• For all $p, q \in P$, $p \leq q$ iff $p_n \leq q_n$ for every $n < \omega$.

**Definition 3.6.** $\ell : P \to \omega$ is defined by letting for all $p = \langle p_n \mid n < \omega \rangle$:

$$\ell(p) := \min\{n < \omega \mid p_n \in Q_{n0}\}.$$  

We already have $P$ and $\ell$; we shall soon see that $1 \models P \models \kappa^+$, so that we now need to introduce a map $e : P \to \mu$. As $\mu^\kappa = \mu$, we shall instead be defining a map $c : P \to H_\mu$. To this end, and as $\mu^\kappa = \mu$ and $2^\mu = \lambda$, we may appeal to the Engelking-Karlowicz theorem [EK65] and fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from $\lambda$ to $\mu$ with the property that, for every function $e : x \to \mu$ with $x \in [\lambda]^{\leq \kappa}$, there exists $i < \mu$ with $e \subseteq e^i$.

**Definition 3.7.** For every $f \in \bigcup_{n<\omega} Q_{n1}$, let $i(f) := \min\{i < \mu \mid f \subseteq e^i\}$.

For every $p = (a, A, f) \in \bigcup_{n<\omega} Q_{n0}$, let $i(p)$ be the least $i < \mu$ such that:

• for all $\alpha \in a$, $e^i(\alpha) = 0$;
• for all $\alpha \in \text{dom}(f)$, $e^i(\alpha) = f(\alpha) + 1$.

Finally, for every condition $p = \langle p_n \mid n < \omega \rangle$ in $P$, let

$$c(p) := \ell(p)^\sim \langle i(p_n) \mid n < \omega \rangle.$$  

Before we turn to the analysis of $(P, \ell, c)$, let us point out the following motivating fact.

**Fact 3.8** (Gitik-Magidor, [GM94]). $P$ is cofinality-preserving, adds no new bounded subsets of $\kappa$, and forces $2^\kappa$ to be $\lambda$.

3.2. **Verification.** We now begin verifying that $(P, \ell, c)$ is indeed $\Sigma$-Prikry.

The following fact is established within the proof of [Git10, Lemma 2.15]:

**Fact 3.9.** Let $p, q \in P$ with $\ell(p) = \ell(q)$. Then $p$ and $q$ are $\leq^0$-compatible iff the two holds:

• for every $n < \omega$, $f^p_n \cup f^q_n$ is a function;
• for every $n \geq \ell(p)$, $\text{dom}(f^p_n) \cup \text{dom}(f^q_n)$ is disjoint from $a^p_n \cup a^q_n$.

It is clear that Clause (1) of Definition 2.3 holds:

**Lemma 3.10.** $(P, \ell)$ is a graded poset. $\square$

Now, we move forward to verify Clause (2). Recall that we denote $P_n := \{p \in P \mid \ell(p) = n\}$.

**Lemma 3.11.** Let $n < \omega$. $P_n := (P_n \cup \{1\}, \leq)$ is $\kappa_n$-directed-closed.

**Proof.** Let $D \in [P_n \cup \{1\}]^{<\kappa_n}$ be a $\leq^0$-directed set, say, $D = \{p^\alpha \mid \alpha < \theta\}$, for some cardinal $\theta < \kappa_n$. For each $\alpha < \theta$, denote $p^\alpha$ by $\langle p^\alpha_m \mid m < \omega \rangle$. Note that, by Fact 3.9, for all $m \geq n$ and $\alpha, \beta < \theta$, $\text{dom}(a^{p^\alpha_m}) \cap \text{dom}(f^{p^\beta_m}) = \emptyset$. Define by recursion $\langle (a_m, A_m) \mid m \geq n \rangle$, where $a_m \in [\lambda]^{<\kappa_m}$ with a $\leq_{E_m}$-maximal element and $A_m \in E_{m,\text{uc}(a_m)}$, as follows:
(1) Let $m \geq n$ and assume that $\langle a_i \mid n \leq i < m \rangle$ has been defined. Set $a^*_m := \{mc(a_i) \mid n \leq i < m\} \cup \bigcup_{\alpha < \theta} a^*_m \alpha$. Since $n \leq m$ and $\theta < \kappa_n$, $a^*_m \in [\lambda]^{<\kappa_m}$. By Lemma 3.2(1) we may find $\delta_m \in \lambda \setminus \bigcup_{\gamma \leq \omega} \text{dom}(f^\gamma_0)$ large enough such that for every $\gamma, \beta \in a^*_m$:

- $\gamma, \beta \leq E_n \delta_m$, and
- if $\gamma \leq E_n \beta$, then $\{\nu \in \kappa_n \mid \pi_{\delta_m, \gamma}(\nu) = \pi_{\delta_m, \beta}(\nu)\} \in E_n, \delta_m$.

Define $a_m := a^*_m \cup \{\delta_m\}$.

(2) Again, appeal to Lemma 3.2 to find $A_m \in E_{m, mc(a_m)}$ with

$$A_m \subseteq \bigcap_{\alpha < \theta} \pi_{mc(a_m), mc(a_m)}^{-1} A^\alpha_m$$

and $A_m$ satisfying Clauses (0)$_m$(e) and (0)$_m$(f) of Definition 3.3.

For every $m < \omega$, set $f_m := \bigcup_{\alpha < \theta} f^\alpha_m$. Finally, for every $m < n$, set $r_m := f_m$, and for every $m \geq n$, set $r_m := (a_m, A_m, f_m)$. Set $r := \langle r_m \mid m < \omega \rangle$. It is not hard to check that, for each $\alpha < \theta$, $r \leq_0 p^{\alpha}$.

Next, we verify Clause (3) of Definition 2.3.

**Lemma 3.12.** Suppose that $p = \langle p_n \mid n < \omega \rangle$ and $q = \langle q_n \mid n < \omega \rangle$ are two conditions, and $c(p) = c(q)$. Then $P^p_0 \cap P^q_0$ is nonempty.

**Proof.** Let $\ell \prec (i_n \mid n < \omega) := c(p)$.

- For all $n \leq \ell$, it follows from $c(p) = c(q)$ that $n < \ell(p) = \ell(q)$ and $p_n \cup q_n \subseteq e^{i_n}$, so that $p_n \cup q_n$ is a function.

- For all $n \geq \ell$, it follows from $i(p_n) = i_n = i(q_n)$ that $e^{i_n}[a^*_n \cup a^*_n] = \{0\}$, $e^{i_n}[-\text{dom}(f^{\alpha}_n) \cup \text{dom}(f^{\beta}_n) \cap \{0\}] = \emptyset$ and $\text{dom}(f^p_n \cap f^q_n) = \text{dom}(f^{\alpha}_n) \cap \text{dom}(f^{\beta}_n)$. So $f^p_n \cup f^q_n$ is a function and $\text{dom}(f^p_n) \cap a^*_n = \text{dom}(f^q_n) \cap a^*_n = \emptyset$.

It thus follows from Fact 3.9 that $P^p_0 \cap P^q_0 \neq \emptyset$. □

The following convention will be applied hereafter.

**Convention 3.13.** For every sequence $\{A_k\}_{i \leq k \leq j}$ such that each $A_k$ is a subset of $\kappa_k$, we shall identify $\prod_{k \in i\ldots j} A_k$ with its subset consisting only of the sequences that are moreover increasing. In addition, for each $p \in P$, we shall refer to $\langle f^p_n \mid n < \ell(p) \rangle$, $\langle f^p_n \mid \ell(p) \leq n < \omega \rangle$ and $\langle a^p_n \mid \ell(p) \leq n < \omega \rangle$, as, respectively, the stem, the $f$-part and the $a$-part of $p$.

**Definition 3.14.** Let $p = \langle f^p_n \mid n < \ell(p) \rangle \prec ((a^p_n, A^p_n, f^p_n) \mid \ell(p) \leq n < \omega)$ in $P$. Define:

- $p^\vdash \emptyset := p$;

- For every $\nu \in A^p_{\ell(p)}$, $p^\vdash (\nu) := q$ where $q = \langle q_n \mid n < \omega \rangle$ is the unique sequence defined as follows:

  $$q_n := \begin{cases} p_n \vdash (\nu), & \text{if } n = \ell(p); \\ p_n, & \text{otherwise.} \end{cases}$$

- By recursion, for all $m \geq \ell(p)$ and $\bar{\nu} = \langle \nu_{\ell(p)}, \ldots, \nu_m, \nu_{m+1} \rangle \in \prod_{n=\ell(p)}^{m+1} A^p_n$, we define $p^\vdash \bar{\nu} := (p^\vdash \bar{\nu} \mid (m + 1)) \vdash (\nu_{m+1})$. 
By the definition of the ordering we have the following:

**Fact 3.15.** If \( p = \langle f_n^p \mid n < \ell(p) \rangle \cap \langle (a_n^p, A_n^p, f_n^p) \mid \ell(p) \leq n < \omega \rangle \) in \( P \) and \( q \leq^m p \), then there exists a unique \( \vec{v} \in \prod_{n=\ell(p)}^{\ell(p)+m-1} A_n^p \) such that \( q \leq^p p \vec{v} \).

In fact, \( \vec{v} = \langle f_i^q(\text{mc}(a_i^p)) \mid \ell(p) \leq i < \ell(q) \rangle \).

By the above fact, given \( n, m < \omega \) and \( q \leq^{n+m} p \), let \( \vec{v} \) be such that \( q \leq^0 p \vec{v} \), and set \( m(p, q) := p^\vec{v}(\vec{v} \restriction n) \). We will soon argue that \( m(p, q) \) indeed coincides with the greatest element of \( \{ r \in P_n^p \mid q \leq^m r \} \). For every \( k < \omega \), set \( W_k(p) := \{ p^\vec{v} \vec{v} \mid \vec{v} \in \prod_{n=\ell(p)}^{\ell(p)+k-1} A_n^p \} \).

Next, we address Clause (4).

**Lemma 3.16.** Let \( p \in P, n, m < \omega \) and \( q \in P_n^p \). The set \( R := \{ r \in P_n^p \mid q \leq^m r \} \) contains a greatest element.

**Proof.** By Fact 3.15, we may let \( \vec{v} \in \prod_{k=\ell(p)}^{\ell(p)+n+m-1} A_k^p \) be such that \( q \leq^0 p \vec{v} \).

It is routine to check that \( p^\vec{v}(\vec{v} \restriction n) \) is the greatest element of \( R \). \( \square \)

Now, to Clause (5).

**Lemma 3.17.** For all \( p \in P \), the set \( W(p) := \{ w(p, q) \mid q \leq p \} \) has size \( \kappa \).

**Proof.** Let \( p \in P, n < \omega \) and \( q \in P_n^p \). By Fact 3.15, we have that \( |W_n(p)| < \kappa_{n+\ell(p)} \), hence \( |W(p)| = \sup_{n<\omega} |W_n(p)| = \kappa < \mu \).

Let us now proceed with the verification of Clause (6).

**Lemma 3.18.** Let \( p' \leq p \) in \( P \). Then \( q \mapsto w(p, q) \) forms an order-preserving map from \( W(p') \) to \( W(p) \).

**Proof.** By Fact 3.15, let \( \vec{\sigma} \in \prod_{k=\ell(p')}^{\ell(p)+k} A_k^p \) be the unique sequence such that \( p' \leq^0 p \vec{\sigma} \). Let \( q, r \in W(p') \) and assume that \( q \leq r \). By the proof of Lemma 3.16, there are \( \vec{v}, \vec{\mu} \) such that \( q = p^\vec{v}\vec{v} \) and \( r = p^\vec{\sigma}\vec{\mu} \). Observe that \( \vec{v} \) must end-extend \( \vec{\mu} \), and so \( w(p, q) = p^\vec{v}\vec{v} \leq p^\vec{\sigma}\vec{\mu} = w(p, r) \). \( \square \)

Our next task is proving that \( (P, \ell, c) \) satisfies the CPP, that is, Clause (7) of Definition 2.3. Some readers\(^8\) may skip directly to Lemma 3.23.

**Definition 3.19.** Given \( m < \omega \) and two conditions \( p, q \in P \), say

- \( p = \langle f_n^p \mid n < \ell(p) \rangle \cap \langle (a_n^p, A_n^p, f_n^p) \mid \ell(p) \leq n < \omega \rangle \);  
- \( q = \langle f_n^q \mid n < \ell(q) \rangle \cap \langle (a_n^q, A_n^q, f_n^q) \mid \ell(q) \leq n < \omega \rangle \),

we shall write \( q \leq^m p \) iff \( q \leq^0 p \) and, for all \( n < \omega \),

\[ \ell(p) \leq n \leq m \implies (a_n^p = a_n^q \text{ and } A_n^p = A_n^q). \]

**Definition 3.20.** For an ordinal \( \delta \leq \kappa \), a sequence of conditions \( \langle p^\alpha \mid \alpha < \delta \rangle \) is said to be a fusion sequence iff, for every pair \( \beta < \alpha < \delta \), \( p^\alpha \sqsubseteq^m (\beta+1) p^\beta \), where \( m(\beta) := \sup \{ m < \omega \mid \kappa_m \leq \beta \} \).

\(^8\)You know who you are!

\(^9\)By convention, \( \sup(\emptyset) := 0 \).
Lemma 3.21 (Fusion Lemma). For every ordinal \( \delta \leq \kappa \) and every fusion sequence \( \langle p^\alpha \mid \alpha < \delta \rangle \), there exists a condition \( p' \) such that, for all \( \beta < \delta \), \( p' \subseteq m(\beta) + 1 \) \( p^\beta \).

Proof. This is a standard fact, so we just briefly go over the main points of the proof. Let \( \langle p^\alpha \mid \alpha < \delta \rangle \) be an arbitrary fusion sequence and set \( \ell \) for the common length of its conditions. Assume \( 0 < \delta \leq \kappa \).

- If \( \delta \) is a successor ordinal, say \( \delta := \beta + 1 \), then, for all \( \gamma \leq \beta \), \( p^\beta \subseteq m(\gamma) + 1 \) \( p^\gamma \). Setting \( p' := p^\beta \) we get the desired condition.

- If \( \delta \) is a limit ordinal, define \( p' := \langle p'_n \mid n < \omega \rangle \) as follows:

\[
p'_n := \begin{cases} 
\bigcup_{\beta < \delta} f^\beta_n, & \text{if } n < \ell; \\
(a_n, A_n, \bigcup_{\beta < \delta} f^\beta_n), & \text{if } n \geq \ell \text{ and } \exists \beta < \delta (n \leq m(\beta) + 1); \\
(a_n, A_n, \bigcup_{\beta < \delta} f^\beta_n), & \text{if } n \geq \ell \text{ and } \forall \beta < \delta (m(\beta) + 1 < n),
\end{cases}
\]

where \((a_n, A_n)\) are constructed as in Lemma 3.11. It is routine to check that \( p' \) is as desired. \(\square\)

Lemma 3.22. Let \( p \in P \) and \( U \) be a 0-open subset of \( P \). Then there is \( q \in P_0^p \) such that, for every \( r \in P^q \cap U \), \( w(q, r) \in U \).

Proof. Fix a bijection \( h : \kappa \to \kappa \) such that, for every \( n < \omega \), \( h^n \kappa_n = \kappa \).

We shall first define a fusion sequence \( \langle p^\alpha \mid \alpha < \kappa \rangle \).

Set \( \ell := \ell(p) \) and \( p_0 := p \). Next, assume that for some \( \alpha < \kappa \), \( \langle p^\beta \mid \beta < \alpha \rangle \) has already been defined and let us show how to construct \( p^\alpha \). By Lemma 3.21, fix a condition \( \bar{p}^\alpha \) such that, for all \( \beta < \alpha \), \( \bar{p}^\alpha \subseteq m(\beta) + 1 \) \( p^\beta \). Let \( \bar{v} := h(\alpha) \). If \( \bar{p}^\alpha \cup \bar{v} \) is not well-defined, that is, \( \bar{v} \notin \prod_{k=\ell}^{\ell+|\bar{v}|-1} A^\alpha_k \), then set \( p^\alpha := \bar{p}^\alpha \). Otherwise, set \( q^\alpha := \bar{p}^\alpha \cup \bar{v} \). There are two cases to consider:

(a) If \( U \cap P_0^{q^\alpha} \) is empty or \( \ell + |\bar{v}| - 1 < m(\alpha) + 1 \), then again set \( p^\alpha := \bar{p}^\alpha \).

(b) Otherwise, pick \( r^\alpha \in U \cap P_0^{q^\alpha} \), and define \( p^\alpha := \langle p_n^\alpha \mid n < \omega \rangle \) by letting, for all \( n < \omega \),

\[
p_n^\alpha := \begin{cases} 
(a_n^\alpha, A_n^\alpha, f_n^\alpha \upharpoonright (\text{dom}(f_n^\alpha) \setminus a_n^\alpha)), & \text{if } \ell \leq n \leq \ell + |\bar{v}| - 1; \\
\alpha, & \text{otherwise.}
\end{cases}
\]

Since \( m(\alpha) + 1 \leq \ell + |\bar{v}| - 1 \), \( p^\alpha \subseteq m(\alpha) + 1 \) \( p^\beta \), hence \( p^\alpha \subseteq m(\beta) + 1 \) \( p^\beta \) for all \( \beta < \alpha \).

Note that if \( p^\alpha \) was defined according to case (b), then \( p^\alpha \cap h(\alpha) = r^\alpha \in U \).

Observe that \( \langle p^\alpha \mid \alpha < \kappa \rangle \) is a fusion sequence and thus, by appealing to Lemma 3.21, we may pick a condition \( q \) which is \( \leq_0 \) below all of them. By shrinking further, we may assume that, for all \( n \geq \ell \), \( A^\alpha_n \cap \kappa_{n-1} = \emptyset \), where by convention \( \kappa_{-1} := 0 \).

We claim that \( q \) is as desired. For if \( r \in P^q \cap U \), and \( \alpha \) is such that \( r \leq_0 q \cap h(\alpha) \), then \( p^\alpha \) must have been defined according to case (b). Then, \( p^\alpha \cap h(\alpha) \) is in the 0-open set \( U \) and so \( w(q, r) = q \cap h(\alpha) \in U \) as well. \(\square\)

We are now ready to complete the verification of the CPP for the EBPF.
Lemma 3.23. Let \( p \in P \) and \( U \) be a 0-open subset of \( P \). For every \( n < \omega \), there is \( q^* \leq^0 p \), such that either \( P^d q^* \cap U = \emptyset \) or \( P^d q^* \subseteq U \).

Proof. Let \( q \leq^0 p \) be given by Lemma 3.22 with respect to \( p \) and \( U \). Set \( \ell := \ell(q) \). Define recursively a \( \leq^0 \)-decreasing sequence of conditions \( \langle q^n | n < \omega \rangle \) such that

1. \( q^0 \leq^0 q \),
2. for each \( n < \omega \), \( q^n := \langle q^n_k | k < \omega \rangle \), where
   \[
   q^n_k := \begin{cases} f^n_k, & \text{if } k < \ell; \\ (a^n_k, A^n_k, f^n_k), & \text{if } k \geq \ell; \end{cases}
   \]
3. for each \( n < \omega \),
   \[
   W_n(q^n) \cap U \neq \emptyset \implies W_n(q^n) \subseteq U.
   \]

Namely, all the \( q^n \)'s have the same stem, \( a \)-parts and \( f \)-parts, and we only shrink the measure one sets so that for each \( n \), either all weak \( n \)-step extensions of \( q^n \) are in \( U \), or none of them are. This is done as in [Git10, Lemma 2.18], so we skip the details.

Now let \( q^* \) be a \( \leq^0 \)-extension of the sequence \( \langle q^n | n < \omega \rangle \). We claim that \( q^* \) is as desired: Let \( n < \omega \) and \( r \in P^d q^* \cap U \). Then by Lemma 3.22, \( w(q^n, r) \in U \). Since \( q^n \) witnesses (3), \( W_n(q^n) \subseteq U \). By the 0-openess of \( U \), \( P^d q^* \subseteq U \), hence \( P^d q^* \subseteq U \). \( \square \)

Corollary 3.24. \( \mathbb{I}_P \models \dot{\mu} = \kappa^+ \).

Proof. Recall that \( \mu = \kappa^+ \) and \( \kappa \) is singular. So, if \( \mathbb{I}_P \not\models \dot{\mu} = \kappa^+ \), then there exists a condition \( p \) in \( P \) such that \( p \models \text{cf}(\mu) < \kappa \). Now, by Lemmas 3.10, 3.11, 3.16 and 3.23, we may appeal to Fact 2.6(2), and infer the existence of \( p' \leq p \) with \(|W(p')| \geq \mu \), contradicting Lemma 3.17. \( \square \)

Altogether, we have established the following:

Corollary 3.25. \( (P, \ell, c) \) is \( \Sigma \)-Prikry. \( \square \)

4. Iteration Scheme

In this section, we present a viable iteration scheme for \( \Sigma \)-Prikry posets. Throughout the section, assume that \( \Sigma = \langle \kappa_n | n < \omega \rangle \) is a non-decreasing sequence of regular uncountable cardinals. Denote \( \kappa := \sup_{n < \omega} \kappa_n \). Also, assume that \( \mu \) is some cardinal satisfying \( \mu^{<\mu} = \mu \), so that \(|H_\mu| = \mu \).

The following convention will be applied hereafter:

Convention 4.1. For all ordinals \( \gamma \leq \alpha \leq \mu^+ \):
1. \( \emptyset_\alpha := \alpha \times \{\emptyset\} \) denotes the \( \alpha \)-sequence with constant value \( \emptyset \);
2. For a \( \gamma \)-sequence \( p \) and an \( \alpha \)-sequence \( q \), \( p \ast q \) denotes the unique \( \alpha \)-sequence satisfying that for all \( \beta < \alpha \):
   \[
   (p \ast q)(\beta) = \begin{cases} q(\beta), & \text{if } \gamma \leq \beta < \alpha; \\ p(\beta), & \text{otherwise}. \end{cases}
   \]
(3) Let $\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha)$ and $\mathbb{P}_\gamma := (P_\gamma, \leq_\gamma)$ be forcing posets such that $P_\alpha \subseteq \alpha^* H_\mu^+$ and $P_\gamma \subseteq \gamma^* H_\mu^+$. Also, assume $p \mapsto p \upharpoonright \gamma$ defines a projection between $\mathbb{P}_\alpha$ and $\mathbb{P}_\gamma$. We denote by $i_\alpha^\gamma : V^{\mathbb{P}_\gamma} \to V^{\mathbb{P}_\alpha}$ the map defined by recursion over the rank of each $\mathbb{P}_\gamma$-name $\sigma$ as follows:

$$i_\alpha^\gamma(\sigma) := \{(i_\gamma^\alpha(\tau), p \upharpoonright \emptyset_\alpha) \mid (\tau, p) \in \sigma\}.$$ 

Our iteration scheme requires three building blocks:

**Building Block I.** We are given a $\Sigma$-Prikry triple $(\mathbb{Q}, \ell, c)$ such that $\mathbb{Q} = (Q, \leq_Q)$ is a subset of $H_\mu^+$, $\mathbb{1}_Q \Vdash \mathbb{Q} \mu = \kappa^+$ and $\mathbb{1}_Q \Vdash \mu \text{ is singular}$.\(^{10}\) To streamline the matter, we also require that $\mathbb{1}_Q$ be equal to $\emptyset$.

**Building Block II.** For every $\Sigma$-Prikry triple $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ such that $\mathbb{P} = (P, \leq)$ is a subset of $H_\mu^+$, $\mathbb{1}_P \Vdash \mathbb{P} \mu = \kappa^+$ and $\mathbb{1}_P \Vdash \mu \text{ is singular}$, every $p^* \in P$, and every $\mathbb{P}$-name $z \in H_\mu^+$, we are given a corresponding $\Sigma$-Prikry triple $(A_\mathbb{P}, \ell_{A_\mathbb{P}}, c_{A_\mathbb{P}})$ such that:

- (a) $(A_\mathbb{P}, \ell_{A_\mathbb{P}}, c_{A_\mathbb{P}})$ admits a forking projection $(\check{\mathbb{Q}}, \pi)$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ that has the mixing property;
- (b) $\mathbb{1}_{A_\mathbb{P}} \Vdash \mathbb{P} \mu = \kappa^+$;
- (c) $A_\mathbb{P} = (A, \unlhd)$ is a subset of $H_\mu^+$.

By Lemma 2.16, we may streamline the matter, and also require that:

- (d) each element of $A$ is a pair $(x, y)$ with $\pi(x, y) = x$;
- (e) for every $a \in A$, $[\pi(a)]_{A_{\mathbb{P}}} = (\pi(a), \emptyset)$;
- (f) for every $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{A_\mathbb{P}}([p]_{A_\mathbb{P}}) = c_{A_\mathbb{P}}([q]_{A_\mathbb{P}})$.

**Building Block III.** We are given a function $\psi : \mu^+ \to H_\mu^+$.

**Goal 4.2.** Our goal is to define a system $(\langle \mathbb{P}_\alpha, \ell_\alpha, c_\alpha, (\check{\mathbb{Q}}_{\alpha, \gamma} \mid \gamma \leq \alpha) \rangle \mid \alpha \leq \mu^+)$ in such a way that for all $\gamma \leq \alpha \leq \mu^+$:

- (i) $\mathbb{P}_\alpha$ is a poset $(P_\alpha, \leq_\alpha)$, $P_\alpha \subseteq \alpha^* H_\mu^+$, and, for all $p \in P_\alpha$, $|B_p| < \mu$,
  where $B_p := \{\beta + 1 \mid \beta \in \text{dom}(p) \& p(\beta) \neq \emptyset\}$;
- (ii) The map $\pi_{\alpha, \gamma} : P_\alpha \to P_\gamma$ defined by $\pi_{\alpha, \gamma}(p) := p|\gamma$ forms a projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$ and $\ell_\alpha = \ell_{\alpha} \circ \pi_{\alpha, \gamma}$;
- (iii) $\mathbb{P}_0$ is a trivial forcing, $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I, and $\mathbb{P}_{\alpha+1}$ is isomorphic to $A$ given by Building Block II when invoked with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ and a pair $(r^*, z)$ which is decoded from $\psi(\alpha)$;
- (iv) If $\alpha > 0$, then $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ is a $\Sigma$-Prikry triple whose greatest element is $\emptyset_\alpha$, $\ell_\alpha = \ell_1 \circ \pi_{\alpha, 1}$, and $\emptyset_\alpha \Vdash \mathbb{P}_\alpha \mu = \kappa^+$;
- (v) If $0 < \gamma < \alpha \leq \mu^+$, then $(\check{\mathbb{Q}}_{\alpha, \gamma}, \pi_{\alpha, \gamma})$ is a forking projection from $(\mathbb{P}_\alpha, \ell_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma)$; in case $\alpha < \mu^+$, $(\check{\mathbb{Q}}_{\alpha, \gamma}, \pi_{\alpha, \gamma})$ is furthermore a forking projection from $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$;
- (vi) If $0 < \gamma \leq \beta \leq \alpha$, then, for all $p \in P_\alpha$ and $r \leq \gamma \mathbb{P}_\alpha r \upharpoonright \gamma, \check{\mathbb{Q}}_{\alpha, \gamma}(p)(r) = (\check{\mathbb{Q}}_{\alpha, \gamma}(p)(r)) \upharpoonright \beta$.

\(^{10}\) At the behest of the referee, we stress that the last hypothesis plays a rather isolated role; see Footnote 12.
Remark 4.3. Note the asymmetry between the case $\alpha < \mu^+$ and the case $\alpha = \mu^+$:

(1) By Clause (i), we will have that $\mathbb{P}_\alpha \subseteq H_{\mu^+}$ for all $\alpha < \mu^+$, but $\mathbb{P}_{\mu^+} \not\subseteq H_{\mu^+}$. Still, $\mathbb{P}_{\mu^+}$ will nevertheless be isomorphic to a subset of $H_{\mu^+}$, as we may identify $P_{\mu^+}$ with $\{p \upharpoonright (\sup(B_p) + 1) \mid p \in P_{\mu^+}\}$.

(2) Clause (v) puts a weaker assertion for $\alpha = \mu^+$. In order to avoid trivialities, let us assume that $\mu^+$-many stages in our iteration $\mathbb{P}_{\mu^+}$ are non-trivial. To see the restriction in Clause (v) is necessary note that, by the pigeonhole principle, there must exist two conditions $p, q \in P_{\mu^+}$ and an ordinal $\gamma < \mu^+$ for which $c_{\mu^+}(p) = c_{\mu^+}(q)$, $B_p \subseteq \gamma$, but $B_q \not\subseteq \gamma$. Now, towards a contradiction, assume there is a map $\mathfrak{h}$ such that $(\mathfrak{h}, \pi_{\mu^+, \gamma})$ forms a forking projection from $(\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$. By Definition 2.7(8), then, $c_\gamma(p \upharpoonright \gamma) = c_\gamma(q \upharpoonright \gamma)$, so that by Definition 2.3(3), we should be able to pick $r \in (P_\gamma)_0^{\mathbb{P}_{\mu^+}} \cap (P_{\mu^+})_0^{\mathbb{P}_{\mu^+}}$, and then by Definition 2.7(8), $\mathfrak{h}(p)(r) = \mathfrak{h}(q)(r)$. Finally, as $B_p \subseteq \gamma$, $p = [p \upharpoonright \gamma]^\mathbb{P}_{\mu^+}$, so that, by Definition 2.7(6), $\mathfrak{h}(p)(r) = [r]^\mathbb{P}_{\mu^+}$. But then $\mathfrak{h}(q)(r) = [r]^\mathbb{P}_{\mu^+}$, contradicting the fact that $B_q \not\subseteq \gamma$.

4.1. Defining the iteration. For every $\alpha < \mu^+$, fix an injection $\phi_\alpha : \alpha \to \mu$. As $|H_\mu| = \mu$, we may also fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from $\mu^+$ to $H_\mu$ such that for every function $e : C \to H_\mu$ with $C \in [\mu^+]^{<\mu}$, there is $i < \mu$ such that $e \subseteq e^i$.

The upcoming definition is by recursion on $\alpha \leq \mu^+$, and we continue as long as we are successful. We shall later verify that the described process is indeed successful.

- Let $P_0 := \{\{\emptyset\}, \leq_0\}$ be the trivial forcing. Let $\ell_0$ and $c_0$ be the constant function $\{\emptyset, \emptyset\}$, and let $\mathfrak{h}_{0,0}$ be the constant function $\{\emptyset, \{\emptyset, \emptyset\}\}$, so that $\mathfrak{h}_{0,0}(\emptyset)$ is the identity map.

- Let $P_1 := (P_1, \leq_1)$, where $P_1 := 1Q$ and $p \leq_1 p'$ iff $p(0) \leq Q p'(0)$. Define $\ell_1$ and $c_1$ by stipulating $\ell_1(p) := \ell(p(0))$ and $c_1(p) = c(p(0))$. For all $p \in P_1$, let $\mathfrak{h}_{1,0}(p) : \emptyset \to \{p\}$ be the constant function, and let $\mathfrak{h}_{1,1}(p)$ be the identity map.

- Suppose $\alpha < \mu^+$ and that $\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, (\mathfrak{h}_{\beta, \gamma} \mid \gamma \leq \beta) \rangle \mid \beta \leq \alpha \rangle$ has already been defined. We now define $(\mathbb{P}_{\alpha+1}, \ell_{\alpha+1}, c_{\alpha+1})$ and $\langle \mathfrak{h}_{\alpha+1, \gamma} \mid \gamma \leq \alpha + 1 \rangle$.

- If $\psi(\alpha)$ happens to be a triple $\langle \beta, r, \sigma \rangle$, where $\beta < \alpha$, $r \in P_\beta$ and $\sigma$ is a $P_\beta$-name, then we appeal to Building Block II with $\langle \mathbb{P}_\alpha, \ell_\alpha, c_\alpha \rangle$, $r^* := r \upharpoonright \emptyset_\alpha$ and $z := i^*_\alpha(\sigma)$ to get a corresponding $\Sigma$-Prikry poset $(A, \ell_\alpha, c_\alpha)$.

- Otherwise, we obtain $(A, \ell_\alpha, c_\alpha)$ by appealing to Building Block II with $\langle \mathbb{P}_\alpha, \ell_\alpha, c_\alpha \rangle$, $r^* := \emptyset_\alpha$ and $z := \emptyset$.

\[1\text{This is a consequence of the fact that } p = (p \upharpoonright \gamma)*\emptyset_{\mu^+} = [p \upharpoonright \gamma]^\mathbb{P}_{\mu^+} \text{. See the discussion at the beginning of Lemma 4.7.}\]
In both cases, we also obtain a forking projection $(\hat{\pi}, \pi)$ from $(A, \ell_A, c_A)$ to $(\mathbb{P}_A, \ell_A, c_A)$. Furthermore, each condition in $A = (\mathbb{P}, \leq)$ is a pair $(x, y)$ with $\pi(x, y) = x$, and, for every $p \in P_A$, $[p]_A = (p, \emptyset)$. Now, define $\mathbb{P}_{\alpha+1} := (P_{\alpha+1}, \leq_{\alpha+1})$ by letting $P_{\alpha+1} := \{x, y \in A \mid (x, y) \in \mathcal{P}_\alpha\}$, and then let $p \leq_{\alpha+1} p'$ iff $(p \upharpoonright \alpha, p(\alpha)) \leq (p' \upharpoonright \alpha, p'(\alpha))$. Put $\ell_{\alpha+1} := \ell_1 \circ \pi_{\alpha+1}$ and define $c_{\alpha+1} : P_{\alpha+1} \to H_\mu$ via $c_{\alpha+1}(p) := c_A(p \upharpoonright \alpha, p(\alpha))$.

Next, let $p \in P_{\alpha+1}$, $\gamma \leq \alpha + 1$ and $r \leq \gamma$ be arbitrary; we need to define $\hat{\pi}_{\alpha+1, \gamma}(p)(r)$. For $\gamma = \alpha + 1$, let $\hat{\pi}_{\alpha+1, \gamma}(p)(r) := r$, and for $\gamma \leq \alpha$, let

\begin{equation}
\hat{\pi}_{\alpha+1, \gamma}(p)(r) := x \upharpoonright y \iff \hat{\pi}(p \upharpoonright \alpha, p(\alpha))(\hat{\pi}_{\alpha, \gamma}(p \upharpoonright \alpha)(r)) = (x, y).
\end{equation}

If $\alpha < \mu^+$ is a nonzero limit ordinal, and that $\langle (\mathbb{P}_\beta, \ell_\beta, c_\beta, (\hat{\pi}_{\beta, \gamma} \mid \gamma < \beta)) \rangle \mid \beta < \alpha \rangle$ has already been defined. Define $\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha)$ by letting $P_\alpha$ be all $\alpha$-sequences $p$ such that $|B_p| < \mu$ and $\forall \beta < \alpha (p \upharpoonright \beta \in P_\beta)$. Let $p \leq_\alpha q$ iff $\forall \beta \leq \alpha (p \upharpoonright \beta \leq q \upharpoonright \beta)$. Let $\ell_\alpha := \ell_1 \circ \pi_{\alpha, 1}$. Next, we define $c_\alpha : P_\alpha \to H_\mu$, as follows.

If $\alpha < \mu^+$, then, for every $p \in P_\alpha$, let

$$c_\alpha(p) := \{((\phi_\alpha(\gamma), c_\gamma(p \upharpoonright \gamma)) \mid \gamma \in B_p\}.$$ 

If $\alpha = \mu^+$, then, given $p \in P_\alpha$, first let $C := \text{cl}(B_p)$, then define a function $e : C \to H_\mu$ by stipulating:

$$e(\gamma) := (\phi_\gamma(C \cap \gamma), c_\gamma(p \upharpoonright \gamma)),$$

and then let $c_\alpha(p) := i$ for the least $i < \mu$ such that $e \subseteq e^i$.

Finally, let $p \in P_\alpha$, $\gamma \leq \alpha$ and $r \leq_\gamma p \upharpoonright \gamma$ be arbitrary; we need to define $\hat{\pi}_{\alpha, \gamma}(p)(r)$. For $\gamma = \alpha$, let $\hat{\pi}_{\alpha, \gamma}(p)(r) := r$, and for $\gamma < \alpha$, let

$$\hat{\pi}_{\alpha, \gamma}(p)(r) := \bigcup\{\hat{\pi}_{\beta, \gamma}(p \upharpoonright \beta)(r) \mid \gamma \leq \beta < \alpha\}.$$

**Convention 4.4.** Even though $(\mathbb{P}_0, \ell_0)$ is not a graded poset, in order to smooth up inductive claims that come later, we define $\leq_0$ to be $\leq_0$, and likewise, for every $p \in P_0$, we interpret $(P_0)_{0 \leq}^p$ as $\{q \in P_0 \mid q \leq_0 p\}$.

**4.2. Verification.** We now verify that for all $\alpha \leq \mu^+$, $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, (\hat{\pi}_{\alpha, \gamma} \mid \gamma \leq \alpha))$ fulfills requirements (i)–(vi) of Goal 4.2. By the recursive definition given so far, it is obvious that Clauses (i) and (iii) hold, so we focus on the rest. We commence with Clause (ii)

**Lemma 4.5.** For all $\gamma \leq \alpha \leq \mu^+$, $\pi_{\alpha, \gamma}$ forms a projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$, and $\ell_\alpha = \ell_\gamma \circ \pi_{\alpha, \gamma}$.

**Proof.** The case $\gamma = \alpha$ is trivial, so assume $\gamma < \alpha \leq \mu^+$ clearly, $\pi_{\alpha, \gamma}$ is order-preserving and also $\pi_{\alpha, \gamma}(\emptyset_\alpha) = \emptyset_\gamma$. Let $q \in P_\alpha$ and $q' \in P_\gamma$ be such that $q' \leq_\gamma \pi_{\alpha, \gamma}(q)$. Set $q^* := q' \ast \emptyset_\alpha$ and notice that $\pi_{\alpha, \gamma}(q^* \gamma) = q'$. Altogether, $\pi_{\alpha, \gamma}$ is indeed a projection. For the second part, recall that, for all $\beta \leq \mu^+$, $\ell_\beta := \ell_1 \circ \pi_{\beta, 1}$, hence $\ell_\alpha = \ell_1 \circ \pi_{\alpha, 1} = \ell_1 \circ (\pi_{\gamma, 1} \circ \pi_{\alpha, \gamma}) = (\ell_1 \circ \pi_{\gamma, 1}) \circ \pi_{\alpha, \gamma} = \ell_\gamma \circ \pi_{\alpha, \gamma}$. □

Next, we deal with an expanded version of Clause (vi).
Lemma 4.6. For all $\gamma \leq \alpha \leq \mu^+$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \upharpoonright \gamma$, if we let $q := \mathfrak{h}_{\alpha, \gamma}(p)(r)$, then:

1. $q \upharpoonright \beta = \mathfrak{h}_{\beta, \gamma}(p \upharpoonright \beta)(r)$ for all $\beta \in [\gamma, \alpha]$;
2. $B_q = B_p \cup B_r$;
3. $q \upharpoonright \gamma = r$;
4. If $\gamma = 0$, then $q = p$;
5. $p = (p \upharpoonright \gamma) * \emptyset_\alpha$ iff $q = r * \emptyset_\alpha$;
6. for all $p' \leq_\alpha p$, if $r \leq_\gamma p' \upharpoonright \gamma$, then $\mathfrak{h}_{\alpha, \gamma}(p')(r) \leq_\alpha \mathfrak{h}_{\alpha, \gamma}(p)(r)$.

Proof. Clause (3) follows from Clause (1) and the fact that $\mathfrak{h}_{\gamma, \gamma}(p \upharpoonright \gamma)$ is the identity function. Clause (5) follows from Clauses (2) and (3).

We now prove Clauses (1), (2), (4) and (6) by induction on $\alpha \leq \mu^+$:

- The case $\alpha = 0$ is trivial, since, in this case, all the conditions under consideration (and their corresponding $B$-sets) are empty, and all the maps under consideration are the identity.
- The case $\alpha = 1$ follows from the fact that, by definition, $\mathfrak{h}_{1, 0}(p)(r) = p$ and $\mathfrak{h}_{1, 1}(p)(r) = r$.
- Suppose $\alpha \geq 2$ is a successor ordinal, say $\alpha = \alpha' + 1$, and that the claim holds for $\alpha'$. Fix arbitrary $\gamma \leq \alpha$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \upharpoonright \gamma$. Denote $q := \mathfrak{h}_{\alpha, \gamma}(p)(r)$. Recall that $P_\alpha = P_{\alpha' + 1}$ was defined by feeding $(P_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a $\Sigma$-Prikry triple $(A, \ell_k, c_k)$ along with a forking projection $(\mathfrak{h}, \pi)$, such that each condition in the poset $A = (A, \leq)$ is a pair $(x, y)$ with $\pi(x, y) = x$. Furthermore, by the definition of $\mathfrak{h}_{\alpha, \gamma}$, $q = \mathfrak{h}_{\alpha, \gamma}(p)(r)$ is equal to $x^{\upharpoonright \gamma}(y)$, where

\[(x, y) := \mathfrak{h}(p \upharpoonright \alpha', p(\alpha'))((\mathfrak{h}_{\alpha', \gamma}(p \upharpoonright \alpha'))(r)).\]

In particular, $q \upharpoonright \alpha' = x = \pi(\mathfrak{h}(p \upharpoonright \alpha', p(\alpha'))((\mathfrak{h}_{\alpha', \gamma}(p \upharpoonright \alpha'))(r)))$, which, by Definition 2.7(5), is equal to $\mathfrak{h}_{\alpha', \gamma}(p \upharpoonright \alpha')(r)$.

1. It follows that, for all $\beta \in [\gamma, \alpha)$,

\[q \upharpoonright \beta = (q \upharpoonright \alpha') \upharpoonright \beta = \mathfrak{h}_{\alpha, \gamma}(p \upharpoonright \alpha')(r) \upharpoonright \beta = \mathfrak{h}_{\beta, \gamma}(p \upharpoonright \beta)(r),\]

where the rightmost equality follows from the induction hypothesis. In addition, the case $\beta = \alpha$ is trivial.

2. To avoid trivialities, assume $\gamma < \alpha$. By Clause (1), $q \upharpoonright \alpha' = \mathfrak{h}_{\alpha, \gamma}(p \upharpoonright \alpha')(r)$. So, by the induction hypothesis, $B_{q\upharpoonright \alpha'} = B_{p\upharpoonright \alpha'} \cup B_r$, and we are left with showing that $\alpha \in B_q$ iff $\alpha \in B_p$. As $q \leq_\alpha p$, we have $B_q \supseteq B_p$, so the forward implication is clear. Finally, if $\alpha \notin B_p$, then $p(\alpha') = \emptyset$, and hence

\[(x, y) = \mathfrak{h}(p \upharpoonright \alpha', \emptyset)((\mathfrak{h}_{\alpha', \gamma}(p \upharpoonright \alpha'))(r)).\]

It thus follows from Clause (e) of Building Block II together with the fact that $\mathfrak{h}$ satisfies Clause (6) of Definition 2.7 that $(x, y) = ((\mathfrak{h}_{\alpha', \gamma}(p \upharpoonright \alpha'))(r), \emptyset)$. Recalling that $q = x^{\upharpoonright \gamma}(y)$, we conclude that $\alpha \notin B_q$, as desired.
Lemma 4.7. Suppose that \( \gamma < \alpha \) and all nonzero \( \beta < \alpha \), so that Clause (7) is covered by Lemma 4.6(6). Clause (3) is obvious, since for all nonzero \( \gamma < \alpha \) and \( p \in P_\gamma \), a straight-forward verification makes clear

\[ (4) \text{ If } \gamma = 0, \text{ then, by the induction hypothesis, } \hat{\alpha}_{\alpha',0}(p \upharpoonright \alpha')(r) = p \upharpoonright \alpha', \text{ so that } \]

\[ (x, y) = \hat{\alpha}(p \upharpoonright \alpha', p(\alpha')) \]

\[ = \hat{\alpha}(p \upharpoonright \alpha', p(\alpha'))(p \upharpoonright \alpha') = (p \upharpoonright \alpha', p(\alpha')) = (x, y), \]

where the rightmost equality follows from Lemma 2.15. Altogether, \( q = x^\gamma(y) = p \).

\[ (6) \text{ To avoid trivialities, assume that } \hat{\alpha}_{\alpha,\gamma}(p')(r) \neq \hat{\alpha}_{\alpha,\gamma}(p)(r), \text{ so that } \gamma < \alpha. \]

Fix \( p' \leq \gamma \) with \( r \leq \gamma \). By the definition of \( \leq_{\alpha'+1} \), proving \( \hat{\alpha}_{\alpha,\gamma}(p')(r) \leq \alpha \) \( \hat{\alpha}_{\alpha,\gamma}(p)(r) \) amounts to verifying that \( (x', y') \leq (x, y) \), where

\[ (x', y') := \hat{\alpha}(p \upharpoonright \alpha', p'(\alpha'))((\hat{\alpha}_{\alpha',\gamma}(p' \upharpoonright \alpha')(r)). \]

Now, by the induction hypothesis, \( \hat{\alpha}_{\alpha',\gamma}(p' \upharpoonright \alpha')(r) \leq \alpha \) \( \hat{\alpha}_{\alpha',\gamma}(p \upharpoonright \alpha')(r) \).

So, since \( \hat{\alpha}(p \upharpoonright \alpha', p(\alpha')) \) is order-preserving, it suffices to prove that

\[ (x', y') \leq \hat{\alpha}(p \upharpoonright \alpha', p(\alpha'))((\hat{\alpha}_{\alpha',\gamma}(p' \upharpoonright \alpha')(r)). \]

Denote \( a := (p \upharpoonright \alpha', p(\alpha')) \) and \( a' := (p' \upharpoonright \alpha', p'(\alpha')) \). Then, by Clause (7) of Definition 2.7, indeed

\[ \hat{\alpha}(a')(\hat{\alpha}_{\alpha',\gamma}(p' \upharpoonright \alpha')(r)) \leq \hat{\alpha}(a)(\hat{\alpha}_{\alpha',\gamma}(p' \upharpoonright \alpha')(r)). \]

\[
\blacktriangleright \text{ Suppose } \alpha \in \text{acc}(\mu^+ + 1) \text{ is an ordinal such that, for all } \alpha' < \alpha, \beta \in [\gamma, \alpha'], \]
\[ p \in P_{\alpha'} \text{ and } r \in P_\gamma \text{ with } r \leq \gamma, \]

\[ \hat{\alpha}_{\beta,\gamma}(p \upharpoonright \beta)(r) = ((\hat{\alpha}_{\alpha',\gamma}(p \upharpoonright \alpha')(r)) \upharpoonright \beta). \]

Fix arbitrary \( \gamma \leq \alpha, p \in P_\alpha \) and \( r \in P_\gamma \) with \( r \leq \gamma \). Denote \( q := \hat{\alpha}_{\alpha,\gamma}(p)(r). \) By our definition of \( \hat{\alpha}_{\alpha,\gamma} \) at the limit stage, we have:

\[ q = \bigcup \{ \hat{\alpha}_{\beta,\gamma}(p \upharpoonright \beta)(r) \mid \gamma \leq \beta < \alpha \}. \]

By the induction hypothesis, \( (\hat{\alpha}_{\beta,\gamma}(p \upharpoonright \beta)(r) \mid \gamma \leq \beta < \alpha) \) is a \( \subseteq \)-increasing sequence, and \( B_{\hat{\alpha}_{\beta,\gamma}(p \upharpoonright \beta)(r)} = B_{p \upharpoonright \beta} \cup B_p \) whenever \( \gamma \leq \beta < \alpha \). It thus follows that \( q \) is a legitimate condition, and Clauses (1), (2), (4) and (6) are satisfied.

Our next task is to verify Clause (v) of Goal 4.2.

Lemma 4.7. Suppose that \( \alpha \leq \mu^+ \) is such that for all nonzero \( \gamma < \alpha, (P_\gamma, c_\gamma, \ell_\gamma) \) is \( \Sigma \)-Prikry. Then, for all nonzero \( \gamma \leq \alpha, (\hat{\alpha}_{\alpha,\gamma}, \pi_{\alpha,\gamma}) \) is a forking projection from \( (P_\alpha, \ell_\alpha) \) to \( (P_\gamma, \ell_\gamma) \). If \( \alpha < \mu^+ \), then \( (\hat{\alpha}_{\alpha,\gamma}, \pi_{\alpha,\gamma}) \) is furthermore a forking projection from \( (P_\alpha, \ell_\alpha, c_\alpha) \) to \( (P_\gamma, \ell_\gamma, c_\gamma) \).

Proof. Let us go over the clauses of Definition 2.7.

Clause (1) is covered by Lemma 4.5, Clause (5) is covered by Lemma 4.6(3), and Clause (7) is covered by Lemma 4.6(6). Clause (3) is obvious, since for all nonzero \( \gamma < \alpha \) and \( p \in P_\gamma \), a straight-forward verification makes clear
that \( p \ast \emptyset_\alpha \) is the greatest element of \( \{ q \in P_\alpha \mid \pi_{\alpha,\gamma}(q) = p \} \). In effect, Clause (6) follows from Lemma 4.6(5).

Thus, we are left with verifying Clauses (2), (4), and (8). The next claim takes care of the first two.

**Claim 4.7.1.** For all nonzero \( \gamma \leq \alpha \) and \( p \in P_\alpha \):

1. \( \frak{h}_{\alpha,\gamma}(p) \) is an order-preserving function from \( (\mathbb{P}_\gamma \downarrow (p \upharpoonright \gamma), \leq_\gamma) \) to \( (\mathbb{P}_\alpha \downarrow p, \leq_\alpha) \);
2. for all \( n, m < \omega \) and \( q \leq_{\alpha}^{n+m} p \), \( m(p, q) \) exists and, furthermore,

\[
m(p, q) = \frak{h}_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma)).
\]

**Proof.** We prove the two clauses by induction on \( \alpha \leq \mu^+ \):

- The case \( \alpha = 1 \) is trivial, since, in this case, \( \gamma = \alpha \).
- Suppose \( \alpha = \alpha' + 1 \) is a successor ordinal and that the claim holds for \( \alpha' \). Let \( \gamma \leq \alpha \) and \( p \in P_\alpha \) be arbitrary. To avoid trivialities, assume \( \gamma < \alpha \). By the induction hypothesis, \( \frak{h}_{\alpha',\gamma}(p \upharpoonright \alpha') \) is an order-preserving function from \( \mathbb{P}_\gamma \downarrow (p \upharpoonright \gamma) \) to \( \mathbb{P}_{\alpha'} \downarrow (p \upharpoonright \alpha') \).

Recall that \( \mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1} \) was defined by feeding \( (\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'}) \) into Building Block II, thus obtaining a \( \Sigma \)-Prikry triple \( (\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}) \) along with the pair \( (\frak{h}, \pi) \). Now, as \( \frak{h}(p \upharpoonright \alpha', p(\alpha')) \) and \( \frak{h}_{\alpha',\gamma}(p \upharpoonright \alpha') \) are both order-preserving, the very definition of \( \frak{h}_{\alpha,\gamma}(p \upharpoonright \gamma) \) and \( \leq_{\alpha'}^{\gamma+1} \) implies that \( \frak{h}_{\alpha,\gamma}(p \upharpoonright \gamma) \) is order-preserving. In addition, as \( (x, y) \) is a condition in \( \mathbb{A} \) iff \( x^\gamma(y) \in P_\alpha \) and as \( \frak{h}(p \upharpoonright \alpha', p(\alpha')) \) is an order-preserving function from \( \mathbb{P}_{\alpha'} \downarrow (p \upharpoonright \alpha') \) to \( \mathbb{A} \downarrow (p \upharpoonright \alpha', p(\alpha')) \), we infer that, for all \( r \leq_{\alpha'}^{\gamma} p \upharpoonright \gamma, \frak{h}_{\alpha,\gamma}(p \upharpoonright \gamma)(r) \) is in \( \mathbb{P}_\alpha \downarrow p \).

Let \( q \leq_{\alpha}^{n+m} p \) for some \( n, m < \omega \). Let

\[
(x, y) := m((p \upharpoonright \alpha', p(\alpha')), (q \upharpoonright \alpha', q(\alpha'))).
\]

Trivially, \( m(p, q) \) exists and is equal to \( x^\gamma(y) \). We need to show that

\[
m(p, q) = \frak{h}_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma)).
\]

By Definition 2.7(4),

\[
(x, y) = \frak{h}(p \upharpoonright \alpha', p(\alpha'))(m(p \upharpoonright \alpha', q \upharpoonright \alpha')).
\]

By the induction hypothesis,

\[
m(p \upharpoonright \alpha', q \upharpoonright \alpha') = \frak{h}_{\alpha',\gamma}(p \upharpoonright \alpha')(m(p \upharpoonright \gamma, q \upharpoonright \gamma)),
\]

and so it follows that

\[
(x, y) = \frak{h}(p \upharpoonright \alpha', p(\alpha'))(\frak{h}_{\alpha',\gamma}(p \upharpoonright \alpha')(m(p \upharpoonright \gamma, q \upharpoonright \gamma))).
\]

Thus, by the definition of \( \frak{h}_{\alpha,\gamma} \) and the above equation, \( \frak{h}_{\alpha,\gamma}(p)(m(p \upharpoonright \gamma, q \upharpoonright \gamma)) \) is indeed equal to \( x^\gamma(y) \).

- Suppose \( \alpha \in \text{acc}(\mu^+ + 1) \) is an ordinal for which the claim holds below \( \alpha \). Let \( \gamma \leq \alpha \) and \( p \in P_\alpha \) be arbitrary. To avoid trivialities, assume \( \gamma < \alpha \). By Lemma 4.6(1), for every \( r \in \mathbb{P}_\gamma \downarrow (p \upharpoonright \gamma) \):

\[
\frak{h}_{\alpha,\gamma}(p)(r) = \bigcup_{\gamma \leq \alpha' < \alpha} \frak{h}_{\alpha',\gamma}(p \upharpoonright \alpha')(r).
\]
As for all $q, q' \in P_\alpha$, $q \leq_\alpha q'$ if $\forall \alpha' < \alpha(q | \alpha' \leq_\alpha q' | \alpha')$, the induction hypothesis implies that $\mathfrak{h}_{\alpha, \gamma}(p)$ is an order-preserving function from $\mathbb{P}_\gamma \downarrow (p | \gamma)$ to $\mathbb{P}_\alpha \downarrow p$.

Finally, let $q \leq_\alpha p$; we shall show that $m(p, q)$ exists and is, in fact, equal to $\mathfrak{h}_{\alpha, \gamma}(p)(m(p | \gamma, q | \gamma))$. By Lemma 4.6(1) and the induction hypothesis,

$$\mathfrak{h}_{\alpha, \gamma}(p)(m(p | \gamma, q | \gamma)) = \bigcup_{\gamma \leq \alpha' < \alpha} m(p | \alpha', q | \alpha'),$$

call it $r$. We shall show that $r$ plays the role of $m(p, q)$.

By the definition of $\leq_\alpha$, it is clear that $q \leq_\alpha^m r \leq_\alpha^m p$, so it remains to show that it is the greatest condition in $(\mathbb{P}_\alpha^m)^n$ to satisfy this. Fix an arbitrary $s \in (\mathbb{P}_\alpha^m)^n$ with $q \leq_\alpha^m s$. For each $\alpha' < \alpha$, $q \upharpoonright \alpha' \leq_\alpha^m s \upharpoonright \alpha' \leq_\alpha^m p \upharpoonright \alpha'$, so that $s \upharpoonright \alpha' \leq_\alpha^m m(p \upharpoonright \alpha', q \upharpoonright \alpha')$, and thus $s \leq_\alpha r$. Altogether this shows that $r = m(p, q)$.

This completes the proof of the claim. $\square$

We are left with verifying Clause (8) of Definition 2.7.

**Claim 4.7.2.** Suppose $\alpha \not= \mu^+$. For all $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$ and all nonzero $\gamma \leq \alpha$:

- $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$, and
- $\mathfrak{h}_{\alpha, \gamma}(p)(r) = \mathfrak{h}_{\alpha, \gamma}(p')(r)$ for every $r \in (\mathbb{P}_\gamma)^p_{\gamma} \cap (\mathbb{P}_\gamma)^{p'}_{\gamma}$.

**Proof.** By induction on $\alpha < \mu^+$:

1. The case $\alpha = 1$ is trivial, since, in this case, $\gamma = \alpha$.

2. Suppose $\alpha = \alpha' + 1$ is a successor ordinal and that the claim holds for $\alpha'$. Fix an arbitrary pair $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$.

Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ into Building Block II, thus obtaining a $\Sigma$-Prikry triple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ along the pair $(\mathfrak{h}, \pi)$. By the definition of $c_{\alpha'+1}$, we have

$$c_{\mathbb{A}}(p \upharpoonright \alpha', p(\alpha')) = c_\alpha(p) = c_\alpha(p') = c_{\mathbb{A}}(p' \upharpoonright \alpha', p'(\alpha')).$$

So, as $(\mathfrak{h}, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$, we have $c_{\alpha'}(p | \alpha') = c_{\alpha'}(p' | \alpha')$, and, for all $r \in (P_{\alpha'})^0_{\alpha'} \cap (P_{\alpha'})^0_{\alpha'}$, $\mathfrak{h}(p \upharpoonright \alpha', p(\alpha'))(r) = \mathfrak{h}(p' \upharpoonright \alpha', p'(\alpha'))(r)$.

Now, as $c_{\alpha'}(p \upharpoonright \alpha') = c_{\alpha'}(p' \upharpoonright \alpha')$, the induction hypothesis implies that $c_{\gamma}(p \upharpoonright \gamma) = c_{\gamma}(p' \upharpoonright \gamma)$ for all nonzero $\gamma \leq \alpha'$. In addition, the case $\gamma = \alpha$ is trivial.

Finally, fix a nonzero $\gamma \leq \alpha$ and $r \in (P_\gamma)^p_{\gamma} \cap (P_\gamma)^{p'}_{\gamma}$, and let us prove that $\mathfrak{h}_{\alpha, \gamma}(p)(r) = \mathfrak{h}_{\alpha, \gamma}(p')(r)$. To avoid trivialities, assume $\gamma < \alpha$. It follows from the definition of $\mathfrak{h}_{\alpha, \gamma}$ that $\mathfrak{h}_{\alpha, \gamma}(p)(r) = x^\gamma(y)$ and $\mathfrak{h}_{\alpha, \gamma}(p')(r) = x^{\gamma'}(y')$, where:

\begin{itemize}
  \item $\langle (x, y) := \mathfrak{h}(p \upharpoonright \alpha', p(\alpha'))(\mathfrak{h}_{\alpha', \gamma}(p \upharpoonright \alpha')(r))$, and
  \item $\langle (x', y') := \mathfrak{h}(p' \upharpoonright \alpha', p'(\alpha'))(\mathfrak{h}_{\alpha', \gamma}(p' \upharpoonright \alpha')(r))$.
\end{itemize}
But we have already pointed out that the induction hypothesis implies that $\hat{\gamma}_{\alpha, \gamma}(p \upharpoonright \alpha')(r) = \hat{\gamma}_{\alpha', \gamma}(p'(\alpha')(r))$, call it, $r'$. So, we just need to prove that $\hat{\gamma}(p \upharpoonright \alpha', p(\alpha'))(r') = \hat{\gamma}(p'(\alpha'))(r')$. But we also have $c_\alpha(p \upharpoonright \alpha, p(\alpha')) = c_\alpha(p) = c_\alpha(p') = c_\alpha(p'(\alpha'))$, so, as $(\hat{\gamma}, \pi)$ is a forking projection from $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A})$ to $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$, Clause (8) of Definition 2.7 implies that $\hat{\gamma}(p \upharpoonright \alpha', p(\alpha'))(r') = \hat{\gamma}(p'(\alpha'))(r')$, as desired.

\begin{itemize}
  \item Suppose $\alpha \in \text{acc}(\mu^+)$ is an ordinal for which the claim holds below $\alpha$. For any condition $q \in \bigcup_{\alpha' \leq \alpha} P_{\alpha'}$, define a function $f_q : B_\mu \rightarrow H_\mu$ via $f_q(\alpha') := c_{\alpha'}(q \upharpoonright \alpha')$. Now, fix an arbitrary pair $p, p' \in P_\alpha$ with $c_\alpha(p) = c_\alpha(p')$. By the definition of $c_\alpha$ this means that

  \[ \{ (\phi_\alpha(\gamma), c_\gamma(p \upharpoonright \gamma)) \mid \gamma \in B_p \} = \{ (\phi_\alpha(\gamma), c_\gamma(p' \upharpoonright \gamma)) \mid \gamma \in B_{p'} \}. \]

  As $\phi_\alpha$ is injective, $f_p = f_{p'}$. Next, let $\gamma \leq \alpha$ be nonzero; we need to show that $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$. The case $\gamma = \alpha$ is trivial, so assume $\gamma < \alpha$.

  Now, if $\text{dom}(f_p) \setminus \gamma$ is nonempty, then for $\alpha' := \min(\text{dom}(f_p) \setminus \gamma)$, we have $c_{\alpha'}(p \upharpoonright \alpha') = f_p(\alpha') = f_{p'}(\alpha') = c_{\alpha'}(p' \upharpoonright \alpha')$, and then the induction hypothesis entails $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$. In particular, if $\text{dom}(f_p)$ is unbounded in $\alpha$, then $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$ for all $\gamma \leq \alpha$.

  Next, suppose that $\text{dom}(f_p)$ is bounded in $\alpha$ and let $\delta < \alpha$ be the least ordinal to satisfy $\text{dom}(f_p) \subseteq \delta$. We already know that $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$ for all $\gamma < \delta$, and now prove by induction that $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$ for all $\gamma \in [\delta, \alpha)$. For a successor ordinal $\gamma$, this follows from Clauses (e) and (f) of Building Block II, and for a limit ordinal $\gamma$, this follows from the fact that the injectivity of $\phi_\gamma$ and the equality $f_\delta \upharpoonright \gamma = f_p = f_{p'} = f_{p' \upharpoonright \gamma}$ implies that $c_\gamma(p \upharpoonright \gamma) = c_\gamma(p' \upharpoonright \gamma)$.

  Finally, fix a nonzero $\gamma \leq \alpha$ and $r \in (P_{\gamma})^{p_\gamma}_0 \cap (P_{\gamma})^{p' \gamma}_0$, and let us prove that $\hat{\phi}_{\alpha, \gamma}(r) = \hat{\phi}_{\alpha', \gamma}(r)$. To avoid trivialities, assume $\gamma < \alpha$. We already know that, for all $\alpha' \in [\gamma, \alpha)$, $c_{\alpha'}(p \upharpoonright \alpha') = c_{\alpha'}(p' \upharpoonright \alpha')$, and so the induction hypothesis implies that $\hat{\phi}_{\alpha', \gamma}(r) = \hat{\phi}_{\alpha', \gamma}(r)$, and then by Lemma 4.6(1):

  \[ \hat{\phi}_{\alpha, \gamma}(r) = \bigcup_{\gamma \leq \alpha' \leq \alpha} \hat{\phi}_{\alpha', \gamma}(p \upharpoonright \alpha')(r) = \bigcup_{\gamma \leq \alpha' \leq \alpha} \hat{\phi}_{\alpha', \gamma}(p' \upharpoonright \alpha')(r) = \hat{\phi}_{\alpha, \gamma}(p')(r), \]

  as desired.
\end{itemize}

This completes the proof of Lemma 4.6.

By now, we have verified all clauses of Goal 4.2 with the exception of Clause (iv). Before we are in conditions to do that, let us verify that $(\hat{\phi}_{\alpha, 1}, \pi_{\alpha, 1})$ has mixing property for every $\alpha \geq 1$. \hfill \Box
Lemma 4.8. Let $1 \leq \alpha \leq \mu^+$, and suppose that, for all nonzero $\gamma < \alpha$, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is a $\Sigma$-Prikry triple admitting a forking projection to $(\mathbb{P}_1, \ell_1, c_1)$, as witnessed by the pair of maps $(\dot{\alpha}_{\gamma,1}, \pi_{\gamma,1})$.

Then $(\dot{\alpha}_{\alpha,1}, \pi_{\alpha,1})$ has the mixing property. That is, for all $p \in P_\alpha$, $p' \leq_1 \pi_{\alpha,1}(p)$ and $m < \omega$, for every $g : W_m(p') \to P_\alpha$ such that $g(r) \leq_\alpha p$ and $\pi_{\alpha,1}(g(r)) = r$ for every $r \in W_m(p')$, there exists $q \in (P_\alpha)_0$ with $\pi_{\alpha,1}(q) = p'$ such that $\dot{\alpha}_{\alpha,1}(q)(r) \leq_\alpha g(r)$ for every $r \in W_m(p')$.

Proof. Notice that, by Lemma 4.7, if, for all nonzero $\gamma < \alpha$, $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma)$ is a $\Sigma$-Prikry triple, then $(\dot{\alpha}_{\alpha,1}, \pi_{\alpha,1})$ is a forking projection from $(\mathbb{P}_\alpha, \ell_\alpha)$ to $(\mathbb{P}_1, \ell_1)$. We shall prove that $(\dot{\alpha}_{\alpha,1}, \pi_{\alpha,1})$ has the mixing property. The proof is by induction on $\alpha \in [1, \mu^+]$.

1. The base case $\alpha = 1$ follows by taking $g := \text{id}$ and $q := p'$, since $\pi_{1,1}$ and $\dot{\alpha}_{1,1}(q)$ are the identity maps.

2. Suppose that $\alpha = \alpha' + 1$ for a nonzero ordinal $\alpha' < \mu^+$ such that $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'})$ is a $\Sigma$-Prikry triple admitting a forking projection to $(\mathbb{P}_1, \ell_1, c_1)$ with the mixing property, as witnessed by $(\dot{\alpha}_{\alpha',1}, \pi_{\alpha',1})$. Suppose that we are given $p, p', m$ and $g : W_m(p') \to P_\alpha$ as in the statement of the lemma.

Derive a function $g' : W_m(p') \to P_{\alpha'}$ via $g'(r) := g(r) \upharpoonright \alpha'$. Since $p' \leq_1 \pi_{\alpha',1}(p \upharpoonright \alpha')$, the hypothesis on $\alpha'$ provides us a condition $p_{\alpha'} \in (P_{\alpha'}^\omega)_{0,1}^{\pi_{\alpha'}(p_{\alpha'})}$ with $\pi_{\alpha',1}(p_{\alpha'}) = p'$ such that, for every $r \in W_m(p')$,

$$\dot{\alpha}_{\alpha',1}(p_{\alpha'})(r) \leq_{\alpha'} g'(r) = g(r) \upharpoonright \alpha'. \tag{1}$$

Claim 4.8.1. There exists $q \leq_\alpha p$ with $\pi_{\alpha,\alpha'}(q) = p_{\alpha'}$ such that, for every $r \in W_m(p')$, $\dot{\alpha}_{\alpha',1}(q)(r) \leq_\alpha g(r)$.

Proof. By Fact 2.9(1), for each $s \in W_m(p_{\alpha'})$, we may let $r_s$ denote the unique element of $W_m(p')$ to satisfy $s = \dot{\alpha}_{\alpha',1}(p_{\alpha'})(r_s)$. Now, recall that, by the definition of $P_\alpha = (P_\alpha, \leq_\alpha)$, we have that $P_\alpha := \{x \upharpoonright \gamma \mid (x, y) \in A\}$ for some poset $(A, \leq)$ given by Building Block II together with maps $\pi : A \to P_{\alpha'}$ and $\dot{\alpha}$ such that $(\dot{\alpha}, \pi)$ is a forking projection having the mixing property. Furthermore, each element of $A$ is pair $(x, y)$ for which $\pi(x, y) = x$, and, for all $q \in P_\alpha$ and $r \leq_1 \pi_{\alpha,1}(q)$, $\dot{\alpha}_{\alpha,1}(q)(r) := x \upharpoonright \gamma$ is defined according to $(*)$ on page 18. Thus, define a function $g_{\alpha'} : W_m(p_{\alpha'}) \to A$ by letting, for each $s \in W_m(p_{\alpha'})$,

$$g_{\alpha'}(s) := \dot{\alpha}(g(r_s) \upharpoonright \alpha', g(r_s)(\alpha'))(\dot{\alpha}_{\alpha',1}(p_{\alpha'})(r_s)).$$

By Equation (1) above, $g_{\alpha'}$ is indeed well-defined. Let $a := (p \upharpoonright \alpha', p(\alpha'))$ so that $a \in A$ and $p_{\alpha'} \leq_\alpha \pi(a)$. For every $s \in W_m(p_{\alpha'})$, as $g(r_s) \leq_\alpha p$, we have

$$g_{\alpha'}(s) \leq (g(r_s) \upharpoonright \alpha', g(r_s)(\alpha')) \leq (p \upharpoonright \alpha', p(\alpha')) = a.$$

Observe that here we are also using Definition 2.7(2) with respect to $\dot{\alpha}(g(r_s) \upharpoonright \alpha', g(r_s)(\alpha'))$. In addition, by Definition 2.7(5), for every condition $s \in W_m(p_{\alpha'})$, $\pi(g_{\alpha'}(s)) = \dot{\alpha}_{\alpha',1}(p_{\alpha'})(r_s) = s$, as a consequence of the choice of $r_s$. Thus, by the mixing property of $(\dot{\alpha}, \pi)$, we may find $b \leq_0 a$ with $\pi(b) = p_{\alpha'}$ such that, for every $s \in W_m(p_{\alpha'})$, $\dot{\alpha}(b)(s) \leq g_{\alpha'}(s)$.
Let \( q := p_{\alpha'} \cdot \langle y^* \rangle \) for the unique \( y^* \) such that \( b = (p_{\alpha'}, y^*) \). To see that \( q \) is as desired, let \( r \in W_m(p') \) be arbitrary.

Let \( s \in W_m(p_{\alpha'}) \) be such that \( rs = r \), and write \( (x_s, y_s) := \cap_b(b)(s) \). Since \( \cap_{\alpha, 1}(q)(r) \) is defined according to equation \((*)\), \( \cap_{\alpha, 1}(q)(r) = x_s \cdot \langle y_s \rangle \). As 

\[
(x_s, y_s) = \cap_b(b)(s) \leq g_{\alpha'}(s) \leq (g(r) \upharpoonright \alpha', g(r)(\alpha')), \n\]

this means that \( \cap_{\alpha, 1}(q)(r) \leq g(r) \), as desired.

Let \( q \) be given by the previous claim. As \( \pi_{\alpha, 1}(q) = \pi_{\alpha', 1}(p_{\alpha'}) = p' \), we are done.

\[\text{Suppose that } \alpha \in \text{acc}(\mu^+ + 1), \text{ and, for every nonzero } \alpha' < \alpha, (\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'}) \text{ is a } \Sigma\text{-Prikry triple admitting a forcing projection to } (\mathbb{P}_1, \ell_1, c_1) \text{ with the mixing property, as witnessed by } (\pi_{\alpha', 1}, \pi_{\alpha', 1}). \]

Suppose that we are given \( p, p', m \) and \( g : W_m(p') \to P_\alpha \) as in the statement of the lemma. Set \( C := \text{cl}(\bigcup_{r \in W_m(p')} B(g(r)) \cup \{1, \alpha\} \). Since \(|W_m(p')| < \mu \) and, for each \( r \), \(|B_r| < \mu \), we have \(|C| < \mu \).

We now turn to define a \( \subseteq \)-increasing sequence \( \langle p_\gamma | \gamma \in C \rangle \in \prod_{\gamma \in C} (P_{\gamma, 0}^{\gamma, 1}) \gamma \) such that \( p_1 = p' \) and, for all \( \gamma \in C \) and \( r \in W_m(p') \),

\[
\cap_{\gamma, 1}(p_\gamma)(r) \leq g(r) \upharpoonright \gamma. \]

The definition is by recursion on \( \gamma \in C \):

- For \( \gamma = 1 \), we clearly let \( p_1 := p' \).
- Suppose \( \gamma > 1 \) is a non-accumulation point of \( C \cap \alpha \). By the definition of \( C \cap \alpha \), this means that there exists \( \beta \) with \( \gamma = \beta + 1 \). Let \( \bar{\beta} := \sup(C \cap \gamma) \), so that \( \beta \leq \bar{\beta} \), and then let \( p_{\beta} := p_{\beta} \ast \theta_{\beta} \). We know that, for every \( r \in W_m(p') \), \( \cap_{\beta, 1}(p_{\beta})(r) \leq \beta (g(r) \upharpoonright \bar{\beta}) \ast \theta_{\beta} = g(r) \upharpoonright \beta \). As the interval \( (\beta, \beta) \) is disjoint from \( \bigcup_{r \in W_m(p')} B(g(r)) \), furthermore, by Lemma 4.6(1) and (2),

\[
\cap_{\gamma, 1}(p_\gamma)(r) = \cap_{\beta, 1}(p_{\beta})(r) \ast \theta_{\beta} \leq \beta (g(r) \upharpoonright \bar{\beta}) \ast \theta_{\beta} = g(r) \upharpoonright \beta. \]

Next, by Claim 4.8.1, we obtain \( q \leq \gamma p \upharpoonright \gamma \) with \( \pi_{\gamma, \beta}(q) = p_{\beta} \) such that for all \( r \in W_m(p') \), \( \cap_{\gamma, 1}(q)(r) \leq g(r) \upharpoonright \gamma \). Thus, \( p_\gamma := q \) is as desired.

- Suppose \( \gamma \in \text{acc}(C) \). Define \( p_\gamma := \bigcup_{\delta \in C \cap \gamma} p_\delta \). By regularity of \( \mu \), we have \(|B_{p_\gamma}| < \mu \), so that \( p_\gamma \in P_\gamma \). As, for all \( \beta \in C \cap \gamma \), \( p_{\beta} \leq \beta p \upharpoonright \beta \), we also have \( p_{\gamma} \leq \gamma p \upharpoonright \gamma \). Combining the definition of \( \cap_{\gamma, 1}(p_\gamma) \), Lemma 4.6(1), and the fact that \( \sup(C \cap \gamma) = \gamma \), it follows that, for each \( r \in W_m(p') \), \( \cap_{\gamma, 1}(p_\gamma)(r) = \bigcup_{\delta \in C \cap \gamma} (\cap_{\delta, 1}(p_\delta))(r) \).

By Equation (2), which was provided by the induction hypothesis, \( \cap_{\gamma, 1}(p_\gamma)(r) \leq g(r) \upharpoonright \gamma \).

- Suppose \( \gamma = \alpha \), but \( \gamma \notin \text{acc}(C) \). In this case, let \( \bar{\alpha} := \sup(C \cap \alpha) \), and then set \( p_\alpha := p_{\bar{\alpha}} \ast \theta_{\alpha} \). As the interval \( (\bar{\alpha}, \alpha] \) is disjoint from \( \bigcup_{r \in W_m(p')} B(g(r)) \), by Lemma 4.6, Clauses (1) and (2), for every \( r \in W_m(p) \),

\[
\cap_{\alpha, 1}(p_\alpha)(r) = \cap_{\alpha, 1}(p_{\bar{\alpha}})(r) \ast \theta_{\alpha} \leq g(r) \upharpoonright \bar{\alpha} \ast \theta_{\alpha} = g(r). \]
Clearly, \( q := p_\alpha \) is as desired. \( \square \)

We are now ready to address Clause (iv) of Goal 4.2.

**Lemma 4.9.** For all nonzero \( \alpha \leq \mu^+ \), \((P_\alpha, \ell_\alpha, c_\alpha)\) is \( \Sigma \)-Prikry with a greatest element \( \emptyset_\alpha \), \( \ell_\alpha := \ell_1 \circ \pi_{\alpha,1} \), and \( \emptyset_\alpha \models_{P_\alpha} \mu = \kappa^+ \).

**Proof.** We argue by induction on \( \alpha \leq \mu^+ \). The base case \( \alpha = 1 \) follows from the fact that \( P_1 \) is isomorphic to \( Q \) given by Building Block I. The successor step \( \alpha = \beta + 1 \) follows from the fact that \( P_{\beta+1} \) was obtained by invoking Building Block II.

Next, suppose that \( \alpha \in \text{acc}(\mu^+ + 1) \) is such that the conclusion of the lemma holds below \( \alpha \). In particular, the hypothesis of Lemma 4.7 is satisfied, so that, for all nonzero \( \beta \leq \gamma \leq \alpha \), \((\mathfrak{h}_{\gamma, \beta}, \pi_{\gamma, \beta})\) is a forking projection from \((P_\gamma, \ell_\gamma)\) to \((P_\beta, \ell_\beta)\). We now go over the clauses of Definition 2.3:

1. The first bullet of Definition 2.1 follows from the fact that \( \ell_\alpha = \ell_1 \circ \pi_{\alpha,1} \). Next, let \( p \in P_\alpha \) be arbitrary. Denote \( \bar{p} := \pi_1(p) \). Since \((P_1, \ell_1, c_1)\) is \( \Sigma \)-Prikry, we may pick \( p' \leq_1 \bar{p} \) with \( \ell_1(p') = \ell_1(\bar{p}) + 1 \). As \((\mathfrak{h}_{1,1}, \pi_{1,1})\) is a forking projection from \((P_1, \ell_1)\) to \((P_1, \ell_1)\), Fact 2.9(2) implies that \( \mathfrak{h}_{1,1}(\bar{p})(p') \) is an element of \((P_\alpha, \ell_\alpha)\).

2. Let \( n < \omega \). To see that \((P_\alpha)_n\) is \( \kappa_n \)-directed-closed, fix an arbitrary directed family \( D \subseteq (P_\alpha)_n \) of size \( < \kappa_n \). Let \( C := \text{cl}((\bigcup_{p \in D} B_p) \cup \{1, \alpha\}) \). We shall define a \( \subseteq \)-increasing sequence \( \langle p_\gamma \mid \gamma \in C \rangle \in \prod_{\gamma \in C} (P_\gamma)_n \) such, for all \( \gamma \in C \), \( p_\gamma \) is a lower bound for \( \{p \mid p \in D\} \). The definition is by recursion on \( \gamma \in C \):

   - For \( \gamma = 1 \), \( \{p \upharpoonright 1 \mid p \in D\} \) is directed. By the induction hypothesis, \((P_1, \ell_1, c_1)\) is \( \Sigma \)-Prikry, and hence we may find a lower bound \( p_1 \in (P_1)_n \) for the set under consideration.
   - Suppose \( \gamma > 1 \) is a non-accumulation point of \( C \cap \alpha \). Let \( \beta := \text{sup}(C \cap \gamma) \), and consider the set \( A_\gamma := \{\mathfrak{h}_{\gamma, \beta}(p \mid \gamma)(p_\beta) \mid p \in D\} \). By the induction hypothesis, \((P_\gamma, \ell_\gamma, c_\gamma)\) is \( \Sigma \)-Prikry and \((\mathfrak{h}_{\gamma, \beta}, \pi_{\gamma, \beta})\) is a forking projection from \((P_\gamma, \ell_\gamma)\) to \((P_\beta, \ell_\beta)\). Clearly, \( |A_\gamma| \leq |D| < \kappa_n \) and also, by Clause (7) of Definition 2.7, \( A_\gamma \) is directed, hence we may find a lower bound \( p_\gamma \in (P_\gamma)_n \) for it. By Lemma 4.6(3), \( q \upharpoonright \beta = p_\beta \), for all \( q \in A_\gamma \). Thus, for each \( p \in D \), \( p_\gamma \upharpoonright \beta \leq \beta p_\beta \leq \beta p \upharpoonright \beta \), yielding the desired lower bound.
   - Suppose \( \gamma \in \text{acc}(C) \). Define \( p_\gamma := \bigcup_{\beta \in (C \cap \gamma)} p_\beta \). By regularity of \( \mu \), we have \( |B_{p_\beta}| < \mu \), so that \( p_\gamma \in P_\gamma \). Now, for all \( p \in D \) and all \( \beta \in C \cap \gamma \), we have \( p_\gamma \upharpoonright \beta = p_\beta \leq \beta p \upharpoonright \beta \). So, \( p_\gamma \) is indeed a bound for \( \{p \mid p \upharpoonright \gamma \in D\} \).
   - Suppose \( \gamma = \alpha \), but \( \gamma \notin \text{acc}(C) \). In this case, let \( \bar{\alpha} := \text{sup}(C \cap \alpha) \), and then set \( p_\alpha := p_{\bar{\alpha}} * \emptyset_\alpha \). As the interval \((\bar{\alpha}, \alpha]\) is disjoint from \( \bigcup_{p \in D} B_p \), for every \( p \in D \),
     \[
     p_\alpha = (p_{\bar{\alpha}} \mid \bar{\alpha}) * \emptyset_\alpha \leq (p \mid \bar{\alpha}) * \emptyset_\alpha = p.
     \]
   Clearly, \( p_\alpha \) is a lower bound for \( D \), as desired.
Claim 4.9.1. Suppose \( p, p' \in P_\alpha \) with \( c_\alpha(p) = c_\alpha(p') \). Then, \((P_\alpha)_0^p \cap (P_\alpha)_0^{p'}\) is nonempty.

Proof. If \( \alpha < \mu^+ \), then since \((h_{\alpha,1}, \pi_{\alpha,1})\) is a forking projection from \((P_\alpha, \ell_\alpha, c_\alpha)\) to \((P_1, \ell_1, c_1)\), we get from Clause (8) of Definition 2.7 that \( c_1(p \upharpoonright 1) = c_1(p' \upharpoonright 1) \), and then by Clause (3) of Definition 2.3, we may pick \( r \in (P_1)_0^{p_1} \cap (P_1)_0^{p'_1} \). In effect, Clause (8) of Definition 2.7 entails \( h_{\alpha,1}(p)(r) = h_{\alpha,1}(p')(r) \). Finally, Fact 2.9(2) implies that \( h_{\alpha,1}(p)(r) \) is in \((P_\alpha)_0^p\) and that \( h_{\alpha,1}(p')(r) \) is in \((P_\alpha)_0^{p'}\). In particular, \((P_\alpha)_0^p \cap (P_\alpha)_0^{p'}\) is nonempty.

From now on, assume \( \alpha = \mu^+ \). In particular, for all nonzero \( \beta < \gamma < \mu^+, (P_\gamma, \ell_\gamma, c_\gamma) \) is a \( \Sigma\)-Prikry triple admitting a forking projection to \((P_\beta, \ell_\beta, c_\beta)\) as witnessed by \((h_{\gamma,\beta}, \pi_{\gamma,\beta})\). To avoid trivialities, assume also that \( |\{1^{\mu^+_0}, p, p'\}| = 3 \). For each \( q \in \{p, p'\} \), let \( C_q := \text{cl}(B_q) \) and define a function \( e_q : C_q \rightarrow H_\mu \) via

\[
e_q(\gamma) := (\phi_q[C_q \cap \gamma], c_q(q \upharpoonright \gamma)).
\]

Write \( i \) for the common value of \( c_{\mu^+}(p) \) and \( c_{\mu^+}(p') \). It follows that, for every \( \gamma \in C_p \cap C_{p'} \), \( e_p(\gamma) = e_i(\gamma) = e_p(\gamma) \), so that \( \phi_q[C_p \cap \gamma] = \phi_q[C_{p'} \cap \gamma] \) and hence \( C_p \cap \gamma = C_{p'} \cap \gamma \). Consequently, \( R := C_p \cap C_{p'} \) is an initial segment of \( C_p \) and an initial segment of \( C_{p'} \).

Let \( \zeta := \max(C_p \cup C_{p'}) \), so that \( p = (p \upharpoonright \zeta) * \emptyset_{\mu^+} \) and \( p' = (p' \upharpoonright \zeta) * \emptyset_{\mu^+} \). Set \( \gamma_0 := \max(\{0\} \cup R) \). By the above analysis, \( C_p \cap (\gamma_0, \zeta) \) and \( C_{p'} \cap (\gamma_0, \zeta) \) are two disjoint closed sets.

If \( \gamma_0 = \zeta \), then \( e_p(\zeta) = e_p(\zeta) \), so that \( c_\zeta(p \upharpoonright \zeta) = c_\zeta(p' \upharpoonright \zeta) \), and hence \( (P_\zeta)_0^{p \upharpoonright \zeta} \cap (P_\zeta)_0^{p' \upharpoonright \zeta} \) is nonempty. Pick \( r \) in that intersection. Then \( r * \emptyset_{\mu^+} \) is an element of \((P_{\mu^+_0})_0^p \cap (P_{\mu^+_0})_0^{p'}\).

Next, suppose that \( \gamma_0 < \zeta \). Consequently, there exists a finite increasing sequence \( (\gamma_j \mid j \leq k) \) of ordinals from \( C_p \cup C_{p'} \) such that \( \gamma_{k+1} = \zeta \) and, for all \( j \leq k \):

(i) if \( \gamma_{j+1} \in C_p \), then \( (\gamma_j, \gamma_j+1] \cap (C_p \cup C_{p'}) \subseteq C_p \);
(ii) if \( \gamma_{j+1} \notin C_p \), then \( (\gamma_j, \gamma_j+1] \cap (C_p \cup C_{p'}) \subseteq C_{p'} \).

We now define a sequence \( \langle r_j \mid j \leq k+1 \rangle \) in \( \prod_{j=0}^{k+1} ((P_{\gamma_j})_0^{p \upharpoonright \gamma_j} \cap (P_{\gamma_j})_0^{p' \upharpoonright \gamma_j}) \), as follows.

- For \( j = 0 \), if \( \gamma_0 \in C_p \cap C_{p'} \), then \( e_p(\gamma_0) = e_{p'}(\gamma_0) \), so that \( c_{\gamma_0}(p \upharpoonright \gamma_0) = c_{\gamma_0}(p' \upharpoonright \gamma_0) \) and we may indeed pick \( r_0 \in (P_{\gamma_0})_0^{p \upharpoonright \gamma_0} \cap (P_{\gamma_0})_0^{p' \upharpoonright \gamma_0} \). If \( \gamma_0 \notin C_p \cap C_{p'} \), then \( \gamma_0 = 0 \), and we simply let \( r_0 := \emptyset \).
- Suppose that \( j < k + 1 \), where \( r_j \) has already been defined. Let \( q := h_{\gamma_{j+1}, \gamma_j}^{\gamma_j}(p \upharpoonright \gamma_{j+1}) \) and \( q' := h_{\gamma_{j+1}, \gamma_j}^{\gamma_{j+1}}(p' \upharpoonright \gamma_{j+1}) \). By Lemma 4.6(2), \( B_q = (B_p \cap \gamma_{j+1}) \cup B_{r_j} \) and \( B_{q'} = (B_{p'} \cap \gamma_{j+1}) \cup B_{r_j} \). In particular, if \( \gamma_{j+1} \in C_p \), then \( (\gamma_j, \gamma_{j+1}] \cap (B_q \cup B_{q'}) \subseteq B_q \), so that \( q' = r_j * \emptyset_{\gamma_{j+1}} \) and \( q \leq_{\gamma_{j+1}} q' \) by Clauses (5) and (6) of Lemma 4.6, respectively. Likewise, if \( \gamma_{j+1} \notin C_p \), then \( q = r_j * \emptyset_{\gamma_{j+1}} \), so that
Let $p \in P_\alpha$, $n, m < \omega$ and $q \in (P_\alpha^\nu)_{n+m}$ be arbitrary. Recalling that $(\mathfrak{n}_\alpha, \pi_\alpha, 1)$ is a forking projection from $(P_\alpha, \ell_\alpha)$ to $(P_1, \ell_1)$, we infer from Fact 2.9(1) that, for every $p \in P_\alpha$, $|W(p)| = |W(p | 1)| < \mu$.

(5) Recalling that $(\mathfrak{n}_1, \ell_1, c_1)$ is $\Sigma$-Prikry, and that $(\mathfrak{n}_\alpha, \pi_\alpha, 1)$ is a forking projection from $(P_\alpha, \ell_\alpha)$ to $(P_1, \ell_1)$, we infer from Fact 2.9(1) that, for every $p \in P_\alpha$, $|W(p)| = |W(p | 1)| < \mu$.

(6) Let $p', p \in P_\alpha$ with $p' \leq \alpha p$. Let $q \in W(p')$ be arbitrary. For all $\gamma < \alpha$, the pair $(\mathfrak{n}_\alpha, \pi_\alpha, 1)$ is a forking projection from $(P_\alpha, \ell_\alpha)$ to $(P_\gamma, \ell_\gamma)$, so that by the special case $m = 0$ of Clause (4) of Definition 2.7,

$$w(p, q) = \mathfrak{n}_\alpha(p)(w(p | \gamma, q | \gamma)).$$

Now, for all $q' \leq \alpha q$, the induction hypothesis implies that, for all $\gamma < \alpha$, $w(p | \gamma, q' | \gamma) \leq \gamma w(p | \gamma, q | \gamma)$. Together with Clause (5) of Definition 2.7, it follows that, for all $\gamma < \alpha$,

$$w(p, q') | \gamma = w(p | \gamma, q' | \gamma) \leq \gamma w(p | \gamma, q | \gamma) = w(p, q) | \gamma.$$  

So, by the definition of $\leq \alpha$, $w(p, q') \leq \alpha w(p, q)$, as desired.

(7) This follows from Fact 2.14, using Lemma 4.8. To complete our proof we shall need the following claim.

Claim 4.9.2. For each $1 \leq \alpha \leq \mu^+$, $1_{P_\alpha} \Vdash \mathfrak{p} = \kappa^+$.

Proof. The case $\alpha = 1$ is given by Building Block I. Towards a contradiction, suppose that $1 < \alpha \leq \mu^+$ and that $1_{P_\alpha} \Vdash \mathfrak{p} = \kappa^+$. As $1_{P_1} \Vdash \mathfrak{p} = \kappa^+$ and $P_\alpha$ projects to $P_1$, this means that there exists $p \in P_\alpha$ such that $p \models P_\alpha \left[ |\mu| \leq |\kappa| \right]$. Since $P_1$ is isomorphic to the poset $Q$ of Building Block I, and since $1_Q \Vdash \mathfrak{q}$ is singular", $1_{P_1} \Vdash \mathfrak{p} \models \kappa$ is singular”.

As $P_\alpha$ projects to $P_1$, in fact $p \models P_\alpha \left[ \text{cf}(\mu) < \kappa \right]$. Thus, Lemma 2.6(2) yields a condition $p' \leq \alpha p$ with $|W(p')| \geq \mu$, contradicting Clause (5) above. 

This completes the proof of Lemma 4.9. 

5. An application

In this section, we present the first application of our iteration scheme. We will be constructing a model of finite simultaneous reflection at a successor of a singular strong limit cardinal $\kappa$ in the presence of $\neg$SCH$_\kappa$.

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12This is the sole part of the whole proof to make use of the fact that the poset given by Building Block I forces $\kappa$ to be singular.
**Definition 5.1.** For cardinals $\theta < \mu = \text{cf}(\mu)$ and stationary subsets $S, \Gamma$ of $\mu$, $\text{Refl}(<\theta, S, \Gamma)$ stands for the following assertion. For every collection $S$ of stationary subsets of $S$, with $|S| < \theta$ and $\sup(\{|\text{cf}(\alpha)| \mid \alpha \in \bigcup S\}) < \mu$, there exists $\delta \in \Gamma \cap E^{\omega}_{\mu}$ such that, for every $S \in S$, $S \cap \delta$ is stationary in $\delta$.

We write $\text{Refl}(<\theta, S)$ for $\text{Refl}(<\theta, S, \mu)$.

A proof of the following folklore fact may be found in [PRS20, §4].

**Fact 5.2.** If $\kappa$ is a singular strong limit cardinal admitting a stationary subset $S \subseteq \kappa^+$ for which $\text{Refl}(<\text{cf}(\kappa)^+, S)$ holds, then $2^\kappa = \kappa^+$.

In particular, if $\kappa$ is a singular strong limit cardinal of countable cofinality for which $\text{SCH}_\kappa$ fails, and $\text{Refl}(<\theta, \kappa^+)$ holds, then $\theta \leq \omega$. We shall soon show that $\theta := \omega$ is indeed feasible.

The following general statement about simultaneous reflection will be useful in our verification later on.

**Proposition 5.3.** Suppose that $\mu$ is non-Mahlo cardinal, and $\theta \leq \text{cf}(\mu)$. For stationary subsets $T, \Gamma, R$ of $\mu$, $\text{Refl}(<2, T, \Gamma) + \text{Refl}(<\theta, \Gamma, R)$ entails $\text{Refl}(<\theta, T \cup \Gamma, R)$.

**Proof.** Given a collection $S$ of stationary subsets of $T \cup \Gamma$, with $|S| < \theta$ and $\sup(\{|\text{cf}(\alpha)| \mid \alpha \in \bigcup S\}) < \mu$, we shall first attach to any set $S \in S$, a stationary subset $S'$ of $\Gamma$, as follows.

1. If $S \cap \Gamma$ is stationary, then let $S' := S \cap \Gamma$.
2. If $S \cap \Gamma$ is nonstationary, then for every (sufficiently thin) club $C \subseteq \mu$, $S \cap C$ is a stationary subset of $T$, and so by $\text{Refl}(<2, T, \Gamma)$, there exists $\alpha \in \Gamma \cap E^{\omega}_{\mu}$ such that $(S \cap C) \cap \alpha$ is stationary in $\alpha$, and in particular, $\alpha \in C$. So, the set $\{\alpha \in \Gamma \mid S \cap \alpha$ is stationary$\}$ is stationary, and, as $\mu$ is non-Mahlo, we may pick $S'$ which is a stationary subset of it and all of its points consists of the same cofinality.

Next, as $|S| < \text{cf}(\mu)$, we have $\sup(\{|\text{cf}(\alpha)| \mid \alpha \in S', S \in S\}) < \mu$, and so, from $\text{Refl}(<\theta, \Gamma, R)$, we find some $\alpha \in R$ such that $S' \cap \alpha$ is stationary for all $S \in S$.

**Claim 5.3.1.** Let $S \in S$. Then $S \cap \alpha$ is stationary in $\alpha$.

**Proof.** If $S' = S$, then $S \cap \alpha = S' \cap \alpha$ is stationary in $\alpha$, and we are done. Next, assume $S' \neq S$, and let $c$ be an arbitrary club in $\alpha$. As $S' \cap \alpha$ is stationary in $\alpha$, we may pick $\delta \in \text{acc}(c) \cap S'$. As $\delta \in S' \subseteq E^{\omega}_{\mu}$, $c \cap \delta$ is a club in $\delta$, and as $\delta \in S'$, $S \cap \delta$ is stationary, so $S \cap c \cap \delta \neq \emptyset$. In particular, $S \cap c \neq \emptyset$. \(\square\)

This completes the proof. \(\square\)

For the rest of this section, we make the following assumptions:

- $\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
\[ 2^\kappa = \kappa^+ \text{ and } 2^\mu = \mu^+. \]

Under these assumptions, [PRS20, Corollary 5.11] and [PRS20, Corollary 7.1] read as follows, respectively:

**Fact 5.4.** If \((\mathcal{P}, \ell, c)\) is a \(\Sigma\)-Prikry notion of forcing such that \(1_\mathcal{P} \Vdash \mathcal{P} \Vdash \tilde{\mu} = \kappa^+\), then \(V^\mathcal{P} \models \text{Refl}(\langle \omega, \Gamma \rangle)\).

**Fact 5.5.** Suppose:

1. \((\mathcal{P}, \ell, c)\) is a \(\Sigma\)-Prikry notion of forcing;
2. \(1_\mathcal{P} \Vdash \mathcal{P} \Vdash \tilde{\mu} = \kappa^+\);
3. \(\mathcal{P} = (P, \leq)\) is a subset of \(H_{\mu^+}\);
4. \(r^* \in P\) forces that \(z\) is a \(\mathcal{P}\)-name for a stationary subset of \((E_\mu^\mu)^V\) that does not reflect in \(\{\alpha < \mu : \omega < \text{cf}^V(\alpha) < \kappa\}\).

Then, there exists a \(\Sigma\)-Prikry triple \((\mathcal{A}, \ell_{\mathcal{A}}, c_{\mathcal{A}})\) such that:

1. \((\mathcal{A}, \ell_{\mathcal{A}}, c_{\mathcal{A}})\) admits a forking projection \((\mathcal{H}, \pi)\) to \((\mathcal{P}, \ell, c)\) that has the mixing property;
2. \(1_{\mathcal{A}} \Vdash_{\mathcal{A}} \tilde{\mu} = \kappa^+\);
3. \(\mathcal{A} = (A, \leq)\) is a subset of \(H_{\mu^+}\);
4. \([r^*]^{\mathcal{A}}\) forces that \(z\) is nonstationary.

Remark 5.6. The above block is obtained as follows.

- If \(r^* \in P\) forces that \(z\) is a \(\mathcal{P}\)-name for a stationary subset of \((E_\mu^\mu)^V\) that does not reflect in \(\Gamma\), then we invoke Fact 5.5.
- Otherwise, let \(\mathcal{A} := (A, \leq)\), where \(A := P \times \{\emptyset\}\) and \((p, q) \leq (p', q')\) iff \(p \leq p'\). Define \(\pi : A \to P\) via \(\pi(x, y) := x\). Define \(\mathcal{H}\) via \(\mathcal{H}(a)(p) := (p, \emptyset)\).
and let $\ell_A := \ell_p \circ \pi$ and $c_A := c_p \circ \pi$. It is straight-forward to verify that $(A, \ell_A, c_A)$ and $(\emptyset, \pi)$ satisfy all the requirements.

**Building Block III.** As $2^\mu = \mu^+$, we fix a surjection $\psi : \mu^+ \to H_{\mu^+}$ such that the preimage of any singleton is cofinal in $\mu^+$.

Now, we appeal to the iteration scheme of Section 4 with these building blocks, and obtain, in return, a $\Sigma$-Prikry triple $(P_{\mu^+}, \ell_{\mu^+}, c_{\mu^+})$.

**Theorem 5.7.** In $V^{P_{\mu^+}}$ all of the following hold true:

1. Any cardinal in $V$ remains a cardinal and retains its cofinality;
2. $\kappa$ is a singular strong limit of countable cofinality;
3. $2^\kappa = \kappa^{++}$;
4. $\text{Refl}(\langle \omega \rangle)$.

**Proof.**

1. By Fact 2.6(1), no cardinal $\leq \kappa$ changes its cofinality; by Fact 2.6(3), $\kappa^+$ is not collapsed, and by Definition 2.3(3), no cardinal $> \kappa^+$ changes its cofinality.

2. In $V$, $\kappa$ is a singular strong limit of countable cofinality, and so by Fact 2.6(1), this remains valid in $V^{P_{\mu^+}}$.

3. In $V$, we have that $2^\kappa = \kappa^+$. In addition, by Remark 4.3(1), $P_{\mu^+}$ is isomorphic to a subset of $H_{\mu^+}$, so that, from $|H_{\mu^+}| = \kappa^{++}$, we infer that $V^{P_{\mu^+}} \models 2^\kappa \leq \kappa^{++}$. Finally, as $P_{\mu^+}$ projects to $P_1$ which is isomorphic to $\mathbb{Q}$, we get that $V^{P_{\mu^+}} \models 2^\kappa \geq \kappa^{++}$. Altogether, $V^{P_{\mu^+}} \models 2^\kappa = \kappa^{++}$.

4. As $\kappa^+ = \mu$ and $\kappa$ is singular, $\text{Refl}(\langle \omega, \kappa^+ \rangle)$ is equivalent to $\text{Refl}(\langle \omega, E_\kappa^\mu \rangle)$.

By Fact 5.4, we already know that $V^{P_{\mu^+}} \models \text{Refl}(\langle \omega, \Gamma \rangle)$. So, by Proposition 5.3, it suffices to verify that $\text{Refl}(\langle 2, (E_\kappa^\mu)^V \rangle, \Gamma)$ holds in $V^{P_{\mu^+}}$.

Let $G$ be $P_{\mu^+}$-generic over $V$ and hereafter work within $V[G]$. Towards a contradiction, suppose that there exists a subset $T$ of $(E_\kappa^\mu)^V$ that does not reflect in $\Gamma$. Fix $r^* \in G$ and a $P_{\mu^+}$-name $\tau$ such that $\tau_G$ is equal to such a $T$ and such that $r^*$ forces $\tau$ to be a stationary subset of $(E_\kappa^\mu)^V$ that does not reflect in $\Gamma$. Furthermore, we may require that $\tau$ be a *nice name*, i.e., each element of $\tau$ is a pair $(\xi, p)$ where $(\xi, p) \in (E_\kappa^\mu)^V \times P_{\mu^+}$, and, for all $\xi \in (E_\kappa^\mu)^V$, the set $\{ p \mid (\xi, p) \in \tau \}$ is an antichain.

As $P_{\mu^+}$ satisfies Clause (3) of Definition 2.3, $P_{\mu^+}$ has the $\mu^+$-cc. Consequently, there exists a large enough $\beta < \mu^+$ such that

$$B_{r^*} \cup \bigcup \{ B_p \mid (\xi, p) \in \tau \} \subseteq \beta.$$ 

Let $r := r^* | \beta$ and set

$$\sigma := \{ (\xi, p | \beta) \mid (\xi, p) \in \tau \}.$$

From the choice of Building Block III, we may find a large enough $\alpha < \mu^+$ with $\alpha > \beta$ such that $\psi(\alpha) = (\beta, r, \sigma)$. As $\beta < \alpha$, $r \in P_\beta$ and $\sigma$ is a $P_\beta$-name, the definition of our iteration at step $\alpha + 1$ involves appealing to Building Block II with $(P_\alpha, \ell_\alpha, c_\alpha), r^* := r \emptyset_\alpha$ and $z := i_\beta^\alpha(\sigma)$. For any...
ordinal $\eta < \mu^+$, denote $G_{\eta} := \pi_{\mu^+, \eta}[G]$. By the choice of $\beta$, and as $\alpha > \beta$, we have

$$\tau = \{((\xi, p \ast \emptyset_{\mu^+}) \mid (\xi, p) \in \sigma\} = \{((\xi, p \ast \emptyset_{\mu^+}) \mid (\xi, p) \in z\},$$

so that, in $V[G]$,

$$T = \tau_G = \sigma_G = z_G.$$  

In addition, $r^* = r^* \ast \emptyset_{\mu^+}$.  

Finally, as $r^*$ forces $\tau$ is a stationary subset of $(E^n_\mu)^V$ that does not reflect in $\Gamma$, $r^*$ forces that $z$ is a stationary subset of $(E^0_\omega)^V$ that does not reflect in $\Gamma$. So, since $\pi_{\mu^+, \alpha + 1}(r^*) = r^* \ast \emptyset_{\alpha + 1} = [r^*]^\mu_{\alpha + 1}$ is in $G_{\alpha + 1}$, Clause (e) of Building Block II entails that, in $V[G_{\alpha + 1}]$, there exists a club in $\mu$ which is disjoint from $T$. In particular, $T$ is nonstationary in $V[G]$, contradicting its very choice.  

Thus, we arrive at the following strengthening of the theorem announced by Sharon in [Sha05].

**Corollary 5.8.** Suppose that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals, converging to a cardinal $\kappa$. Then there exists a forcing extension where the following properties hold:

1. $\kappa$ is a singular strong limit cardinal of countable cofinality;
2. $2^\kappa = \kappa^{++}$, hence SCH fails;
3. Refl$(<\omega, \kappa^+)$ holds.

**Proof.** Let $L$ be the inverse limit of the iteration $\langle L_n; \check{Q}_n \mid n < \omega \rangle$, where $L_0$ is the trivial forcing and for positive integer $n$, if $L_{\check{Q}_n} \models L_n \models "\kappa_{n-1}$ is supercompact", then $L_{\check{Q}_n} \models \"\hat{Q}_n is a Laver preparation for $\kappa_n\". After forcing with $L$, each $\kappa_n$ remains supercompact and, moreover, becomes indestructible under $\kappa_n$-directed-closed forcing. Also, the cardinals and cofinalities of interest are preserved.

Working in $V^L$, set $\mu := \kappa^+$, $\lambda := \kappa^{++}$ and $C := \text{Add}(\lambda, 1)$. Finally, work in $W := V^{L_{\check{C}}}$. Since $\kappa$ is singular strong limit of cofinality $\omega < \kappa_0$ and $\kappa_0$ is supercompact, $2^\kappa = \kappa^+$. Also, thanks to the forcing $C$, $2^\mu = \mu^+$. Altogether, in $W$, all the following hold:

- $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-Indestructible supercompact cardinals;
- $\kappa := \sup_{n < \omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$.

Now, appeal to Theorem 5.7.  

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