SIGMA-PRIKRY FORCING III:
DOWN TO $\aleph_\omega$

ALEJANDRO POVEDA, ASSAF RINOT, AND DIMA SINAPOVA

Abstract. We prove the consistency of the failure of the singular cardinals hypothesis at $\aleph_\omega$ together with the reflection of all stationary subsets of $\aleph_{\omega+1}$. This shows that two classic results of Magidor (from 1977 and 1982) can hold simultaneously.

1. Introduction

Many natural questions cannot be resolved by the standard mathematical axioms (ZFC); the most famous example being Hilbert’s first problem, the continuum hypothesis (CH). At the late 1930’s, Gödel constructed an inner model of set theory [Göd40] in which the generalized continuum hypothesis (GCH) holds, demonstrating, in particular, that CH is consistent with ZFC. Then, in 1963, Cohen invented the method of forcing [Coh63] and used it to prove that ¬CH is, as well, consistent with ZFC.

In an advance made by Easton [Eas70], it was shown that any reasonable behavior of the continuum function $\kappa \mapsto 2^\kappa$ for regular cardinals $\kappa$ may be materialized. In a review on Easton’s paper for AMS Mathematical Reviews, Azriel Lévy writes:

The corresponding question concerning the singular $\aleph_\alpha$’s is still open, and seems to be one of the most difficult open problems of set theory in the post-Cohen era. It is, e.g., unknown whether for all $n(n < \omega \rightarrow 2^{\aleph_n} = \aleph_{n+1})$ implies $2^{\aleph_\omega} = \aleph_{\omega+1}$ or not.

A preliminary finding of Bukovský [Buk65] (and independently Hechler) suggested that singular cardinals may indeed behave differently, but it was only around 1975, with Silver’s theorem [Sil75] and the pioneering work of Galvin and Hajnal [GH75], that it became clear that singular cardinals obey much deeper constraints. This lead to the formulation of the singular cardinals hypothesis (SCH) as a (correct) relativization of GCH to singular cardinals, and ultimately to Shelah’s pcf theory [She92, She00]. Shortly after Silver’s discovery, advances in inner model theory due to Jensen (see [DJ75]) provided a covering lemma between Gödel’s original model of GCH and many other models of set theory, thus establishing that any consistent

Date: November 13, 2020. This is a preliminary preprint.
failure of SCH must rely on an extension of ZFC involving large cardinals axioms.

Compactness is the phenomenon where if a certain property holds for every strictly smaller substructure of a given object, then it holds for the object itself. Countless results in topology, graph theory, algebra and logic demonstrate that the first infinite cardinal is compact. Large cardinals axioms are compactness postulates for the higher infinite.

A crucial tool for connecting large cardinals axioms with singular cardinals was introduced by Prikry in [Pri70]. Then Silver (see [Men76]) constructed a model of ZFC whose extension by Prikry’s forcing gave the first universe of set theory with a singular strong limit cardinal \( \kappa \) such that \( 2^{\kappa} > \kappa^+ \). Shortly after, Magidor [Mag77a] proved that the same may be achieved at level of the very first singular cardinal, that is, \( \kappa = \aleph_\omega \). Finally, in 1977, Magidor answered the question from Lévy’s review in the affirmative:

**Theorem 1** (Magidor, [Mag77b]). Assuming the consistency of a supercompact cardinal and a huge cardinal above it, it is consistent that \( 2^{\aleph_n} = \aleph_{n+1} \) for all \( n < \omega \), and \( 2^{\aleph_\omega} = \aleph_{\omega+2} \).

Later works of Gitik, Mitchell, and Woodin pinpointed the optimal large cardinal hypothesis required for Magidor’s theorem (see [Git02, Mit10]).

Note that Theorem 1 is an incompactness result; the values of the power-set function are small below \( \aleph_\omega \), and blow up at \( \aleph_\omega \). In a paper from 1982, Magidor obtained a result of an opposite nature, asserting that stationary reflection — one of the most canonical forms of compactness — may hold at the level of the successor of the first singular cardinal:

**Theorem 2** (Magidor, [Mag82]). Assuming the consistency of infinitely many supercompact cardinals, it is consistent that every stationary subset of \( \aleph_{\omega+1} \) reflects.\(^1\)

Ever since, it remained open whether Magidor’s compactness and incompactness results may co-exist.

The main tool for obtaining Theorem 1 (and the failures of SCH, in general) is Prikry-type forcing (see Gitik’s survey [Git10]), however, adding Prikry sequences at a cardinal \( \kappa \) typically implies the failure of reflection at \( \kappa^+ \). On the other hand, Magidor’s proof of Theorem 2 goes through Lévy-collapsing \( \omega \)-many supercompact cardinals to become the \( \aleph_n \)’s, and in any such model SCH would naturally hold at the supremum, \( \aleph_\omega \).

Various partial progress to combine the two results was made along the way. Cummings, Foreman and Magidor [CFM01] investigated which sets can reflect in the classical Prikry generic extension. In his 2005 dissertation [Sha05], Sharon analyzed reflection properties of extender-based Prikry forcing (EBPF, due to Gitik and Magidor [GM94]) and devised a way to kill

\(^1\)That is, for every subset \( S \subseteq \aleph_{\omega+1} \), if for every ordinal \( \alpha < \aleph_{\omega+1} \) (of uncountable cofinality), there exists a closed and unbounded subset of \( \alpha \) disjoint from \( S \), then there exists a closed and unbounded subset of \( \aleph_{\omega+1} \) disjoint from \( S \).
one non-reflecting stationary set, again in a Prikry-type fashion. He then described an iteration to kill all non-reflecting stationary sets, but unfortunately, that proof was incomplete.\footnote{Specifically, Clause (4) in the proof of Claim 3.29 cannot be assumed.}

In the other direction, works of Solovay [Sol74], Foreman, Magidor and Shelah [FMS88], Velči\v{k}ovi\v{c} [Vel92], Todor\v{c}evi\v{c} [Tod93], Foreman and Todor\v{c}evi\v{c} [FT05], Moore [Moo06], Viale [Via06], Rinot [Rin08], Shelah [She08], Fuchino and Rinot [FR11], and Sakai [Sak15] add up to a long list of compactness principles that are sufficient to imply the SCH.

Recently, the authors found a repair to Sharon’s proof. In [PRS19], we introduced a new class of Prikry-type forcing called $\Sigma$-Prikry and showed that many of the standard Prikry-type forcing for violating SCH at the level of a singular cardinal of countable cofinality fits into this class. In addition, we verified that Sharon’s forcing for killing a single non-reflecting stationary set fits into this class. Then, in [PRS20], we devised a general iteration scheme for $\Sigma$-Prikry forcing. From this, we constructed a model of the failure of SCH at $\kappa$ with stationary reflection at $\kappa^+$; we first violate the SCH using EBPF and then carry out an iteration of length $\kappa^{++}$ of the $\Sigma$-Prikry posets to kill all non-reflecting stationary subsets of $\kappa^+$.

Independently, and around the same time, Ben-Neria, Hayut and Unger [OHU19] also obtained the consistency of the failure of SCH at $\kappa$ with stationary reflection at $\kappa^+$. Their proof differs from ours in quite a few aspects; we mention just two of them. First, instead of EBPF, they violate SCH by using Gitik’s very recent forcing [Git19a] which is also applicable to cardinals of uncountable cofinality. Second, they cleverly avoid the need to carry out iterated forcing, by invoking iterated ultrapowers, instead. An even simpler proof was then given by Gitik in [Git19c].

Still, in all of the above, the constructions are for a singular cardinal $\kappa$ that is very high up; more precisely, $\kappa$ is a limit of inaccessible cardinals. Obtaining a similar construction for $\kappa = \aleph_\omega$ is quite more difficult, as it involves interleaving collapses. This makes key parts of the forcing no longer closed, and closure is an essential tool to make use of the indestructibility of the supercompact cardinals when proving reflection.

In this paper, we extend the machinery developed in [PRS19, PRS20] to support interleaved collapses, and show that this new framework captures Gitik’s EBPF with interleaved collapses [Git19b]. The new class is called $(\Sigma, S)$-Prikry. Finally, by running our iteration of $(\Sigma, S)$-Prikry forcings over a suitable ground model, we establish that Magidor’s compactness and incompactness results can indeed co-exist:

**Main Theorem.** Assuming the consistency of infinitely many supercompact cardinals, it is consistent that all of the following hold:

(1) $2^{\aleph_0} = \aleph_{n+1}$ for all $n < \omega$;
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$;
(3) every stationary subset of $\aleph_{\omega+1}$ reflects.
1.1. **Organization of this paper.** In Section 2, we introduce the concepts of nice exact projection and suitability for reflection.

In Section 3, we introduce the class of \((\Sigma, \vec{S})\)-Prikry forcing and prove some of their main properties.

In Section 4, we prove that Gitik’s Extender Based Prikry Forcing with Collapses (EBPFC) fits into the \((\Sigma, \vec{S})\)-Prikry framework. Here we also analyze the preservation of cardinals in the corresponding generic extension and show that EBPFC is suitable for reflection.

In Section 5, we introduce the notion of exact forking projection, a strengthening of the concept of forking projection from Part I of this series. We show that a graded poset admitting an exact forking projection to a \((\Sigma, \vec{S})\)-Prikry poset is not far from being \((\Sigma, \vec{S})\)-Prikry on its own. The section concludes with a sufficient condition for exact forking projections to preserve suitability for reflection.

In Section 6, we revisit the functor \(A(\cdot, \cdot)\) from Part I of this series, improving the main result of [PRS19, §6]. Specifically, we prove that, for every \((\Sigma, \vec{S})\)-Prikry forcing \(P\) and every \(P\)-name \(\dot{T}\) for a fragile stationary set, the said functor produces a \((\Sigma, \vec{S})\)-Prikry forcing \(A(P, \dot{T})\) admitting an exact forking projection to \(P\) and killing the stationarity of \(\dot{T}\).

In Section 7, we improve one of the main result from Part II of this series, showing that, modulo necessary variations, the very same iteration scheme from [PRS20, §4] is also adequate for \((\Sigma, \vec{S})\)-Prikry forcings.

In Section 8, we present the primary application of our framework. The proof of the Main Theorem may be found there.

1.2. **Notation and conventions.** Our forcing convention is that \(p \leq q\) means that \(p\) extends \(q\). We write \(P \downarrow q\) for \(\{p \in P \mid p \leq q\}\). Denote \(E^\mu_\theta := \{\alpha < \mu \mid \text{cf}(\alpha) = \theta\}\). The sets \(E^{\mu}_{<\theta}\) and \(E^{\mu}_{\geq \theta}\) are defined in a similar fashion. For a stationary subset \(S\) of a regular uncountable cardinal \(\mu\), we write \(\text{Tr}(S) := \{\delta \in E^\mu_{<\omega} \mid S \cap \delta\text{ is stationary in }\delta\}\). \(H_\nu\) denotes the collection of all sets of hereditary cardinality less than \(\nu\). For every set of ordinals \(x\), we denote \(\text{cl}(x) := \{\sup(x \cap \gamma) \mid \gamma \in \text{Ord}, x \cap \gamma \neq \emptyset\}\), and \(\text{acc}(x) := \{\gamma \in x \mid \sup(x \cap \gamma) = \gamma > 0\}\). We write \(\text{CH}_\mu\) to denote \(2^\mu = \mu^+\) and \(\text{GCH}_{<\nu}\) as a shorthand for \(\text{CH}_\mu\) holds for every infinite cardinal \(\mu < \nu\).

For a sequence of maps \(\vec{\omega} = (\omega_n \mid n < \omega)\) and yet another map \(\pi\) such that \(\text{Im}(\pi) \subseteq \bigcap_{n < \omega} \text{dom}(\omega_n)\), we let \(\vec{\omega} \bullet \pi\) denote \((\omega_n \circ \pi \mid n < \omega)\).

2. **Nice projections and reflection**

**Definition 2.1.** For two notions of forcing \(P = (P, \leq)\) and \(S = (S, \leq)\) with maximal elements \(1_P\) and \(1_S\), respectively, we say that a map \(\omega : P \to S\) is a **nice projection from** \(P\) to \(S\) iff all of the following hold:

1. \(\omega(1_P) = 1_S\);
2. for any pair \(q \leq p\) of elements of \(P\), \(\omega(q) \leq \omega(p)\);
Then the first item above $q$ left with checking the converse inequality. Towards this first note that by $s$ for all $p \in P$ and $s \preceq \varpi(p)$.

The map $\varpi$ is said to be an exact nice projection if, in addition, $\varpi(p + s) = s$ for all $p \in P$ and $s \preceq \varpi(p)$.

**Example 2.2.** If $P$ is a product of the form $S \times T$, then the map $(s, t) \mapsto s$ forms an exact nice projection from $P$ to $S$.

Note that the composition of (resp. exact) nice projections is again (resp. exact) nice projection.

**Definition 2.3.** Let $P = (P, \preceq)$ and $S = (S, \preceq)$ be two notions of forcing and $\varpi : P \to S$ be a nice projection. For an $S$-generic filter $H$, we define the quotient forcing $P/H := (P/H, \preceq_{P/H})$ as follows:

- $P/H := \{ p \in P \mid \varpi(p) \in H \}$;
- for all $p, q \in P/H$, $q \preceq_{P/H} p$ iff there is $s \in H$ with $s \preceq \varpi(q)$ such that $q + s \preceq p$.

**Remark 2.4.** In a slight abuse of notation, we tend to write $P/S$ when referring to a quotient as above, without specifying the generic for $S$ or the map $\varpi$. By standard arguments, $P$ is isomorphic to a dense subposet of $S \ast P/S$ (see [Abr10, p. 337]).

**Definition 2.5.** Given a poset $P = (P, \preceq)$ and a map $\varpi$ with $\dom(\varpi) \supseteq P$, we derive a poset $P^{\varpi} := (P, \preceq^{\varpi})$ by letting

\[ p \preceq^{\varpi} q \text{ iff } ((p \preceq q) \text{ and } (\varpi(p) = \varpi(q))). \]

**Lemma 2.6.** Suppose that we are given $P, S$ and a nice projection $\varpi$ as in Definition 2.1. Let $p \in P$ and $s \preceq \varpi(p)$.

1. For every $q \preceq p + s$, there is $p' \leq^{\varpi} p$ such that $q = p' + \varpi(q)$;
2. For $s = \varpi(p)$, the map $(p', s') \mapsto p' + s'$ forms a projection from $(P^{\varpi} \downarrow p) \times (S \downarrow s)$ onto $P \downarrow p$.

**Proof.** (1) Let $q \preceq p + s$. Following Remark 2.4, we can write $p$ and $q$ as pairs $(s_p, i_p)$ and $(s_q, i_q)$. Now define an element $p' = (s_p, i_{p'})$ in $P$, where $i_{p'}$ is an $S$-name for a condition in the quotient such that the following are true:

- $s_q \Vdash_S i_{p'} = i_q$, and
- $s_p \Vdash_S i_{p'} \leq^{\varpi}_{S} i_p$.

(For example we can ensure that any condition below $s_p$ incompatible with $s_q$, forces that $i_{p'} = i_p$).

Let us verify that $p' + \varpi(q) = q$. Clearly, $q \preceq p' + \varpi(q)$, and so we are left with checking the converse inequality. Towards this first note that by the first item above $q \preceq p'$. Now suppose $u = (s_u, i_u) \leq p'$ with $s_u \preceq \varpi(q)$. Then $s_u \preceq s_q$ and so it forces “$i_u \leq i_{p'} = i_q$,” hence $u \preceq q$. By definition of $p' + \varpi(q)$, it follows that $p' + \varpi(p') \preceq q$, as wanted.

---

By convention, a greatest element, if exists, is unique.
Therefore, \( p \) 

**Proof.** By hypothesis, \( p \) such that

\[
\text{Claim 2.8.1.}
\]

For notational convenience, denote \( p \) always the case, for instance, in iterated forcing.

\[ 6 \] ALEJANDRO POVEDA, ASSAF RINOT, AND DIMA SINAPOVA

\[ \begin{align*}
\text{Evidently, } Q'' & \text{ is a subset of } Q'. \text{ Thus, the greatest element of } Q'' \text{ is } \leq \text{ the greatest element of } Q'. \text{ That is, } p'' + s'' \leq p' + s', \text{ as desired.} \quad \square
\end{align*} \]

**Lemma 2.7.** Suppose that \( \varpi : \mathbb{P} \to \mathbb{S} \) is a nice projection. Let \( p \in \mathbb{P} \) and set \( S := \varpi(p) \). For any condition \( a \in \mathbb{S} \) such that \( \varpi(a) \leq a \) for \( a \in \mathbb{S} \) are as in the above lemma. Let \( G \) be \( \mathbb{P} \)-generic with \( p \in G \).

Next, let \( H \times G^* \) be \((\mathbb{S} \downarrow s) \times (\mathbb{P} \downarrow p))\)-generic over \( V[G] \). For each \( a \in H \) with \( a \leq s \), let \( G_a \) be the \((\mathbb{P} \downarrow p), \leq_a\)-generic filter obtained from \( G^* \). Then:

- For any \( a \in H \), \( V[G] \subseteq V[H \times G_a] \subseteq V[H \times G^*] \), and \( G \supseteq G_a \supseteq G^* \);
- For any pair \( a' \leq a \) of elements of \( H \), \( V[H \times G_{a'}] \subseteq V[H \times G_a] \), and \( G_{a'} \supseteq G_a \);
- \( G^* = \bigcap_{a \in H} G_a \), and
- \( G \cap \mathbb{P}^* = \bigcup_{a \in H} G_a \).

**Proof.** For notational convenience, denote \( \mathbb{P}^* := (\mathbb{P} \downarrow p, \leq_{\mathbb{P}}) \) and \( \mathbb{S}^* := (\mathbb{S} \downarrow s, \subseteq) \). Note that \( G^* \subseteq \mathbb{P}^* \cap G \).

The first two items follow from Lemma 2.7. For the others, we will use the following claim:

**Claim 2.8.1.** Let \((a_0, p_0), (a_1, p_1) \in (\mathbb{S} \times \mathbb{P}^*) / G \). Then there exists \( a \in H \) such that \( p_0 \) and \( p_1 \) are \( \leq_a \)-compatible.

**Proof.** By hypothesis, \( p_0 + a_0, p_1 + a_1 \in G \). Fix \( q \in G \) such that \( q \leq p_i + a_i \) for every \( i < 2 \). Let \( a := \varpi(q) \). Then, for every \( i < 2 \), \( q + a = q \leq p_i + a_i \). Therefore, \( p_0 \) and \( p_1 \) are \( \leq_a \)-compatible.

\[ \square \]

\[ 4 \] Strictly speaking, \( \leq_a \) is reflexive and transitive, but not asymmetric. But this is also always the case, for instance, in iterated forcing.
Now, for the third item, suppose that \( r^* \in \bigcap_{a \in H} G_a \), but \( r^* \notin G^* \). In \( V[H] \), consider \( E := \{(a, r) \in (S^* \times P^*)/H \mid r \downarrow_a r^* \vee r \leq_{P/H} r^* \} \). This is dense, so let \( (a, r) \in E \cap (H \times G^*) \). By assumption, we cannot have \( r \leq r^* \), because that would mean that \( r \leq_{P/H} r^* \) and we have supposed that \( r^* \notin G^* \).

So, for some \( a \in H, r \downarrow a r^* \). But since \( r \in G^* \subseteq G_a \), that is a contradiction with \( r^* \in G_a \).

For the last statement, suppose that \( r^* \in G \cap P^* \), and consider \( D := \{(a, r) \in (S^* \times P^*)/H \mid r \leq a r^* \vee r \downarrow_{P/H} r^* \} \). \( D \) is a dense set in \( S^* \times P^*/H \), so let \( (a, r) \in D \cap (H \times G^*) \). We claim that \( r^* \in G_a \). That is because, since both \( r, r^* \in G \), it must be that \( r \leq a r^* \). And since \( r \in G^* \subseteq G_a \), we get that \( r^* \in G_a \).

\[ \text{Claim 2.9.1.} \quad \text{Let } V[G] \models T \subseteq E_{< \delta}^\kappa \text{ is a stationary set, but that } T \text{ is nonstationary in } V[H \times G^*]. \]

\[ \text{Proof.} \quad \text{Since } \langle P^*/G \rangle_a \text{ is } \delta \text{-closed, by Shelah’s theorem [She79], for all } a \in H, \langle P^*/G \rangle_a \text{ preserves stationary subsets of } E_{< \delta}^\kappa. \text{ Now, if } T \text{ was stationary is } V[H][G_a] \text{, then since } |S| < \kappa, \text{ for some } b \in H, \]

\[ T' := \{ \alpha < \kappa \mid (\exists r \in G_a) (a, r) \models_{S^* \times P_a} \alpha \in \check{T} \} \]

is a stationary set lying in \( V[G_a] \). So, \( T' \) will remain stationary in \( V[G^*] \). Since \( |S| < \kappa \), \( T' \) is also stationary in \( V[H \times G^*] \). This is in contradiction with \( T' \subseteq T \) and our assumption that \( T \) is nonstationary in \( V[H \times G^*] \). \( \Box \)

Then for every \( a \in H \), let \( C_a \) be a club in \( V[H \times G_a] \), disjoint from \( T \). Since \( S \) is a small forcing, we may assume that \( C_a \subseteq V[G_a] \). Let \( \check{C}_a \) be a \( P_a \)-name for this club such that

- \( p \Vdash_{P_a} \check{C}_a \) is a club, and
- \( (a, p) \Vdash_{(S^* \times P_a)/G} \check{C}_a \cap T = \emptyset. \)

(Here we identify \( \check{C}_a \) with a \( (S \times P_a)/G \)-name in the natural way.)

Since \( S \) is a small forcing, we may fix some \( a \in H \), such that

\[ T_a := \{ \alpha < \kappa \mid \exists r \in P[\check{\varpi}(r) = a \& r \Vdash \check{a} \in \check{T}] \} \]

is stationary. Note that \( T_a \in V \).
Claim 2.9.2. There is a condition $p^* \leq_a p$ and an ordinal $\gamma < \kappa$ such that $(a, p^*) \Vdash_{\mathcal{S}^* \times P_a} \gamma \in \mathcal{C}_a \cap \check{T}$.

Proof. This is a standard argument (cf. [She79]), but we spell it out. Work in $V$. Since $T_a$ is in $I[\kappa]$ and is a stationary set consisting of singular ordinals, we may find a continuous $\mathcal{E}$-chain of elementary submodels of $H_\theta$ (for a large enough regular cardinal $\theta$), $(M_i \mid i \leq \chi)$ such that:

- $M_0$ contains all the above objects under discussion, including $\check{C}_a$;
- for every $i \leq \chi$, $\gamma_i := M_i \cap \kappa$ is in $\kappa$;
- $\gamma_\chi \in T_a$.

Write $\gamma := \gamma_\chi$. As $\gamma \in T_a \subseteq T \subseteq E^\kappa_{\leq \delta}$, $\text{cf}(\gamma) = \chi < \delta$. Also, since $\gamma \in T_a$, we may fix some $r \in P$ with $\check{\varphi}(r) = a$ such that $r \Vdash \check{\gamma} \in \check{T}$. Let $r^*$ be a condition in $P^*$ such that $r^* + a = r$. Now, construct a $\leq^\mathcal{S}$-decreasing sequence of conditions in $P^*$ below $r^*$, $(p_i \mid i \leq \chi)$, such that, for all $i < \chi$:

- $p_i \in N_{i+1}$, and
- for some ordinal $\alpha \in N_i \setminus N_i$, $p_i + a \Vdash P \check{\alpha} \in \check{C}_a$.

Let $p^* := p_\chi$, so that $p^* + a \Vdash P \check{\gamma} \in \check{C}_a$. As $p^* + a \leq r^* + a = r$, $p^* \leq_a r^* \leq_a r$ and so $p^* \Vdash_{P_a} \check{\gamma} \in \check{T}$. Then $(a, p^*) \Vdash_{\mathcal{S}^* \times P_a} \check{\gamma} \in \check{T} \cap \check{C}_a$. \qed

Since the above argument can be carried out below any condition in $P^*$, by density, we get a contradiction. \qed

Definition 2.10. For stationary subsets $\Delta, \Gamma$ of a regular uncountable cardinal $\mu$, $\text{Refl}(\Delta, \Gamma)$ asserts that for every stationary subset $T \subseteq \Delta$, there exists $\gamma \in \Gamma \cap E^\mu_{\leq \omega}$ such that $T \cap \gamma$ is stationary in $\gamma$.

We end this section by establishing a sufficient condition for $\text{Refl}(\ldots)$ to hold in generic extensions; this will play a crucial role at the end of Section 5.

Definition 2.11. For infinite cardinals $\tau < \sigma < \kappa < \mu$, we say that $(P, S, \varphi)$ is suitable for reflection with respect to $(\tau, \sigma, \kappa, \mu)$ if all the following hold:

1. $P$ and $S$ are nontrivial notions of forcing;
2. $\varphi : P \rightarrow S$ is a nice projection and $P^{\varphi}$ is $\sigma$-directed-closed;
3. In any forcing extension by $P$ or $S \times P^{\varphi}$, $|\mu| = \text{cf}(\mu) = \kappa = \sigma^+$;
4. For any $s \in S \setminus \{\check{1}_S\}$, there is a cardinal $\delta$ with $\tau^+ < \delta < \sigma$, such that $S \downarrow s \cong Q \times \text{Col}(\delta, <\sigma)$ for some notion of forcing $Q$ of size $\delta$.

Lemma 2.12. Let $(P, S, \varphi)$ be suitable for reflection with respect to $(\tau, \sigma, \kappa, \mu)$. Suppose $\sigma$ is a supercompact cardinal indestructible under forcing with $P^{\varphi}$. Then $V^P \models \text{Refl}(E^\mu_{\leq \tau}, E^\tau_{< \sigma^+})$.

Proof. By Definition 2.11(3), it suffices to prove that $V^P \models \text{Refl}(E^\kappa_{\leq \tau}, E^\kappa_{< \sigma^+})$.

Let $G$ be $P$-generic. In $V[G]$, let $T$ be a stationary subset of $E^\kappa_{\leq \tau}$. Suppose for simplicity that this is forced by the empty condition.

Claim 2.12.1. There exists $p \in G$, such that, setting $s := \varphi(p)$, the quotient $(\langle S \downarrow s \rangle \times (P^{\varphi} \downarrow p))/G$ preserves the stationarity of $T$.  

Proof. Using Clauses (1) and (2) of Definition 2.11, let us pick any $\bar{p} \in G$ for which $\bar{s} := \pi(\bar{p})$ strictly extends $\mathbb{I}_\sigma$. Back in $V$, using Clause (4) of Definition 2.11, fix a cardinal $\delta$ with $\tau^+ < \delta < \sigma$, a notion of forcing $\mathbb{Q}$ of size $< \delta$, and an isomorphism $\iota$ from $\mathbb{S} \downarrow s$ to $\mathbb{Q} \times \text{Col}(\delta, < \sigma)$. Let $\iota_0, \iota_1$ denote the unique maps to satisfy $\iota(s') = (\iota_0(s'), \iota_1(s'))$. By Example 2.2, $\iota_0$ and $\iota_1$ are exact nice projections. Set $\pi := (\iota_0 \circ \pi) \upharpoonright (\mathbb{P} \downarrow \bar{p})$, so that $\pi$ is a nice projection from $\mathbb{P} \downarrow \bar{p}$ to $\mathbb{Q}$.

Since $\mathbb{Q}$ is a poset of size $< \delta < \kappa$, in $V[G]$, we may now find some condition $q$ in $\mathbb{Q}$, such that

$$ T_q := \{ \alpha < \kappa \mid \exists p \in G(p \subseteq \bar{p} \land \pi(p) = q \land p \Vdash \bar{\alpha} \in \bar{T} \} $$

is stationary. By possibly shrinking $T$, we may assume that $T = T_q$. Fix $p \in G$ such that $p \subseteq \bar{p}$ and $\pi(p) = q$. We will show that $p$ is as desired.

**Subclaim 2.12.1.1.** Set $s := \varpi(p)$. Then:

(i) $(\mathbb{S} \downarrow s) \times (\mathbb{P}^{\sigma} \downarrow p)$ projects onto $(\mathbb{Q} \downarrow q) \times (\mathbb{P}^{\pi} \downarrow p)$ that projects onto $\mathbb{P} \downarrow p$;

(ii) $(\mathbb{S} \downarrow s) \times (\mathbb{P}^{\sigma} \downarrow p)$ projects onto $(\mathbb{Q} \downarrow q) \times (\mathbb{P}^{\pi} \downarrow p)$ that projects onto $\mathbb{P}^{\pi} \downarrow p$;

(iii) $(\mathbb{S} \downarrow s) \times (\mathbb{P}^{\sigma} \downarrow p)$ projects onto $\text{Col}(\delta, < \sigma) \times (\mathbb{P}^{\sigma} \downarrow p)$ that projects onto $\mathbb{P}^{\pi} \downarrow p$.

*Proof.* (i) For the first part, the map $(s', p') \mapsto (\iota_0(s'), p')$ is such a projection, since $\pi$ factors through $\varpi$. For the second part, we appeal to Lemma 2.6(1), since $p$ is a condition in $\mathbb{P} \downarrow \bar{p}$, $q = \pi(p)$ and $\pi$ is a nice projection from $\mathbb{P} \downarrow \bar{p}$ to $\mathbb{Q}$.

(ii) For the second part, the map $(q', p') \mapsto p'$ is such a projection.

(iii) For the first part, the map $(s', p') \mapsto (\iota_1(s'), p')$ is such a projection. For the second part, the map $(q', p') \mapsto p'$ is such a projection, since $\pi$ factors through $\varpi$.

By Definition 2.11(3), in all forcing extensions with posets from Clause (i), $\kappa$ is a cardinal which is the double successor of $\sigma$. But then, since $\mathbb{Q}$ is small, it follows from the second part of Clause (ii) that $\kappa$ is the double successor of $\sigma$ in forcing extensions by $\mathbb{P}^{\pi} \downarrow p$. Altogether, in all forcing extensions with posets from the preceding subclaim, $\kappa$ is the double successor of $\sigma$.

Next, let $G_q \times G^*$ be $((\mathbb{Q} \downarrow q) \times (\mathbb{P}^{\pi} \downarrow p))/G$-generic over $V[G]$. By Lemma 2.9, $T$ remains stationary in $V[G_q \times G^*]$. As $\mathbb{Q}$ is small, we may fix $T' \subseteq T$ such that $T'$ is in $V[G^*]$ and moreover stationary in $V[G^*]$. As established earlier, $V[G^*] \models \kappa = \sigma^{++}$. So, $T'$ is a stationary subset of $E_{\sigma^{++}}^{<\tau} \subseteq E_{\sigma^{++}}^{<\tau}$. By [She91, Lemma 4.4], $E_{\sigma^{++}}^{<\tau} \in I[\sigma^{++}]^{\mathbb{V}[G^*]}$, and thus $T' \subseteq I[\sigma^{++}]^{\mathbb{V}[G^*]}$. In $V$, by Definition 2.11(2), $\mathbb{P}^{\sigma}$ is $\delta$-closed, and hence also $\mathbb{P}^{\pi}$ is $\delta$-closed, so that, in $V[G^*]$, the quotient $(\text{Col}(\delta, < \sigma) \times (\mathbb{P}^{\sigma} \downarrow p))/G^*$ is $\delta$-closed. It follows that $(\text{Col}(\delta, < \sigma) \times (\mathbb{P}^{\sigma} \downarrow p))/G^*$ preserves the stationarity of $T'$. 

Finally, since $S \downarrow s \cong \mathbb{Q} \times \text{Col}(\delta, <\sigma)$, the quotient
\[(S \downarrow s) \times (\mathbb{P}^\omega \downarrow p))/((\text{Col}(\delta, <\sigma) \times (\mathbb{P}^\omega \downarrow p))
\]
is isomorphic to $\mathbb{Q}$ which is a small of forcing. So $T'$ (and hence also $T$) remain stationary in the extension by $(S \downarrow s) \times (\mathbb{P}^\omega \downarrow p)$.

Let $p \in G$ be given by the claim. Let $s := \varpi(p)$, and let $H$ be the generic filter for $S$ induced by $\varpi$. Let $G^*$ be a generic filter for $\mathbb{P}^\omega$ such that $p \in G^*$ and $H \times G^*$ is generic for $((S \downarrow s) \times (\mathbb{P}^\omega \downarrow p))/G$. By the choice of $p$, $T$ is still stationary in $V[G^*][H]$. Also, by Definition 2.11(3), $T \subseteq (E^\omega_{\leq \tau} V[G^*][H]$. Using that $\sigma$ is a supercompact indestructible under $\mathbb{P}^\omega$, let (in $V[G^*]$)

\[j : V[G^*] \rightarrow M\]

be a $\kappa$-supercompact embedding with $\text{crit}(j) = \sigma$. We shall want to lift this embedding to $V[G^*][H]$.

Work below the condition $s$ that we fixed earlier. Recall that $S \downarrow s \cong \mathbb{Q} \times \text{Col}(\delta, <\sigma)$ for some poset $\mathbb{Q}$ of size $< \delta$ with $\tau^+ < \delta < \sigma$. So, $H$ may be seen as a product of two corresponding generics, $H = H_0 \times H_1$. For the ease of notation, denote $C := \text{Col}(\delta, <\sigma)$.

Since $\mathbb{Q}$ has size $< \delta < \text{crit}(j)$, we can lift $j$ to an embedding

\[j : V[G^*][H_0] \rightarrow M'.\]

Then we lift $j$ again to get

\[j : V[G^*][H] \rightarrow N\]
in an outer generic extension of $V[G^*][H]$ by $j(\mathbb{C})/H_1$. Since $j(\mathbb{C})/H_1$ is $\tau^+$-closed in $M''[H_1]$ and this latter is closed under $\kappa$-sequences in $V[G^*][H]$, $j(\mathbb{C})/H_1$ is $\tau^+$-closed in $V[G^*][H]$.

Set $\gamma := \sup(j''\kappa)$. Evidently, $j(T) \cap \gamma = j''\gamma$. Note that, by virtue of the collapse $j(\mathbb{C})$, $N \models |\kappa| = \delta$ & $\text{cf}(\gamma) \leq \text{cf}(|\kappa|) = \delta$.

On one hand, by [She91] and Definition 2.11(3),

\[(E_{\leq \tau}^\sigma V[G^*][H]) \subseteq (E_{<\sigma^+}^\tau V[G^*][H]) \in I[\sigma^+] V[G^*][H].\]

On the other hand, by [She79] and since $j(\mathbb{C})/H_1$ is $\tau^+$-closed in $V[G^*][H]$, it follows that $j(\mathbb{C})/H_1$ preserves stationary subsets of $(E_{<\sigma^+}^\tau V[G^*][H])$, hence it preserves the stationarity of $T$. A standard argument now shows that this implies that $j(T) \cap \gamma$ is stationary in $N$. Thus, $N \models \exists \alpha \in E_{<\gamma}^{j(\kappa)} (j(T) \cap \alpha \text{ is stationary in } \alpha)''$.

So, by elementarity, in $V[G^*][H]$, $T$ reflects at a point of cofinality $< \sigma^+$. Since reflection is downwards absolute, it follows that $T$ reflects at a point of cofinality $< \sigma^+$ in $V[G]$, as wanted.

---

5Actually, at a point of cofinality $< \sigma$. 

3. $(\Sigma, \mathcal{S})$-Prikry forcings

We commence by recalling a few concepts from [PRS19, §2].

**Definition 3.1.** A graded poset is a pair $(\mathbb{P}, \ell)$ such that $\mathbb{P} = (P, \leq)$ is a poset, $\ell : P \to \omega$ is a surjection, and, for all $p \in P$:

- For every $q \leq p$, $\ell(q) \geq \ell(p)$;
- There exists $q \leq p$ with $\ell(q) = \ell(p) + 1$.

**Convention 3.2.** For a graded poset as above, we denote $P_n := \{ p \in P \mid \ell(p) = n \}$ and $\mathbb{P}_n := (P_n \cup \{1\}, \leq)$. $\mathbb{P}_{\geq n}$ and $\mathbb{P}_{> n}$ are defined analogously.

We also write $P^n := \{ q \in P \mid q \leq p, \ell(q) = \ell(p) + n \}$, and sometimes write $q \leq^np$ (and say the $q$ is an $n$-step extension of $p$) rather than writing $q \in P^n$.

A subset $U \subseteq P$ is said to be 0-open set iff, for all $r \in U$, $P^0_r \subseteq U$.

Now, we define the $(\Sigma, \mathcal{S})$-Prikry class, a class broader than $\Sigma$-Prikry from [PRS19, Definition 2.3].

**Definition 3.3.** Suppose:

$(\alpha)$ $\Sigma = \{ \sigma_n \mid n < \omega \}$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$;

$(\beta)$ $\mathcal{S} = \{ S_n \mid n < \omega \}$ is a sequence of notions of forcing, $S_n = (S_n, \leq_n)$, with $|S_n| < \sigma_n$;

$(\gamma)$ $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $1$;

$(\delta)$ $\mu$ is a cardinal such that $1 \models P \mu = \check{\kappa}^+$;

$(\varepsilon)$ $\ell : P \to \omega$ and $c : P \to \mu$ are functions;\(^6\)

$(\zeta)$ $\varpi = \{ \varpi_n \mid n < \omega \}$ is a sequence of functions.

We say that $(\mathbb{P}, \ell, c, \varpi)$ is $(\Sigma, \mathcal{S})$-Prikry iff all of the following hold:

$(1)$ $(\mathbb{P}, \ell)$ is a graded poset;

$(2)$ For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{1\}, \leq)$ is $\aleph_1$-closed;

$(3)$ For all $p, q \in P$, if $c(p) = c(q)$, then $P^p_0 \cap P^q_0$ is non-empty;

$(4)$ For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{ r \leq^n p \mid q \leq^m r \}$ contains a greatest element which we denote by $m(p, q)$.\(^7\) In the special case $m = 0$, we shall write $w(p, q)$ rather than $0(p, q)$;\(^8\)

$(5)$ For all $p \in P$, the set $W(p) := \{ w(p, q) \mid q \leq p \}$ has size $\mu$;

$(6)$ For all $p' \leq p$ in $P$, $q \mapsto w(p, q)$ forms an order-preserving map from $W(p')$ to $W(p)$;

$(7)$ Suppose that $U \subseteq P$ is a 0-open set. Then, for all $p \in P$ and $n < \omega$, there is $q \leq^0 p$, such that, either $P^q_0 \cap U = \emptyset$ or $P^n_0 \subseteq U$;

$(8)$ For all $n < \omega$, $\varpi_n$ is an exact nice projection from $\mathbb{P}_{\geq n}$ to $S_n$, such that, for any integer $k \geq n$, $\varpi_n | \mathbb{P}_k$ is again an exact nice projection;

---

\(^6\)In some applications $c$ will be a function from $P$ to some canonical structure of size $\mu$, such as $H_\mu$ (assuming $\mu^{<\mu} = \mu$).

\(^7\)By convention, a greatest element, if exists, is unique.

\(^8\)Note that $w(p, q)$ is the weakest extension of $p$ above $q$. 
Convention 3.4. We derive yet another ordering $\leq\overset{\sigma}{\sim}$ of the set $P$, letting $\leq\overset{\sigma}{\sim} := \bigcup_{n<\omega} \leq\overset{\omega}{\sim}$. Simply put, this means that $q \leq\overset{\sigma}{\sim} p$ iff $(p = \mathbb{1})$ or, $(q \leq^0 p$, $\ell(p) = \ell(q)$ and $\varpi_\ell(p) = \varpi_\ell(q))$.

Convention 3.5. We say that $(P, \ell, c)$ has the Linked$_0$-property if it witnesses Clause (3) above. Similarly, we will say that $(P, \ell)$ has the Complete Prikry Property (CPP) if it witnesses Clause (7).

Any $\Sigma$-Prikry triple $(P, \ell, c)$ can be regarded as a $(\Sigma, \vec{S})$-Prikry forcing $(\mathbb{P}, \ell, c, \vec{S})$ by letting $\vec{S} := \langle (n, \{1_p\}) \mid n < \omega \rangle$ and $\vec{S}$ be the sequence of trivial projections $p \mapsto 1_p$. In particular, all the forcings from [PRS19, §3] are examples of $(\Sigma, \vec{S})$-Prikry forcings. In Section 4, we will add a new example to this list by showing that Gitik's EPBFC (The long Extender-Based Prikry forcing with Collapses [Git19b]) is $(\Sigma, \vec{S})$-Prikry.

Throughout the rest of the section, assume that $(\mathbb{P}, \ell, c, \vec{S})$ is a $(\Sigma, \vec{S})$-Prikry quadruple. We shall spell out some basic features of the components of the quadruple, and work towards proving Lemma 3.12 that explains how bounded sets of $\kappa$ are added to generic extensions by $\mathbb{P}$.

Lemma 3.6 (The $p$-tree). Let $p \in P$.

1. For every $n < \omega$, $W_n(p)$ is a maximal antichain in $\mathbb{P} \downarrow p$;
2. Every two compatible elements of $W(p)$ are comparable;
3. For any pair $q \leq q' \in W(p)$, $q' \in W(q)$;
4. $c \upharpoonright W(p)$ is injective.

Proof. The proof of [PRS19, Lemma 2.8] goes through. \hfill \Box

Proposition 3.7. For every condition $p$ in $\mathbb{P}$ and an ordinal $\alpha < \kappa$, there exists an extension $p' \leq p$ such that $\sigma_{\ell(p')} > \alpha$.

Proof. Let $p$ and $\alpha$ be as above. Since $\alpha < \kappa = \sup_{n<\omega} \sigma_n$, we may find some $n < \omega$ such that $\alpha < \sigma_n$. By Definition 3.3(1), $(\mathbb{P}, \ell)$ is a graded poset, so by possibly iterating the second bullet of Definition 3.1 finitely many times, we may find an extension $p' \leq p$ such that $\ell(p') \geq n$. As $\Sigma$ is non-decreasing, $p'$ is as desired. \hfill \Box

Proposition 3.8. Let $n < \omega$. The poset $\mathcal{S}_n$ is $\aleph_1$-closed.

Proof. Let $\langle s_m \mid m < \omega \rangle$ be a $\leq_n$-decreasing sequence of conditions in $\mathcal{S}_n$. For each $m < \omega$, set $p_m := 1+s_m$. Observe that this is well-defined as $s_m \leq_n \varpi_n(1) = 1_{s_n}$. Clearly, $\langle p_m \mid m < \omega \rangle$ is a $\leq$-decreasing sequence of conditions in $\mathbb{P}_n$ so that appealing to Definition 3.3(2) we find a lower bound $q \in P_n$. Evidently, $\varpi_n(q)$ forms a $\leq_n$-lower bound for our original sequence $\langle s_m \mid m < \omega \rangle$. \hfill \Box

The next proposition provides a very useful strengthening of the CPP:
Lemma 3.9. Suppose that $U \subseteq P$ is a 0-open set. Then, for all $p \in P$, $n < \omega$, and $s \leq_{\ell(p)} \varpi_{\ell(p)}(p)$, there are $q \leq_{\omega} p$ and $t \leq_{\ell(p)} s$, such that, either $P_n^{q+t} \cap U = \emptyset$ or $P_n^{q+t} \subseteq U$.

Proof. Let $p \in P$, $n < \omega$, and $s \leq_{\ell(p)} \varpi_{\ell(p)}(p)$, be arbitrary. Appealing to Definition 3.3(7) of the current corollary from our previous work, as follows.

First, note that the proof of Lemma 2.6 from Part I (that is, [PRS19])

Proof. Let $n < \omega$, and $s \leq_{\ell(p)} \varpi_{\ell(p)}(p)$, be arbitrary. Appealing to Definition 3.3(7) with the condition $p' := p+s$, one finds a condition $q' \leq_{0} p'$ such that, either $P_n^q \subseteq U$ or $P_n^q \cap U = \emptyset$. By Lemma 2.6(1), $q' = q + \varpi_{\ell(p)}(q')$ for some $q \leq_{\omega} p$. Clearly, $\ell(q') = \ell(p)$ and $\varpi_{\ell(p)}(q') \leq_{\ell(p)} s$, so that $q$ and $t := \varpi_{\ell(p)}(q')$ are as desired. \hfill \□

As in the context of $\Sigma$-Prikry forcings, also here, the CPP implies the Prikry Property (PP) and the Strong Prikry Property (SPP).

Corollary 3.10. Let $p \in P$ and $s \leq_{\ell(p)} \varpi_{\ell(p)}(p)$.

(1) Suppose $\varphi$ is a sentence in the language of forcing. Then there is $p' \leq_{\omega} p$ and $s' \leq_{\ell(p)} s$ such that $p' + s'$ decides $\varphi$;

(2) Suppose $D \subseteq P$ is a 0-open set which is dense below $p$. Then there are $p' \leq_{\omega} p$, $s' \leq_{\ell(p)} s$ and $n < \omega$ such that $P_n^{p'+s'} \subseteq D$.\footnote{Note that if $D$ is moreover open, then $P_n^{q+t} \subseteq D$ for all $m \geq n$.}

Proof. First, note that the proof of Lemma 2.6 from Part I (that is, [PRS19]) goes through provided that the cardinal $\theta$ there satisfies $\log(\theta) < \omega_1$. The proof of Corollary 2.7(1) there relies on $\theta = 3$, and the proof Corollary 2.7(2) there relies on $\theta = 2$, hence, they go through as well. Thus, we can derive the current corollary from our previous work, as follows.

(1) By Corollary 2.7(1) of Part I, let us fix $q \leq_{0} (p + s)$ that decides $\varphi$. By Lemma 2.6(1), then, there is $p' \leq_{\omega} p$ such that $q = p' + \varpi(q)$. So $p'$ and $s' := \varpi_{\ell(p)}(q)$ are as sought.

(2) By Corollary 2.7(2) of Part I, let us fix $q \leq_{0} (p + s)$ and $n < \omega$ such that $P_n^q \subseteq D$. By Lemma 2.6(1), then, there is $p' \leq_{\omega} p$ such that $q = p' + \varpi(q)$. So $p'$ and $s' := \varpi_{\ell(p)}(q)$ are as sought. \hfill \□

Working a bit more we can obtain the following:

Lemma 3.11. Let $p \in P$. Set $n := \ell(p)$ and $s := \varpi_n(p)$.

(1) Suppose $\varphi$ is a sentence in the language of forcing. Then there is $q \leq_{\omega} p$ such that $D_{\varphi,q} := \{t \leq_n s \mid (q + t \models_p \varphi) \text{ or } (q + t \not\models_p \neg\varphi)\}$ is dense in $S_n \downarrow s$;

(2) Suppose $D \subseteq P$ is a 0-open set which is dense below $p$. Then there is $q \leq_{\omega} p$ such that $U_{D,q} := \{t \leq_n s \mid \exists m < \omega \ P_{m+n}^{q+t} \subseteq D\}$ is dense in $S_n \downarrow s$.

Proof. (1) By Definition 3.3(\(^\beta\)), let us fix some cardinal $\theta < \sigma_n$ along with an injective enumeration $\langle s_{\alpha} \mid \alpha < \theta \rangle$ of the conditions in $S_n \downarrow s$, such that $s_0 = s$. We will construct by recursion two sequences of conditions $\vec{p} = \langle p^\alpha \mid \alpha < \theta \rangle$ and $\vec{s} = \langle s^\alpha \mid \alpha < \theta \rangle$ for which all of the following hold:
Proposition 3.7. Thus, let us suppose that \( p \).

\[ \text{Proof.} \]

\[ \text{Claim 3.11.1. There are sequences } \vec{p} \text{ and } \vec{s} \text{ as above.} \]

\[ \text{Proof.} \]

\[ \text{We construct the two sequences by recursion on } \alpha < \theta. \text{ For the base case, appeal to Corollary 3.10(1) with } p \text{ and } s, \text{ and retrieve } p^0 \leq \pi p \text{ and } s^0 \leq \pi s \text{ such that } p^0 + s^0 \text{ indeed decides } \varphi. \]

\[ \text{Assume } \alpha = \beta + 1 \text{ and that } \langle p^\gamma \mid \gamma \leq \beta \rangle \text{ and } \langle s^\gamma \mid \gamma \leq \beta \rangle \text{ have been already defined. Since } s_\alpha \leq \pi s = \pi_n(p^\beta), \text{ it follows that } p^\beta + s_\alpha \text{ is a legitimate condition in } P_n. \text{ Appealing to Corollary 3.10(1) with } p^\beta \text{ and } s_\alpha, \text{ we retrieve } p^\alpha \leq \pi p^\beta \text{ and } s^\alpha \leq \pi s_\alpha \text{ such that the condition } p^\alpha + s^\alpha \text{ decides } \varphi. \]

\[ \text{Assume } \alpha \in \text{acc}(\theta) \text{ and that the sequences } \langle p^\beta \mid \beta < \alpha \rangle \text{ and } \langle s^\beta \mid \beta < \alpha \rangle \text{ have already been defined. Appealing to Definition 3.3(9), we may let } p^* \text{ be a } \leq \pi \text{-lower bound for } \langle p^\beta \mid \beta < \alpha \rangle. \text{ Finally, obtain } p^\alpha \text{ and } s^\alpha \text{ by appealing to Corollary 3.10(1) with } p^* \text{ and } s_\alpha. \]

This completes the proof of Clause (1). For the proof of Clause (2) one mimics the above argument but appealing to Corollary 3.10(2) rather than Corollary 3.10(1).

We now arrive at the main result of the section:

\[ \text{Lemma 3.12 (Analysis of bounded sets).} \]

\[ (1) \text{ If } p \in P \text{ forces that } \dot{a} \text{ is a } \mathbb{P}\text{-name for a bounded subset } a \text{ of } \sigma_{\ell(p)}, \text{ then } a \text{ is added by } \mathbb{S}_{\ell(p)}. \text{ In particular, If } \dot{a} \text{ is a } \mathbb{P}\text{-name for a bounded subset } a \text{ of } \kappa, \text{ then, for any large enough } n < \omega, a \text{ is added by } \mathbb{S}_n; \]

\[ (2) \mathbb{P} \text{ preserves } \kappa. \text{ Moreover, if } \kappa \text{ is a strong limit, it remains so;} \]

\[ (3) \text{For every regular cardinal } \nu \geq \kappa, \text{ if there exists } p \in P \text{ for which } p \forces \text{cf}(\nu) < \kappa, \text{ then there exists } q \leq \pi p \text{ with } |W(q)| \geq \nu;^1 \]

\[ (4) \text{Suppose } 1 \forces \text{``} \kappa \text{ is singular}. \text{ Then } \mu = \kappa^+ \text{ if and only if, for all } p \in P, |W(p)| \leq \kappa. \]

\[ \text{Proof.} \]

\[ (1) \text{The } \text{“in particular” part follows from the first part together with Proposition 3.7. Thus, let us suppose that } p \text{ is a given condition forcing that } \dot{a} \text{ is a name for a subset } a \text{ of some cardinal } \theta < \sigma_{\ell(p)}. \]

\[ ^1 \text{For future reference, we point out that this fact relies only on clauses (1), (5), (7), (8) and (9) of Definition 3.3.} \]
For each $\alpha < \theta$, denote the sentence “$\check{\alpha} \in \check{a}$” by $\varphi_\alpha$. Set $n := \ell(p)$ and $s := \varpi_n(p)$. Combining Definition 3.3(9) with Lemma 3.11(1), we may recursively obtain a $\leq^{\varpi_n}$-decreasing sequence of conditions $\vec{\rho} = \langle p^\alpha \mid \alpha < \theta \rangle$, such that, for each $\alpha < \theta$, $p^\alpha \leq^{\varpi_n} p$ and $D_{\vec{\rho}, p^\alpha}$ is dense in $S_n \downarrow s$. Then, we utilize the $\sigma_n$-closure of $\mathbb{P}^\varpi_n$ once more and find $q \in P_n$ which is $\leq^{\varpi_n}$-below all elements of $\vec{\rho}$. Altogether, for every $\alpha < \theta$,

$$D_{\vec{\varphi}, q} = \{ t \leq_n s \mid (q + t \Vdash p \varphi_\alpha) \text{ or } (q + t \Vdash p \lnot \varphi_\alpha) \}$$

is dense in $S_n \downarrow s$.

Now, let $G$ be a $\mathbb{P}$-generic filter with $p \in G$. Let $H_n$ be the $S_n$-generic filter induced by $\varpi_n$ from $G$, and work in $V[H_n]$. It follows that, for every $\alpha < \theta$, for some $t \in H_n$, either $(q + t \Vdash p \check{\alpha} \in \check{a})$ or $(q + t \Vdash p \check{\alpha} \notin \check{a})$. Set

$$b := \{ \alpha < \theta \mid \exists t \in H_n[q + t \Vdash p \check{\alpha} \in \check{a}] \}.$$

As $q \leq^{\varpi_n} p$, we infer that $\varpi_n(q) = \varpi_n(p) = s \in H_n$, so that $q \in P/H_n$.

**Claim 3.12.1.** $q \Vdash p/H_n b = \check{a}_{H_n}$.

**Proof.** Clearly, $q \Vdash p/H_n b \subseteq \check{a}_{H_n}$. For the converse, let $\alpha < \theta$ and $r \leq^{p/H_n} q$ be such that $r \Vdash p/H_n \check{\alpha} \in \check{a}_{H_n}$. By the very Definition 2.3, there is $t_0 \in H_n$ with $t_0 \leq_n \varpi_n(r)$ such that $r + t_0 \leq q$. By extending $t$ if necessary, we may moreover assume that $r + t_0 \Vdash p \check{\alpha} \in \check{a}$. Set $q_0 := r + t_0$.

By the choice of $q$, there is $t_1 \in H_n$ such that $q + t_1 \Vdash p \check{\alpha} \in \check{a}$. Set $q_1 := q + t_1$. Let $t \in H_n$ be such that $t \leq_n t_0, t_1$. Recalling Definition 3.3(9), $\varpi_n$ is exact, so $t \leq_n \varpi_n(q_0), \varpi_n(q_1)$. By Lemma 2.6, $q_0 + t_1$ witnesses the compatibility of $q_0$ and $q_1$, hence $q + t_1 \Vdash p \check{\alpha} \in \check{a}$, and thus $\alpha \in b$. \hfill \Box

Altogether, $\check{a}_G \in V[H_n]$.

(2) If $\kappa$ were to be collapsed, then, by Clause (1), it would have been collapsed by $S_n$ for some $n < \omega$. However, $S_n$ is a notion of forcing of size $< \sigma_n \leq \kappa$.

Next, suppose towards a contradiction that $\kappa$ is strong limit cardinal, and yet, for some $\mathbb{P}$-generic filter $G$, for some $\theta < \kappa$, $V[G] \models 2^\theta \geq \kappa$. For each $n < \omega$, let $H_n$ be the $S_n$-generic filter induced by $\varpi_n$ from $G$. Using Clause (1), for every $a \in \mathcal{P}^{V[G]}(\theta)$, we fix $n_a < \omega$ such that $a \in V[H_{n_a}]$.

- If $\kappa$ is regular, then there must exist some $n < \omega$ for which $|\{a \in \mathcal{P}^{V[G]}(\theta) \mid n_a = n\}| \geq \kappa$. However $S_n$ is a notion of forcing of some size $\lambda < \kappa$, and so by counting nice names, we see it cannot add more than $\theta^\lambda$ many subsets to $\theta$, contradicting the fact that $\kappa$ is strong limit.

- If $\kappa$ is not regular, then $\Sigma$ is not eventually constant, and $\text{cf}(\kappa) = \omega$, so that, by König’s lemma, $V[G] \models 2^\theta \geq \kappa^+$. It follows that exists some $n < \omega$ for which $|\{a \in \mathcal{P}^{V[G]}(\theta) \mid n_a = n\}| > \kappa$, leading to the same contradiction.

(3) Suppose $\theta, \nu$ are regular cardinals with $\theta < \kappa \leq \nu$, $\check{f}$ is a $\mathbb{P}$-name for a function from $\theta$ to $\nu$, and $p \in P$ is a condition forcing that the image of $\check{f}$ is cofinal in $\nu$. Denote $n := \ell(p)$ and $s := \varpi_n(p)$. By Definition 3.3(1), we may extend $p$ and assume that $\sigma_n > \theta$. 
For all $\alpha < \theta$, set $D_\alpha := \{ r \leq p \mid \exists \beta < \nu, r \vDash P \hat{f}(\alpha) = \hat{\beta} \}$. As $D_\alpha$ is $0$-open and dense below $p$, by combining the $\sigma_n$-closure of $P^{\aleph_\alpha}_n$ (cf. Definition 3.3(9)) with Lemma 3.11(2), we may recursively define a $\leq^{\aleph_0}$-decreasing sequence of conditions $(q^\alpha \mid \alpha \leq \theta)$ below $p$ such that, for every $\alpha < \theta$, $U_{D_\alpha, q^\alpha}$ is dense in $\mathcal{S}_n \downarrow s$. Set $q := q^\theta$, and note that, for each $\alpha < \theta$,

$$U_{D_\alpha, q} := \{ t \leq_n s \mid \exists m < \omega [P^{\omega+1}_m \subseteq D_\alpha] \}$$

is dense in $\mathcal{S}_n \downarrow s$. In particular, the above sets are non-empty. For each $\alpha < \theta$, let us fix $t_\alpha \in U_{D_\alpha, q}$ and $m_\alpha < \omega$ witnessing this. We now show that $|W(q)| \geq \nu$. Let $A_\alpha := \{ \beta < \nu \mid \exists r \in P^{m_\alpha+1}_\alpha [r \vDash P \hat{f}(\alpha) = \hat{\beta}] \}$. By Lemma 3.6(1), we have

$$A_\alpha = \{ \beta < \nu \mid \exists r \in W_{m_\alpha}(q + t_\alpha) [r \vDash P \hat{f}(\alpha) = \hat{\beta}] \}.$$ 

Let $A := \bigcup_{\alpha < \theta} A_\alpha$. Then

$$|A| \leq \sum_{m < \omega, t \leq_n s} |W_m(q + t)| \leq \max\{ \aleph_0, |S_n| \} \cdot |W(q)|.$$

Observe that, for each $t \leq_n s$, $|W(q + t)| \leq |W(q)|$. Also, by clauses $(\alpha)$ and $(\beta)$ of Definition 3.3 and our assumption on $\nu$, $\max\{ \aleph_0, |S_n| \} < \sigma_n < \nu$. It follows that if $|W(q)| < \nu$, then $|A| < \nu$, and so $\sup(A) < \nu$. Thus, $q$ forces that the range of $f$ is bounded below $\nu$, which leads us to a contradiction with respect to our initial assumption. Therefore, $|W(q)| \geq \nu$, as desired.

Next, suppose that, for all $p \in P$, $|W(p)| \leq \kappa$. Towards a contradiction, suppose that there exist $p \in P$ forcing that $\kappa^+$ is collapsed. Denote $\nu := \kappa^+$. As $1 \vDash P \kappa$ is singular", this means that $p \vDash \text{cf}(\nu) < \kappa$, contradicting Clause (2) of this lemma. \[\square\]

4. Extender Based Prikry Forcing with collapses

In this section we present Gitik’s notion of forcing from [Git19b], and analyze its properties. Gitik came up with this notion of forcing in September 2019, during the week of the 15th International Workshop on Set Theory in Luminy, after being asked by the second author whether it is possible to interleave collapses in the Extender Based Prikry Forcing (EBPF) with long extenders [GM94, §3]. Unlike the exposition of this forcing from [Git19b], the exposition here shall not assume the GCH.

**Setup 4.** Throughout this section our setup will be as follows:

- $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of cardinals;
- $\kappa_{-1} := \aleph_0$, $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \mu^+$;
- $\mu^{\leq \mu} = \mu$ and $\lambda^{< \lambda} = \lambda$;
- for each $n < \omega$, $\kappa_n$ is $(\lambda + 1)$-strong;
- $\Sigma := \langle \sigma_n \mid n < \omega \rangle$, where, for each $n < \omega$, $\sigma_n := (\kappa_{n-1})^+$;\[11\]

\[11\]In particular, $\sigma_0 = \aleph_1$.  

In particular, we are assuming that, for each \( n < \omega \), there is a \((\kappa_n, \lambda + 1)\)-extender \( E_n \) whose associated embedding \( j_n : V \to M_n \) is such that \( M_n \) is a transitive class, \( \kappa_n M_n \subseteq M_n, V_{\lambda+1} \subseteq M_n \) and \( j_n(\kappa_n) > \lambda \). For each \( n < \omega \), and each \( \alpha < \lambda \), set

\[
E_{n, \alpha} := \{ X \subseteq \kappa_n \mid \alpha \in j_n(X) \}.
\]

Note that \( E_{n, \alpha} \) is a non-principal \( \kappa_n \)-complete ultrafilter over \( \kappa_n \), provided that \( \alpha \geq \kappa_n \). Moreover, in the particular case of \( \alpha = \kappa_n \), \( E_{n, \kappa_n} \) is also normal. For ordinals \( \alpha < \kappa_n \) the measures \( E_{n, \alpha} \) are principal so the only reason to consider them is for a more neat presentation.

For each \( n < \omega \), we shall consider an ordering \( \leq E_n \) over \( \lambda \), as follows:

**Definition 4.1.** For each \( n < \omega \), set

\[
\leq E_n := \{ (\beta, \alpha) \in \lambda \times \lambda \mid \beta \leq \alpha, \land \exists f \in \kappa_n \kappa_n j_n(f)(\alpha) = \beta \}.
\]

It is routine to check that \( \leq E_n \) is reflexive, transitive and antisymmetric, hence \((\lambda, \leq E_n)\) is a partial order. In case \( \beta \leq E_n \alpha \), we shall fix in advance a witnessing map \( \pi_{\alpha, \beta} : \kappa_n \to \kappa_n \). In the special case where \( \alpha = \beta \), by convention, \( \pi_{\alpha, \alpha} := \text{id} \). Observe that \( \leq E_n \mid (\kappa_n \times \kappa_n) \) is exactly the \( \epsilon \)-order over \( \kappa_n \) so that when we refer to \( \leq E_n \) we will really be speaking about the restriction of this order to \( \lambda \setminus \kappa_n \).

The following lemma lists some key features of the poset \((\lambda, \leq E_n)\) as presented in [Git10, §2]. A proof that does not assume the GCH may be found in [PRS20, §3].

**Lemma 4.2.** Let \( n < \omega \).

1. For every \( a \in [\lambda]^{<\kappa_n} \), there are \( \lambda \)-many \( \alpha < \lambda \) above \( \text{sup}(a) \) such that for every \( \gamma, \beta \in x \):
   - \( \gamma, \beta \leq E_n \alpha \);
   - if \( \gamma \leq E_n \beta \), then \( \{ \nu \in \kappa_n \mid \pi_{\alpha, \gamma}(\nu) = \pi_{\beta, \gamma}(\pi_{\alpha, \beta}(\nu)) \} \in E_{n, \alpha} \).

2. For all \( \gamma < \beta \), \( \gamma \leq E_n \alpha \), and \( \beta \leq E_n \alpha \),
   \[
   \{ \nu \in \kappa_n \mid \pi_{\alpha, \gamma}(\nu) < \pi_{\alpha, \beta}(\nu) \} \in E_{n, \alpha}.
   \]

3. For all \( \alpha, \beta < \lambda \) with \( \beta \leq E_n \alpha \), \( \pi_{\alpha, \beta} : \kappa_n \to \kappa_n \) is a projection map, such that for each \( A \in E_{n, \alpha} \), \( \pi_{\alpha, \beta} A \in E_{n, \beta} \).  \( \square \)

**Remark 4.3.** For future reference, it is worth mentioning that the clauses of the above lemma also apply to the poset \( (\mu, \leq E_n \mid \mu \times \mu) \). For details, see [Git10, §2]. In particular, every \( a \in [\lambda]^{<\kappa_n} \) can be extended to \( a^+ \) such that \( \kappa_n, \mu \in a^+ \) and \( a^+ \cap \mu \) has a \( \leq E_n \)-greatest element in \( a^+ \cap \mu \).

**Definition 4.4.** For each \( n < \omega \), let \( s_n^* : \kappa_n \to \text{\#}^{\kappa_n} \) be a function such that \( j_{E_n}(s_n^*)(\kappa_n) = \langle \kappa_k \mid k < \omega \rangle \). Let \( s_n : \kappa_n \to \kappa_n \) be given by \( s_n(\rho) := \text{sup}(s_n^*(\rho))^\kappa_n \).

The meaning of \( s_n \) is that if \( \rho \) is an indiscernible corresponding to \( \kappa_n \), then \( s_n(\rho) \) is an indiscernible corresponding to \( \kappa_n^+ \); in other words, \( s_n \) is a function representing \( \kappa_n^+ \) in the ultrapower \( M_n \).
Remark 4.5. For each $n < \omega$, and for all $\alpha \geq \kappa_n$,
\[ \{ \rho < \kappa_n \mid (\kappa_{n-1})^+ < \rho < s_n(\rho) < \kappa_n \text{ \& } \rho \text{ inaccessible} \} \in E_{n,\alpha}. \]

We will only use these measures of the extenders, and by restricting to a measure one set, we assume that this is always the case for all $\rho < \kappa_n$ that we ever consider. Similarly, we may also assume that $s_n(\rho)$ is regular (actually the successor of a singular) and that $s_n(\rho^p)^{<\rho^p} = s_n(\rho^p)$.

4.1. The forcing. Before giving the definition of Gitik’s forcing we shall first introduce the basic building block modules $Q_{n0}$ and $Q_{n1}$.

Definition 4.6. For each $n < \omega$, define $Q_{n1}$, $Q_{n0}$ and $Q_n$ as follows:

(0)\(Q_{n1} := (Q_{n1}, \leq_{n1})\) is the set of conditions $p := (f^p, \rho^p, h_0^p, h_1^p, h_2^p)$, where

1. $f^p$ is a function from some $x \in [\lambda]^{<\kappa}$ to $\kappa_n$;
2. $\rho^p < \kappa_n$ inaccessible;
3. $h_0^p \subseteq \text{Col}(\sigma_n, <\rho^p)$;
4. $h_1^p \subseteq \text{Col}(\rho^p, s_n(\rho^p))$;
5. $h_2^p \subseteq \text{Col}(s_n(\rho^p)^{++}, <\kappa_n)$.

The ordering $\leq_{n1}$ is defined as follows: $q \leq_{n1} p$ iff $f^q \supseteq f^p$, $\rho^q = \rho^p$, and for $i < 3$, $h_1^q \supseteq h_i^p$.

(1)\(Q_{n0} := (Q_{n0}, \leq_{n0})\) is the set of conditions $p := (a^p, A^p, f^p, F_0^p, F_1^p, F_2^p)$, where:

1. $(a^p, A^p, f^p)$ is in the $n$-module $Q_{n0}^{a^p}$ from the Extender Based Prikry Forcing (EBPF) as defined in [Git10, Definition 2.6]. Moreover, we require that $\kappa_n, \mu \in a^p$ and that $a^p \cap \mu$ contains a $\leq_{\kappa_n}$-greatest element denoted by $mc(a^p \cap \mu)$;\(^\text{12}\)
2. For $i < 3$, $\text{dom}(F_i^p) = \pi_{mc(a^p), mc(a^p \cap \mu)}[A^p]$, and for each $\nu \in \text{dom}(F_i^p)$, setting $\nu_0 := \pi_{mc(a^p \cap \mu)}(\kappa_n(\nu))$, we have:
   a. $F_0^p(\nu) \subseteq \text{Col}(\sigma_n, <\nu_0)$;
   b. $F_1^p(\nu) \subseteq \text{Col}(\nu_0, s_n(\nu_0))$;
   c. $F_2^p(\nu) \subseteq \text{Col}(s_n(\nu_0)^{++}, <\kappa_n)$

The ordering $\leq_{n0}$ is defined as follows: $q \leq_{n0} p$ iff $(a^q, A^q, f^q) \leq_{Q_{n0}} (a^p, A^p, f^p)$ as in [Git10, Definition 2.7], and for each $\nu \in \text{dom}(F_i^q)$, $F_i^q(\nu) \supseteq F_i^p(\nu')$, where $\nu' = \pi_{mc(a^q \cap \mu), mc(a^p \cap \mu)}(\nu)$.

(2)\(Q_n := (Q_{n0} \cup Q_{n1}, \leq_n)\) where the ordering $\leq_n$ is defined as follows: for each $p, q \in Q_n$, $q \leq_n p$ if:

1. Either $p, q \in Q_{ni}$, some $i \in \{0, 1\}$, and $q \leq_{ni} p$, or
2. $q \in Q_{n1}$, $p \in Q_{n0}$, and, for some $\nu \in A^p$, $q \leq_{n1} p^{\rho^p}(\nu')$, where

\[ p^{\rho^p}(\nu') := (f^p \cup \{ (\beta, \pi_{mc(a^p), b}(\nu)) \mid \beta \in a^p \}, \nu_0, F_0^p(\nu'), F_1^p(\nu'), F_2^p(\nu'), \]

and $\nu' = \pi_{mc(a^p), mc(a^p \cap \mu)}(\nu)$.\(^\text{12}\)

\(^{12}\)Recall that $(a^p, A^p, f^p) \in Q_{n0}$ in particular implies that $a^p$ contains a $\leq_{\kappa_n}$-greatest element. This will be denoted by $mc(a^p)$. 

Notation 4.7. For each conditions \( p, q \in Q_n \), we shall write \( q \leq_n p \) iff \( q \leq_n p \) as witnessed by clause (2) of Definition 4.6.

Remark 4.8. One main difference compared to the short-extender EBPF with collapses (see [Git11]) is that the domains of the guiding collapsing functions here can be measure one sets from the ultrafilters \( E_n \) for \( \kappa_n \leq \alpha < \mu = \kappa^+ \). Previously, the domains were taken only from the normal measures \( E_n \). Consequently, in the Prikry lemma, the short EBPF needed a diagonalization argument relying on normality. Here, this is not used.

Having all necessary building blocks, we can now define the poset \( \mathbb{P} \).

Definition 4.9. Gitik’s Extender Based Prikry Forcing with collapses (EBPFC) is the poset \( \mathbb{P} := (P, \leq) \) defined by the following clauses:

- Conditions in \( P \) are sequences \( p = \langle p_n \mid n < \omega \rangle \in \prod_{n<\omega} Q_n \).
- For all \( p \in P \),
  - There is \( n < \omega \) such that \( p_n \in Q_{n0} \);
  - For every \( n < \omega \), if \( p_n \in Q_{n0} \) then \( p_m \in Q_{m0} \) and \( a^p_n \subseteq a^{p_m} \), for every \( m \geq n \).
- For all \( p, q \in P \), \( p \leq q \) iff \( p_n \leq q_n \), for every \( n < \omega \).

Definition 4.10. \( \ell : P \to \omega \) is defined by letting for all \( p = \langle p_n \mid n < \omega \rangle \),

\[
\ell(p) := \min\{n < \omega \mid p_n \in Q_{n0}\}.
\]

Notation 4.11. Given \( p \in P \), \( p = \langle p_n \mid n < \omega \rangle \), we will typically write \( p_n = (f_n, p_n, h_n^{0p}, h_n^{1p}, h_n^{2p}) \) for \( n < \ell(p) \), and \( p_n = (a_n, A_n, f_n, F_n, F_n^{1p}, F_n^{2p}) \) for \( n \geq \ell(p) \). Also, we shall denote \( \text{mc}(a_n^p \cap \mu) \) by \( \alpha_{p_n} \).

We already have \( \mathbb{P} \) and \( \ell \) and we shall soon see that \( \mathbb{P} \models \mu = \kappa^+ \). Now we introduce sequences \( \vec{S} = \langle S_n \mid n < \omega \rangle \) and \( \vec{\omega} = \langle \omega_n \mid n < \omega \rangle \), and a map \( c : P \to H_\mu \) such that \( (\mathbb{P}, \ell, c, \vec{\omega}) \) will witness the clauses of Definition 3.3. Let us begin by defining the function \( c \).

As in [PRS20, §3], using \( \mu^\kappa = \mu \) and \( 2^\mu = \lambda \), we fix a sequence of functions \( \langle e^i \mid i < \mu \rangle \) from \( \lambda \) to \( \mu \) such that, for all \( x \in [\lambda]^\kappa \) and every function \( e : x \to \mu \), there exists \( i < \mu \) with \( e \subseteq e^i \).

Definition 4.12. For every condition \( p = \langle p_n \mid n < \omega \rangle \) in \( \mathbb{P} \), define a sequence of indices \( \langle i(p_n) \mid n < \omega \rangle \) as follows:

\[
i(p_n) := \begin{cases} 
\min\{i < \mu \mid f \subseteq e^i\} & \text{if } n < \ell(p); \\
\min\{i < \mu \mid e^i \upharpoonright a_n^p = \vec{0} \upharpoonright e^i \upharpoonright \text{dom}(f_n^p) = f_n^p + 1\} & \text{if } n \geq \ell(p).
\end{cases}
\]

Define the map \( c : P \to H_\mu \), by letting for any condition \( p = \langle p_n \mid n < \omega \rangle \):

\[
c(p) := (\ell(p), p^p_n \mid n < \ell(p)), (i(p_n) \mid n < \omega), (\vec{h}_n^p \mid n < \ell(p)), (\vec{G}_n^{1p} \mid n \geq \ell(p)),
\]

where \( \vec{h}_n^p := (h_n^{ip} \mid i < 3) \) and \( \vec{G}_n^{1p} := (j(F_n^{ip})(\alpha_{p_n}) \mid i < 3) \).

We now define the sequence \( \vec{S} \):
Definition 4.13. For each \( n < \omega \), set
\[
S_n := \begin{cases} 
\{1\}, & \text{if } n = 0; \\
\{(\langle \rho_k^p, h_k^1, h_k^2 \rangle \mid k < n) \mid p \in P_n\}, & \text{if } n \geq 1.
\end{cases}
\]
For \( n \geq 1 \) and \( s, t \in S_n \), we write \( s \leq_n t \) iff there are \( p, q \in P_n \) with \( p \leq q \) witnessing, respectively, that \( s \) and \( t \) are in \( S_n \). Denote \( \mathbb{S}_n := (S_n, \leq_n) \) and set \( \tilde{\mathbb{S}} := (\mathbb{S}_n \mid n < \omega) \).

Remark 4.14. Observe that \(|S_n| < \sigma_n\). Moreover, for each \( s \in S_n \setminus \{1_{S_n}\} \), \( S_n \downarrow s \cong \text{Col}(\delta, <\kappa_n) \times \mathbb{Q} \) with \( \mathbb{Q} \) is a notion of a forcing of size \( <\delta \) such that \( \sigma_{n-1} < \delta < \kappa_n-1 \). Specifically, if \( p \in P_n \) is the condition from which \( s \) arises, then \( \delta = s_n-1(\rho_{n-1})^{++} \) and \( \mathbb{Q} \) is a product \( \mathbb{R} \times \text{Col}(\sigma_{n-1}, <\rho_{n-1}) \times \text{Col}(\rho_{n-1}, s_n(\rho_{n-1})) \), where \( \mathbb{R} \) is a notion of forcing of size \( <\kappa_n-2 \).

Also, by combining Easton’s lemma with a counting of nice names, if the GCH holds below \( \kappa \), then \( S_n \downarrow s \) preserves this behavior of the power set function, for each \( s \in S_n \setminus \{1_{S_n}\} \).

Finally, we define the sequence \( \tilde{\omega} \):

Definition 4.15. For each \( n < \omega \), define \( \omega_n : P_{\geq n} \to S_n \) as follows:
\[
\omega_n(p) := \begin{cases} 
\{1\}, & \text{if } n = 0; \\
\{(\langle \rho_k^p, h_k^1, h_k^2 \rangle \mid k < n) \mid p \in P_n\}, & \text{if } n \geq 1.
\end{cases}
\]
Set \( \tilde{\omega} := (\omega_n \mid n < \omega) \).

The next lemma collects some useful properties about the \( n0 \)-modules of \( EBPFC \) (i.e., the \( Q_{n0} \)'s) and reveals some of their connections with the corresponding modules of \( EBPF \) (i.e., then \( Q_{n0}^* \)'s).

Lemma 4.16. Let \( n < \omega \). All of the following hold:

(1) \( P_n \) projects to \( Q_{n0} \), and this latter projects to \( Q_{n0}^* \).

(2) \( Q_{n0}^* \) is \( \kappa_n \)-directed-closed, while \( Q_{n0} \) is \( \sigma_n \)-directed-closed.

(3) \( S_n \) satisfies the \( (\kappa_n-1)\)-cc.

Proof. (1) It is routine to check that \( p \mapsto (a_n^p, A_n^p, f_n^p, F_n^0, F_n^1, F_n^{2p}) \) defines a projection between \( P_n \) and \( Q_{n0} \). Similarly, \( q_n : (a, A, f, F^0, F^1, F^2) \mapsto (a, A, f) \) is a projection between \( Q_{n0} \) and \( Q_{n0}^* \).

(2) The argument for the \( \kappa_n \)-directed-closedness of \( Q_{n0}^* \) is essentially given in [Git10, §2] (or in [PRS20, Lemma 3.11]). Let \( D \subseteq Q_{n0} \) be a directed set of size \( <\sigma_n \) and denote by \( q_n \) the projection between \( Q_{n0} \) and \( Q_{n0}^* \) given in the proof of item (1). Clearly, \( q_n[D] \) is a directed subset of \( Q_{n0}^* \) of size \( <\sigma_n \), so that we may let \( (a, A, f) \) be a \( \leq_{Q_{n0}} \)-lower bound for it. By \( \leq_{Q_{n0}} \)-extending \( (a, A, f) \) if necessary we may assume that \( \kappa_n, \mu \in a \) and that \( a \cap \mu \) contains a \( \leq_{E_n} \)-greatest element. Set \( \alpha := \text{mc}(a \cap \mu) \). For each \( i < 3 \) and each \( \nu \in \text{mc}(\alpha)[a] \), define \( F^i(\nu) := \bigcup_{p \in D} F^i_p(\pi_{\alpha,\nu}(\nu)) \) (cf. Notation 4.11). Observe that this is well-defined as the conditions in \( D \) are pairwise

\[\text{By Setup 4, in the case } n = 1, \kappa_1 = \aleph_0, \text{ so that } \mathbb{R} \text{ is trivial.}\]
compatible. Finally, it is not hard to check that \((a,A,f,F^0,F^1,F^2)\) is a legitimate condition in \(Q_{n0}\) extending all \(p \in D\).

(3) This is immediate from the definition of \(S_n\) (cf. Definition 4.13). \(\square\)

4.2. **EBPFC is \((\Sigma,\overline{S})\)-Prikry.** We verify that \((P,\ell,c,\overline{\sigma})\) is \((\Sigma,\overline{S})\)-Prikry by going over the clauses of Definition 3.3. It is clear that clauses \((\alpha)\)–\((\zeta)\) are satisfied. Thus, we are left with verifying the rest of the clauses. To this aim we will take advantage of the following notation:

**Convention 4.17.** For every sequence \(\{A_k\}_{i \leq k \leq j}\) such that each \(A_k\) is a subset of \(\kappa_k\), we shall identify \(\prod_{k=i}^j A_k\) with its subset consisting only of the sequences that are moreover increasing.

**Definition 4.18.** Let \(p = (p_n \mid n < \omega) \in P\). Define:
- \(p^\frown \emptyset := p\);
- For every \(\nu \in A^p_{\ell(p)}\), \(p^\frown \langle \nu \rangle\) is the unique condition \(q = (q_n \mid n < \omega)\), such that for each \(n < \omega\):
  \[
  q_n = \begin{cases} 
  p_n, & \text{if } n \neq \ell(p) \\
  p_{\ell(p)}^\frown \langle \nu \rangle, & \text{otherwise}.
  \end{cases}
  \]
- Inductively, for all \(m \geq \ell(p)\) and \(\nu = (\nu_{\ell(p)}, \ldots, \nu_m, \nu_{m+1}) \in \prod_{n=\ell(p)}^{m+1} A^p_n\), set \(p^\frown \nu := (p^\frown \nu \mid (m+1))^\frown \langle \nu_{m+1} \rangle\).

**Fact 4.19.** Let \(p,q \in P\).
- \(q \leq^0 p\) iff \(\ell(p) = \ell(q)\) and \(q \leq^0 p\), for each \(n < \omega\);
- \(q \leq p\) iff there is \(\nu \in \prod_{n=\ell(p)}^{\ell(q)-1} A^p_n\) such that \(q \leq^0 p^\frown \langle \nu \rangle\);
- \(\nu\) in the above item is uniquely determined by \(q\). More precisely, for each \(\ell(p) \leq n < \ell(q)\), \(\nu_n = f_n^p(\text{mc}(a^n_q))\).

By the very definition of EBPFC (cf. Definition 4.9) and the function \(\ell\) (cf. Definition 4.10), \((P,\ell)\) is a graded poset, hence \((P,\ell,c,\overline{\sigma})\) witnesses Clause (1). The next lemma takes care of Clause (2).

**Lemma 4.20.** For all \(n < \omega\), \(P_n\) is \(\aleph_1\)-closed.

**Proof.** This follows from Lemma 4.16 and the fact that all of the collapses are at least countably closed. \(\square\)

We now verify that the function \(c\) of Definition 4.12 witnesses Clause (3):

**Lemma 4.21.** For all \(p,q \in P\), if \(c(p) = c(q)\), then \(P^p_0 \cap P^q_0\) is non-empty.

**Proof.** Let \(p,q \in P\) and assume that \(c(p) = c(q)\). By definition of \(c\), \(\ell(p) = \ell(q)\) and \(\rho^p_n = \rho^q_n\), for each \(n < \ell(p)\). Set \(\ell := \ell(p)\) and \(\rho_n := \rho^p_n\), for each \(n < \ell\). Also, \(c(p) = c(q)\) entails \(\overline{h}^p_n = \overline{h}^q_n\) for each \(n < \ell\) and \(\overline{G}^p_n = \overline{G}^q_n\), for each \(n \geq \ell\). For \(n < \ell\), set \(\overline{h}_n := \overline{h}^p_n\) and denote \(\overline{h}_n = (h^0_n, h^1_n, h^2_n)\). For \(n < \omega\) let us now define \(r_n \in Q_{\aleph^\ell(n)}\) as follows.\(^{14}\)

\(^{14}\) Here \(\chi_\ell\) stands for the characteristic function on the set \(\ell = \{0,\ldots, \ell - 1\}\).
If $n < \ell$, $c(p) = c(q)$ implies $i = i(p_n) = i(q_n)$, and so $f^p_0 \cap f^q_0 \subseteq e^i$. Set $p^r_n := p_n$, $f^p_n := f^p_0 \cup f^p_0$ and, for $i < 3$, $h^r_n := h^r_n$. Finally, set $r_n := (p^r_n, f^p_n, h^r_n, h^r_n, h^r_n)$. Clearly, $r_n \in Q_{n1}$ and $r_n \leq n p_n, q_n$.

If $n \geq \ell$, set $a^r_{n-1} := \emptyset$ and argue by recursion as follows: Since $c(p) = c(q)$ implies $i = i(p_n) = i(q_n)$, arguing as in [PRS20, Lemma 3.12] it follows that $a^r_n \cap \text{dom}(f^r_0) = a^r_n \cap \text{dom}(f^r_0) = \emptyset$. This implies that $(a^r_n, A^r_n, f^r_n)$ and $(a^r_n, A^r_n, f^r_n)$ are two compatible conditions in $Q^*_{n0}$. Let $(a^r_n, A^r_n, f^r_n) \in Q^*_{n0}$ witnessing this and being such that $a^r_{n-1} \subseteq a^r_n$. By enlarging $a^r_n$ if necessary, we may assume that $\kappa_n, \mu \in a^r_n$ and that $a^r_n \cap \mu$ has an $E_n$-maximal element which we denote by $\alpha_n$ (cf. Remark 4.3).

Let us now define the $F$-part of $r_n$. Since $G^p_n = G^q_n$, for each $i < 3$, $j_n(F^p_n(\alpha_{p_n})) = j_n(F^q_n(\alpha_{q_n})$. Also $j_n(F^p_n(\alpha_{p_n})) = j_n(F^p_n \circ \pi_{\alpha_n, \alpha_{p_n}}(\alpha_{r_n})$. Similarly, the same is true for $j_n(F^q_n(\alpha_{q_n}))$. Thus,

$$\forall \nu \in E_{n, \alpha_n} \nu (F^p_n(\pi_{\alpha_{r_n}, \alpha_{p_n}}(\nu))) = F^q_n(\pi_{\alpha_{r_n}, \alpha_{q_n}}(\nu)))$$

holds. By shrinking $A^r_n$ if necessary, we define $F^r_n$ with domain $\pi_{\alpha_n, \alpha_{p_n}}(A^r_n)$ as $\nu \mapsto F_n(\pi_{\alpha_n, \alpha_{p_n}}(\nu))$. Clearly, $r_n := (a^r_n, A^r_n, f^r_n, F^r_n, F^r_n, F^r_n) \in Q_n$, and $r_n \leq n p_n, q_n$. It is routine to check that $r := \langle r_n \mid n < \omega \rangle$ is a condition in $P_\ell$ that witnesses the compatibility of $p$ and $q$. Thus, $P^p_0 \cap P^q_0$ is non-empty.

The verification of Clauses (4), (5) and (6) are the same as in Lemma 3.16, Lemma 3.17 and Lemma 3.18 of [PRS20], respectively. It is worth saying that regarding Clause (5) we can actually prove that $|W(p)| \leq \kappa$, for each $p \in P$.

Before verifying Clause (7), let us dispose with the verification of Clause (8):

**Lemma 4.22.** For all $n < \omega$, the map $\varpi_n$ is an exact nice projection from $\mathbb{P}_{\geq n}$ to $\mathbb{S}_n$ such that, for all $k \geq n$, $\varpi_n|\mathbb{P}_k$ is again an exact nice projection.

**Proof.** Fix some $n < \omega$. By definition, the map $\varpi_n$ fulfils (1) of Definition 2.1. Also, it is not hard to check that $\varpi_n$ is order-preserving.

Let $p \in \mathbb{P}_{\geq n}$ and $s \leq n \varpi_n(p)$. By definition, $s = \langle (p^0_k, h^0_k, h^1_k, h^2_k) \mid k < n \rangle$, for some $q \leq ^0 p$. Define $r := \langle r_k \mid k < \omega \rangle$ as follows:

$$r_k := \begin{cases} (p^0_k, h^0_k, h^1_k, h^2_k), & \text{if } k < n; \\ p_k, & \text{otherwise.} \end{cases}$$

It is not hard to check that $r \leq ^0 p$ and clearly $\varpi_n(q) = s$. Actually, by Definition 4.6(2), $q$ is the greatest such condition, so that $q = p + s$. This shows that $\varpi_n$ is an exact nice projection, as wanted.

The same proof shows that $\varpi_n|\mathbb{P}_k$ is again an exact nice projection, for each $k \geq n$.

**Remark 4.23.** Let $n \leq \omega$ and $p \in \mathbb{P}_n$. By the very definition of $\varpi_n$ and $\mathbb{S}_n$, it is evident that the map from Lemma 2.6 yields an isomorphism between $\mathbb{P}_n \downarrow p$ and $(\mathbb{S}_n \downarrow \varpi_n(p)) \times (\mathbb{P}_n \downarrow p)$. 


Let \( \leq^{\sigma} \) denote the ordering defined in Convention 3.4. Before we can prove Clause (7), we need some notation and a couple of technical lemmas.

**Definition 4.24.** For \( p, q \in P \) and \( m \geq 1 \), we shall write \( q \sqsubseteq m p \) iff \( q \leq^{\sigma} p \) and \( q_{\ell(p)+k} = p_{\ell(p)+k} \), for all \( k < m \). A sequence of conditions is called an \( m \)-fusion sequence if it is \( \sqsubseteq m \)-decreasing.

**Remark 4.25.** If \( \langle r^\alpha \mid \alpha < \sigma_{\ell(r^\nu)} \rangle \) is an \( m \)-fusion sequence then it has a \( \sqsubseteq m \)-lower bound. This can be proved by a similar argument to Lemma 4.29.

**Lemma 4.26 (Diagonalization Lemma).** Let \( p \in P \) and \( U \) be a 0-open set. There is \( q \sqsubseteq 0 p \) such that, for each \( n < \omega \), if \( r \in P_n^p \cap U \) then \( w(q, r) \in U \).

**Proof.** The strategy is to find a \( \leq^{\sigma} \)-decreasing sequence \( \langle q_m \mid m < \omega \rangle \) such that for each \( m < \omega \) the following property holds:

\[
(*)_m \quad q_m \leq^0 p \quad \text{and} \quad r \in P_m^{q_m} \cap U \quad \text{then} \quad w(q_m, r) \in U.
\]

Once this sequence is constructed, we may let \( q^* \) be a \( \leq^{\sigma} \)-lower bound for this sequence. Clearly, \( q^* \) yields the desired condition.

We will build this sequence by induction on \( m < \omega \). Suppose \( m = 0 \) and let \( q_0 \in P_0^p \cap U \), if such condition exists; otherwise, set \( q_0 := p \). Clearly, \( q_0 \) witnesses \((*)_0\). Now, suppose \( m = 1 \) and set \( \ell := \ell(p) \). Let \( \langle \nu_\alpha \mid \alpha < \sigma_\ell \rangle \) be an enumeration of \( A_\ell^p \).

**Claim 4.26.1.** There is a sequence of conditions \( \langle q^\nu_\alpha \mid \alpha < \sigma_\ell \rangle \) below \( q_0 \) such that, for each \( \alpha < \sigma_\ell \), the following properties hold:

1. \( q^\nu_\alpha \leq^0 q_0 \cap \langle \nu_\alpha \rangle \);
2. If \( P_0^\nu_\alpha \cap U \neq \emptyset \), then \( q^\nu_\alpha \in U \);
3. For each \( n < \ell \) and \( \alpha < \beta < \sigma_\ell \), \( f_n^{\nu_\alpha} \subseteq f_n^{\nu_\beta} \);
4. For each \( n > \ell \) and \( \alpha < \beta < \sigma_\ell \), \( g_n^{\nu_\alpha} \leq w \emptyset 0 q^{\nu_\alpha} \).

**Proof.** We will define the sequence \( \langle q^{\nu_\alpha} \mid \alpha < \sigma_\ell \rangle \) and, simultaneously, a 1-fusion sequence \( \langle r^{\nu_\alpha} \mid \alpha < \sigma_\ell \rangle \) \( \sqsubseteq 1 \)-below \( q_0 \) such that the following hold:

(a) For all \( \alpha < \sigma_\ell \) and \( n < \ell \), \( f_n^{q^{\nu_\alpha}} = f_n^{q^{\nu_0}} \);
(b) For all \( \alpha < \sigma_\ell \) and \( n > \ell \), \( g_n^{q^{\nu_\alpha}} = r_n^{\nu_0} \);

Assume this construction has been carried for all \( \beta < \alpha \). Set \( s_0 := q_0 \) and let \( s_0 \) be a \( \sqsubseteq 1 \)-lower bound for \( \langle r^{\nu_\beta} \mid \beta < \alpha \rangle \), in case \( \alpha > 0 \). Observe that this is latter choice is plausible, as \( \alpha < \sigma_\ell \) and \( \langle r^{\nu_\beta} \mid \beta < \alpha \rangle \) is a 1-fusion sequence \( \sqsubseteq 1 \)-below \( p \) (cf. Remark 4.25).

Define \( q^{\nu_0} \) and \( r^{\nu_0} \) as follows. If \( P_0^{s_0} \cap \langle q^{\nu_0} \rangle \cap U \) is non empty let \( q^{\nu_0} \in P_0^{s_0} \cap \langle q^{\nu_0} \rangle \cap U \). Otherwise, \( q^{\nu_0} := s_0 \cap \langle q^{\nu_0} \rangle \). Define \( r^{\nu_0} := \langle r^{\nu_0} \mid n < \omega \rangle \) by

\[
r^{\nu_0}_n := \begin{cases} (f_n^{q^{\nu_0}}, \rho_n^p, h_n^0, h_n^1, h_n^2), & \text{if } n < \ell; \\
p_\ell, & \text{if } n = \ell; \\
g_n^{q^{\nu_0}}, & \text{if } n > \ell.
\end{cases}
\]

It is not hard to check that \( r^{\nu_0} \sqsubseteq r^{\nu_\beta} \) and that (a)-(b) are true for \( \alpha \).
After this recursion we obtain sequences \( \langle q^{\alpha} \mid \alpha < \sigma_\ell \rangle \) and \( \langle r^{\alpha} \mid \alpha < \sigma_\ell \rangle \) as above. Let us now check that the first of these witnesses \((1)-(4)\) of the statement. Certainly \((1)\) is true by construction and \((3)\) and \((4)\) follow from \((a)-(b)\) and from the fact that \( \langle r^{\alpha} \mid \alpha < \sigma_\ell \rangle \) is \( \leq 0 \)-decreasing. For \((2)\), assume that \( P^0_\alpha \cap U \neq \emptyset \). Then, since \( q^{\alpha} \leq 0 s_\alpha \backslash (\nu_\alpha) \), \( P^0_\alpha \cap (\nu_\alpha) \cap U \neq \emptyset \). By construction, \( q^{\alpha} \in U \), as wanted.

Let \( \langle q^{\alpha} \mid \alpha < \sigma_\ell \rangle \) be as in the above claim. For each \( n < \ell \), set \( f_n := \bigcup_{\alpha < \sigma_\ell} f^{q^\alpha}_n \). Refining \( A^\alpha_\ell \) if necessary, for each \( n < \ell \) and \( i < 3 \), we may also find a function \( h^n_i \) such that \( h^{i q^{\alpha}}_n = h_i^{q^\alpha} \), for all \( \alpha < \sigma_\ell \). For \( n < \ell \), set \( f^{q^n}_i := f_n \) and \( \rho_i^{q^n} := \rho^{q^n}_\alpha \) and \( q^{1}_{q^n} := (f^{q^n}_i, \rho^{q^n}_\alpha, h^{i n}_n, h^{i q^{\alpha}}_n) \), and for \( n > \ell \), we appeal to Lemma 4.16(2) to obtain \( q^{1}_{q^n} \in a^{1}_\ell \) be a \( \leq n_0 \)-lower bound for the sequence \( \langle q^{\alpha} \mid \alpha < \sigma_\ell \rangle \).

Now we define \( (a^{q^n}_i, A^{q^n}_i, f^{q^n}_i) \) by diagonalizing \( \langle f^{q^\alpha}_i \mid \alpha < \sigma_\ell \rangle \), i.e., as
\[
a^{q^n}_i := a^{q^n}_i, \quad A^{q^n}_i := A^{q^n}_i, \quad f^{q^n}_i := f^{q^n}_i \cup \bigcup_{\alpha < \sigma_\ell} (f^{q^\alpha}_i \upharpoonright \text{dom}(f^{q^\alpha}_i) \setminus a^{q^n}_i).
\]
Set \( \alpha_0 := \text{mc}(a^{q^n}_i \cap \mu) \). For \( \delta \in \pi_{mc(a^{q^n}_i), \alpha_0}[A^{q^n}_i] \), we arrange that \( \langle h^{2 \nu} \mid \pi_{mc(a^{q^n}_i), \alpha_0}(\nu) = \delta \rangle \) is a \( \leq \text{Col}(s(\delta))^{+, < \kappa_\ell} \)-decreasing, say with lower bound \( h_\delta \). We can do this since \( s(\delta)^+ \) corresponds to \( \kappa^{+3} \), which is above all the generators of \( E_\ell \). Define \( F^{2 \nu}_\ell (\delta) := h_\delta \).

It remains to define \( F^{i \nu}_\ell \) for \( i < 2 \). For each \( \nu \in A^{q^n}_i \), set \( t(\nu) := \langle h^{0 \nu}_\ell, h^{1 \nu}_\ell \rangle \) and note that \( j_{E_\ell}(t(\text{mc}(a^1_\ell))) \in V^{M_{E_\ell}}_{\kappa^{+1}_\ell} \). We use now the following claim from [Git19b]:

\textbf{Claim 4.26.2.} Let \( \alpha < \lambda \) and \( r : \kappa_\ell \to V_{\kappa_\ell} \) be such that \( j_{E_\ell}(r)(\alpha) \in V^{M_{E_\ell}}_{\kappa^{+1}_\ell} \). Then there are \( \alpha' < \mu \) and \( r' : \kappa_\ell \to V_{\kappa^{+1}_\ell} \), such that \( j_{E_\ell}(r)(\alpha) = j_{E_\ell}(r')(\alpha') \).

\textbf{Proof.} Let \( E' := E_\ell \upharpoonright \mu \), i.e. \( E' := \{ E_\ell, \beta \mid \beta < \mu \} \). Clearly, \( E' \) is a \( (\kappa_\ell, \mu) \)-extender so that \( k : M_{E'} \to M_{E_\ell} \) given by \( k(j_{E'}(f)(\alpha)) = j_{E_\ell}(f)(\alpha) \) is an elementary embedding. Set \( \theta := \mu \). Since all \( \beta < \mu \) are generators of \( E' \), \( \text{crit}(k) = (\mu^+)^{M_{E'}} \) and this is above \( \mu \), hence \( j_{E_\ell}(r)(\alpha) \in M_{E'} \). Then for some \( \alpha' < \mu \) and \( r' : \kappa_\ell \to V_{\kappa^{+1}_\ell} \), we have that
\[
j_{E_\ell}(r)(\alpha) = j_{E'}(r')(\alpha') = k(j_{E'}(r')(\alpha')) = j_{E_\ell}(r')(\alpha'). \quad \square
\]

Applying the claim to \( \text{mc}(a^{q^n}_i) \) and \( t \), and possibly by shrinking \( A^{q^n}_i \) and extending \( \alpha_0 \), we find a function \( t' \) with domain \( \pi_{mc(a^{q^n}_i), \alpha_0}[A^{q^n}_i] \), such that for all \( \nu \in A^{q^n}_i \), \( t(\nu) = t'(\pi_{mc(a^{q^n}_i), \alpha_0}(\nu)) \). Set \( \langle F^{0 \nu}_\ell (\delta), F^{1 \nu}_\ell (\delta) \rangle := t'(\delta) \), for every \( \delta \in \pi_{mc(a^{q^n}_i), \alpha_0}[A^{q^n}_i] \). Finally,
\[
q^{1}_{q^n} := (a^{q^n}_i, A^{q^n}_i, f^{q^n}_i, F^{0 q^n}_\ell, F^{1 q^n}_\ell, F^{2 q^n}_\ell).
\]

Define \( q^n := \langle q^n \mid n < \omega \rangle \). Notice that, by construction, for each \( \alpha < \sigma_\ell \), \( q^{\alpha} \subseteq (\nu_\alpha) \leq 0 q^{\alpha} \). We claim that \( q := \langle q^{\alpha} \mid n < \omega \rangle \) is as desired.

\textsuperscript{15}This is possible because all the Lévy collapses involved are of cardinality \( < \sigma_\ell \).
4.27 Let \( q_0 \leq q_1 \leq \alpha \) and by 0-openness of \( q_0 \), it follows from Claim 4.26.1(2) that \( q_0^\omega \in U \). Hence, by 0-openness of \( U \), \( w(q_1, r) = q_1^\omega \) is \( q_1 \) and \( \rho \). This completes the argument for \( m = 1 \).

For the general case assume by induction that \((*)_m\) holds and argue as before but below \( q_m \). This procedure will yield \( q_{m+1} \) satisfying \((*)_{m+1}\). □

Remark 4.27. Actually, in the conclusion of the above lemma, we get a condition \( q \approx p \).

We are finally in conditions to verify Clause (7):

**Lemma 4.28.** Let \( p \in P \) and \( U \) be a 0-open subset of \( P \). For every \( n < \omega \), there is \( q^n \leq p \), such that either \( P_n^q \cap U = \emptyset \) or \( P_n^q \subseteq U \). In particular, \( (P, \ell, c, \vec{\gamma}) \) witnesses Clause (7).

**Proof.** Let \( q \leq p \) be as in the conclusion of Lemma 4.26 and set \( \ell := \ell(p) \). We shall argue by induction that there is a \( \leq 0 \)-decreasing sequence of conditions \( \langle q^n : n < \omega \rangle \), \( q^0 := q \), such that for each \( 1 \leq n < \omega \), either \( W_n(q^n) \subseteq U \) or \( W_n(q^n) \cap U = \emptyset \). Let us simply describe how to do this for \( n = 2 \), as the case \( n = 1 \) is easy and the case \( n \geq 3 \) is similar. Assume that \( q^1 \) has been already defined. For all \( \nu \in A^q_1 \), define

\[
A^+_\nu := \{ \delta \in A^q_{\ell+1} \mid q^1 \land (\nu, \delta) \in U \} \quad \text{and} \quad A^-\nu := A^q_{\ell+1} \setminus A^+_\nu.
\]

For each \( \nu \in A^q_1 \), set \( A_\nu := A^+\nu \) as in \( E_{\ell+1, \text{mc}(q^1_{\ell+1})} \), and \( A^-\nu := A^-\nu \) otherwise. Let \( A^+ := \{ \nu \in A^q_1 \mid A_\nu = A^+\nu \} \), and \( A^- := \{ \nu \in A^q_1 \mid A_\nu = A^-\nu \} \). Let \( A_\ell := A^+ \) if \( A^+ \in E_{\ell, \text{mc}(q^1)} \), and \( A_\ell := A^- \) otherwise. Let \( q^2 \leq 0 q^1 \) be such that \( A_{\ell+1}^q := A_\ell \) and \( A_{\ell+1}^q := \bigcap_{\nu \in A^q_1} A_\nu \). Then \( q^2 \) is as desired.

Once we have defined \( \langle q^n : n < \omega \rangle \), let \( q^* \) be a \( \leq 0 \)-lower bound (cf. Definition 3.3(2)). Let \( n < \omega \) and \( r \in P_n^q \cap U \). By Lemma 4.26, \( w(q, r) \in U \), and by 0-openness of \( U \), \( w(q, r) \in U \), hence \( W_n(q^n) \subseteq U \). Thus, again by 0-openness, \( W_n(q^n) \subseteq U \). It then follows that \( P_n^q \subseteq U \). □

The following lemma takes care of Clause (9):

**Lemma 4.29.** For each \( n < \omega \), \( P_n^{\sigma_0} \) is \( \sigma_n \)-directed-closed.

**Proof.** Since \( P_0^{\sigma_0} = \{ 1 \} \) the result is clearly true. Let \( n \geq 1 \) and \( D \subseteq P_n^{\sigma_n} \) be a directed set of size \( < \sigma_n \). By definition,

\[
\varpi_n[D] = \{ (s_k^p, h_{k}^0, h_{k}^1, h_{k}^2) : k < n \},
\]

for some \( p \in D \). Arguing similarly to the proof of Lemma 4.20 but with respect to \( \varpi_n[D] \) we obtain lower bound \( q \) for \( D \) such that \( \varpi_n(q) \subseteq \varpi_n[D] \). Thus, \( q \) is a \( \leq \bar{\sigma} \)-lower bound for \( D \). □

The proof of the following is identical to [PRS20, Corollary 3.24].
Corollary 4.30. $\mathbf{1}_P \models_P \mu = \kappa^+$. □

Combining all the previous lemmas we finally arrive at the desired result:

Corollary 4.31. $(\mathbb{P}, \ell, c, \vec{\omega})$ is $(\Sigma, \vec{S})$-Prikry. □

We close this section proving some relevant properties that hold after forcing with $\mathbb{P}$:

Proposition 4.32. All of the following hold in $V^P$:

1. All cardinals $\geq \kappa$ are preserved;
2. $\kappa = \aleph_\omega$, $\mu = \aleph_{\omega+1}$ and $\lambda = \aleph_{\omega+2}$;
3. $\aleph_\omega$ is a strong limit cardinal;
4. $\text{GCH}_{\leq \kappa}$, provided that $V \models \text{GCH}_{<\kappa}$;
5. $2^{\aleph_\omega} = \aleph_{\omega+2}$, hence the $\text{SCH}_{\aleph_\omega}$ fails.

Proof. (1) The fact that $\kappa$ is preserved follows from Lemma 3.12(2). Likewise, $\kappa^+$ is preserved by combining Lemma 3.12(4) with the fact that $|W(p)| \leq \kappa$, for each $p \in \mathbb{P}$ (see page 22). Finally, cardinals $\geq \kappa^+$ are preserved by Lemma 4.21.

(2) Let $G$ a $\mathbb{P}$-generic over $V$. By standard arguments,

$$\text{CARD}^{|G|} \cap \kappa = \bigcup_{n<\omega} \{\sigma_n, \rho_n, (s_n(\rho_n)^+)^V, (s_n(\rho_n)^{++})^V, \kappa_n\},$$

where, for each $n < \omega$, $\rho_n = \rho^{p_n}$, for some (all) $p \in G \cap P_{>n}$. Thus, in $V[G]$, $\kappa$ becomes the $\omega$-successor of $\sigma_0 = \aleph_1$, i.e., $\kappa = \aleph_\omega$. By Corollary 4.30, $\mathbf{1}_P \models \mu = \aleph_{\omega+1}$. Finally, since $\lambda$ is preserved it is forced to be $\aleph_{\omega+2}$.

(3) This follows combining Lemma 3.12(2) and Clause (2).

(4) Suppose that $V \models \text{GCH}_{<\kappa}$. Towards a contradiction, suppose that $p \models \text{CH}_{\aleph_n}$, for some $p \in P$ and $n < \omega$. Then, using Lemma 3.12(1), there are $m \in \omega \setminus n$ and $s \leq_n \omega_n(p)$ such that $s \models \text{CH}_{\aleph_n}$. But by Remark 4.14, $\mathbf{1}_P \models \text{CH}_{\aleph_n}$. Finally, since $\lambda$ is preserved it is forced to be $\aleph_{\omega+2}$.

(5) Let $Q$ be denote EBPF defined with respect to the sequence of extenders $\langle E_{n, \alpha} \mid n < \omega, \alpha < \lambda \rangle$. By [Git10, Theorem 2.23], $Q$ forces $“2^\kappa \geq \kappa^{++}“$. It is not hard to check that $\mathbb{P}$ projects onto $Q$, hence $\mathbb{P}$ forces the same. Actually, by counting nice names, $\mathbf{1}_P \models “\text{GCH}_{\aleph_\omega}“$. Now item (1) and (2) combined yield the desired result. □

4.3. EBPF is suitable for reflection. In this section we show that $(\mathbb{P}_n, \mathbb{S}_n, \varpi_n)$ is suitable for reflection with respect to a relevant sequence of cardinals. Our setup will be the same as the one from page 16 and we will also rely on the notation established in page 19. The main result of the section is Corollary 4.37, which will be preceded by a series of technical lemmas. The first one is essentially due to Sharon [Sha05], but we give details for completeness.

Lemma 4.33. For each $n < \omega$, $V^{Q^*_n} \models |\mu| = \text{cf}(\mu) = \kappa_n$. 
Proof. By Lemma 4.16, $Q_{n0}^*$ preserves cofinalities $\leq \kappa_n$, and by the Linked0- property (see [PRS20, Lemma 3.12]) it preserves cardinals $\geq \mu^+$.

Next we show that $Q_{n0}^*$ collapses $\mu$ to $\kappa_n$. For each condition $p \in Q_{n0}^*$, denote $p \coloneqq (a^P, A^P, f^P)$. Let $G$ be $Q_{n0}^*$-generic and set $a \coloneqq \bigcup_{p \in G} a^P$. By a density argument, $\text{otp}(a \cap \mu) = \kappa_n$, and so $\mu$ is collapsed. By a result of Shelah (see, e.g. [CFM01, Fact 4.5]), this implies that $V^{Q_{n0}^*} \models \text{cf}(|\mu|) = \text{cf}(\mu)$.

□

Lemma 4.34. For each $n < \omega$, $V^{Q_{n0}} \models |\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^+$.

Proof. Observe that for each $p \in Q_{n0}$ and $i < 3$, $F^p_i$ represents a condition in some Lévy collapse as calculated in $M_{n,\alpha_p} \simeq \text{Ult}(V, E_{n,\alpha_p})$. Specifically, $[F^0]\mathcal{E}_{n,\alpha_p}$, $[F^1]\mathcal{E}_{n,\alpha_p}$ and $[F^2]\mathcal{E}_{n,\alpha_p}$ are conditions in $\text{Col}(\sigma_n, < \kappa_n)^{M_{n,\alpha_p}}$, $\text{Col}(\kappa_n, \kappa^+_n)^{M_{n,\alpha_p}}$ and $\text{Col}(\kappa^+_n, < j_n(\kappa_n))^{M_{n,\alpha_p}}$, respectively. The first of these forcings is nothing but $\text{Col}(\sigma_n, < \kappa_n)^V$.

Set $C_n \coloneqq \{\langle F^1_i, F^2_i \rangle \mid p \in Q_{n0}\}$ and define $\square$ as follows:

$\langle F^1_i, F^2_i \rangle \subseteq \langle F^1_j, F^2_j \rangle$ iff $\forall i \in \{1, 2\}$ $j_n(F^1_i)(\alpha_p) \geq j_n(F^1_j)(\alpha_p)$.

Clearly, $\square$ is transitive. Set $C_n \coloneqq (C_n, \square)$.

Claim 4.34.1. $C_n$ is $\kappa_n$-directed closed. Furthermore, if $D \subseteq C_n$ is a directed set of size $< \kappa_n$ and $\alpha < \lambda \leq \kappa$ is $\leq E_n$-above all the ordinals in $\{\alpha_p \mid p \in D\}$, then there is $\square$-lower bound $\langle F^1, F^2 \rangle \in C_n$ for $D$ with $\text{dom}(F^1) = \text{dom}(F^2) \in E_n, \alpha$.

Proof. Let $D \subseteq C_n$ be a directed set of size $< \kappa_n$. Let $\alpha$ be an ordinal in $\kappa^+ \setminus \bigcup_{p \in D} \text{dom}(F^p)$ which is $\leq E_n$-above all the ordinals in $\bigcup_{p \in D} (a^P \cap \kappa^+)$. This is possible because the poset $\langle \kappa^+, \leq E_n \rangle_{\kappa^+}^{\kappa^+}$ is $\kappa_n$-directed, $|\bigcup_{p \in D} (a^P \cap \kappa^+)| < \kappa_n$ and $|\bigcup_{p \in D} \text{dom}(F^p)| \leq \kappa$ (cf. Remark 4.3). Similarly, we may let $\beta$ an ordinal in $\lambda \setminus \{\alpha\} \cup \bigcup_{p \in D} \text{dom}(f^p)$ which is $\leq E_n$-above all ordinals in $\{\alpha\} \cup \bigcup_{p \in D} \mathcal{E}_n$.

Recall that this is possible because the poset $\langle \lambda, \leq E_n \rangle$ is $\kappa_n$-directed.

Set $b \coloneqq \{\alpha, \beta\} \cup \bigcup_{p \in D} \mathcal{E}_n$. For each $i \in \{1, 2\}$ and $p \in D$, $j_n(F^p_i)(\alpha_p) \coloneqq j_n(F^p \circ \pi_{\alpha, \alpha_p})(\alpha) \in M_{n, \alpha}$. Thus, for each $i \in \{1, 2\}$, $D_i \coloneqq \{j_n(F^p_i)(\alpha_p) \mid p \in D\}$ is a directed family of size $< \kappa_n$ of conditions in some $\kappa_n$-directed closed Lévy collapse defined in $M_{n, \alpha}$. Let $q^i$ be a $\square$-lower bound for $D_i$ in $M_{n, \alpha}$. Clearly, $q_i = j_n(F^i)(\alpha)$, for some function $F^i$ with $\text{dom}(F^i) \in E_n, \alpha$. Set $B := \pi^{1}_{\beta, \alpha}[\text{dom}(F^0) \cap \text{dom}(F^1)]$. For each $\nu \in \pi_{\beta, \alpha}[B]$, define $G^0(\nu) := 1_{\text{Col}(\sigma_n, < \kappa_n, \nu)}$, and for $i \in \{1, 2\}$, $G^i(\nu) := F^i(\nu)$.

Clearly, $q := (b, B, \emptyset, G^0, G^1, G^2) \in Q_{n0}$, hence $\langle G^1, G^2 \rangle = \langle F^1_q, F^2_q \rangle \in C_n$ is the desired $\square$-lower bound for $D$.

Let $G$ be a $Q_{n0}$-generic filter over $V$ and denote by $G^*$ the $Q_{n0}^*$-generic filter generated by $G$ and the projection $g_n$ of Lemma 4.16(1). By Lemma 4.33,

\footnote{Here we use that $\kappa_n M_{n, \alpha_p} \subseteq M_{n, \alpha_p}$.}
$V[G^\ast] \models |\mu| = \text{cf}(\mu) = \kappa_n$. It thus remains to check that $\kappa_n$ is preserved and turned into $(\sigma_n)^{+}$.

**Claim 4.34.2.** If $Q_{n_0}/G^\ast$ and $Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$ are forcing equivalent over $V[G^\ast]$, then $V[G] \models |\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^{+}$.

**Proof.** Since $Q_{n_0}^{\ast}$ is $\kappa_n$-directed closed, the collapses $Col(\sigma_n, \langle \kappa_n \rangle)^V$ and $Col(\sigma_n, \langle \kappa_n \rangle)^{V[G^\ast]}$ are the same. Also, $\mathbb{C}_n$ is $\kappa_n$-directed closed over $V[G^\ast]$. By Easton’s lemma, $Col(\sigma_n, \langle \kappa_n \rangle)^V$ is $\kappa_n$-cc over $V[G^\ast]$, hence $Col(\sigma_n, \langle \kappa_n \rangle)^V$ forces “$|\mu| = \kappa_n = (\sigma_n)^{+}$” over $V[G^\ast]$.

Appealing again to Easton’s lemma we have that $\mathbb{C}_n$ is $\kappa_n$-distributive in any generic extension of $V[G^\ast]$ given by $Col(\sigma_n, \langle \kappa_n \rangle)^V$. Altogether, $Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$ yields a generic extension of $V[G^\ast]$ where “$|\mu| = \kappa_n = (\sigma_n)^{+}$” is true. Since cardinals above $\mu$ are preserved, we also have that in this extension $\text{cf}(\mu) = \kappa_n$. The claim now follows using the forcing equivalence between $Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$ and $Q_{n_0}/G^\ast$ over $V[G^\ast]$. \[\square\]

**Claim 4.34.3.** The map $e: Q_{n_0}/G^\ast \rightarrow Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$ defined by $p \mapsto \langle j_n(F_0^p)(\alpha_p), \langle F_1^p, F_2^p \rangle \rangle$ is a dense embedding. In particular, $Q_{n_0}/G^\ast$ and $Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$ are forcing equivalent.

**Proof.** Fix $p, q \in Q_{n_0}/G^\ast$.

- **Assume** $p \leq_{Q_{n_0}/G} q$. By definition, for each $i < 3$ and each ordinal $\nu \in \text{dom}(F_0^p)$,

  \[F_i^p(\nu) \supseteq F_i^{q}(\pi_{\alpha_p,\alpha_q}(\nu)).\]

  Since $\alpha_q \leq E_n \alpha_p$, $j_n(\pi_{\alpha_p,\alpha_q})(\alpha_p) = \alpha_q$, hence $j_n(F_0^p)(\alpha_p) \supseteq j_n(F_0^q)(\alpha_q)$. This shows that $e(p) \leq_{Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n} e(q)$.

- **Assume** $p \not\leq_{Q_{n_0}/G} q$. Since $\varrho_n(p), \varrho_n(q) \in G^\ast$ we may let $r \in G^\ast$ be such that $r \leq_{Q_{n_0}} \varrho_n(p), \varrho_n(q)$. Enlarging $r$ if necessary, we may further assume that $a^r \cap \mu$ contains a $\leq_{E_n}$-greatest element $\alpha_r$. For some $i < 3$,

  \[\{ \nu < \alpha_r \mid F_i^p(\pi_{\alpha_r,\alpha_p}(\nu)) \cup F_i^{q}(\pi_{\alpha_r,\alpha_q}(\nu)) \text{ is not a function} \} \in E_{n,\alpha_r},\]

  for otherwise it would be easy to cook up a condition $\tilde{r} \leq_{Q_{n_0}/G} p, q$. Thus, for some $i < 3$, $j_n(F_i^p)(\alpha_p) \cup j_n(F_i^q)(\alpha_q)$ is not a function. This shows that $e(p)$ and $e(q)$ are incompatible, as desired.

We are now left with checking that the range of $e$ is dense in the product $Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$. Actually we prove that $e$ is surjective.

Fix $\langle c, \langle F_1, F_2 \rangle \rangle \in Col(\sigma_n, \langle \kappa_n \rangle)^V \times \mathbb{C}_n$. By definition of $\mathbb{C}_n$, there is $p \in Q_{n_0}$ such that $F_i = F_i^p$, for $i \in \{1, 2\}$. Also, since $Col(\sigma_n, \langle \kappa_n \rangle)^V = Col(\sigma_n, \langle \kappa_n \rangle)^{M_n,\alpha_p}$ we may let a function $F^0$ with $\text{dom}(F^0) \in E_{n,\alpha_p}$ such that $j_n(F^0)(\alpha_p) = c$. Set $q := (a^p, A^p, f^p, F^0, F_1^p, F_2^p)$. Clearly, $q \in Q_{n_0}$ and $e(q) = \langle c, \langle F_1, F_2 \rangle \rangle$, as wanted. \[\square\]

This completes the proof.

**Lemma 4.35.** For each positive $n < \omega$, $\prod_{i<n} Q_i$ is isomorphic to a product of $S_n$ with some $\mu$-directed-closed forcing.
Proof. If $n = 0$ there is nothing to prove. For $n \geq 1$, it is routine to check that the map $p \mapsto \langle (f^p_i)_{i < n}, (\langle \rho^p_i, h^0_{p_i}, h^1_{p_i}, h^2_{p_i} \rangle \mid i < n) \rangle$ defines the desired isomorphism. \hfill \Box

Lemma 4.36. For each $n < \omega$, $V^{P_n} \models |\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^+.$

Proof. Observe that $P_n$ is the subforcing of conditions $p \in \prod_{i < n} Q_{i1} \times \prod_{i \geq n} Q_{i0}$ satisfying $a^{p_i} \subseteq a^{p_{i+1}}$, for each $i \geq n$. Certainly, this a dense subposet of the product. Thus, both forcing produce the same generic extension. On the other hand, Lemma 4.34 yields

$$V^{Q_{00}} \models |\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^+.$$ 

Since $Q_{n0}$ is $\sigma_n$-directed-closed, Easton’s lemma, Lemma 4.16 and Lemma 4.35 together imply that $Q_{n0}$ forces $\prod_{i < n} Q_{i1}$ to be a product of a $(\kappa_{n-1})$-cc forcing times a $\kappa_n$-distributive forcing. Similarly, $Q_{n0}$ forces $\prod_{i > n} Q_{i0}$ to be $\kappa_n$-directed-closed. Thereby, forcing with $\prod_{i \neq n} Q_{i\chi(i)}$ over $V^{Q_{00}}$ preserves the above cardinal configuration and thus the result follows. \hfill \Box

As a consequence of the above we get the main result of the section:

Corollary 4.37. For each $n \in \omega \setminus 2$, $(P_n, S_n, \mathcal{S}_n)$ is suitable for reflection with respect to the sequence $(\sigma_{n-1}, \kappa_{n-1}, \kappa_n, \mu)$.

Proof. We go over the clauses of Definition 2.11. Clause (1) is obvious. Clause (2) follows from Lemma 4.22 and Lemma 4.29. Clause (4) follows from Remark 4.14. Now, let us address Clause (3). That $P_n$ forces "$|\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^+"$ follows from Lemma 4.36.\footnote{Recall that $\sigma_n = (\kappa_{n-1})^+$.} Then, by Remark 4.23, $S_n \times P_n$ forces the same. \hfill \Box

We conclude this section, establishing two more facts that will be needed for the proof of the Main Theorem in Section 8.

Definition 4.38. For every $n < \omega$, let $T_n := S_n \times \text{Col}((\kappa_{n-1})^+, \kappa_n)$, and let $\psi_n : P_n \rightarrow T_n$ be the map defined via

$$\psi_n(p) := \begin{cases} (\mathcal{S}_n(p), j_n(F^{p_{n0}})(\alpha_{p_n})), & \text{if } \ell(p) > 0; \\ (1_{\mathcal{S}_n}, \emptyset), & \text{otherwise.} \end{cases}$$

Lemma 4.39. Let $n < \omega$.

(1) $T_n$ is a $\kappa_n$-cc poset of size $\kappa_n$;
(2) $\psi_n$ defines an exact nice projection;
(3) $(P_n)_{\psi_n}$ is $\kappa_n$-directed-closed;
(4) for each $p \in P_n$, $P_n \downarrow p$ and $(T_n \downarrow \psi_n(p)) \times ((P_n)_{\psi_n} \downarrow p)$ are forcing equivalent.

Proof. (1) $T_n$ is the product of a small forcing with $\text{Col}((\kappa_{n-1})^+, \kappa_n)$, and $\kappa_n$ is strongly inaccessible.
Let us go over the clauses of Definition 2.1. Clearly, \( \psi_n(\mathbb{1}_p) = \langle \mathbb{1}_{\kappa_n}, \emptyset \rangle \), and so Clause (1) holds. Using that \( \mathcal{w}_n \) is order-preserving, it is routine to check that so is \( \psi_n \). Thus, Clause (2) follows.

Let \( p \in P_n \) and \( t \leq_{S_n \times \text{Col}((\kappa_n-1)^+, \kappa_n)} \psi_n(p) \), say \( t = \langle s, c \rangle \). By definition, \( s \leq_n \mathcal{w}_n(p) \) and \( c \geq j_n(F^{0p_n})(\alpha_{p_n}) \). On one hand, since \( \mathcal{w}_n \) is a nice projection, the condition \( q := p + s \) exists. On the other hand, let \( F \) be a function with domain \( \pi_{\text{mc}(\alpha_{p_n}), \alpha_{q_n}}[A^{q_n}] \) such that \( j_n(F)(\alpha_{q_n}) = c \). Let \( r \) be a condition in \( \mathbb{P}_n \) with the same entries as \( q \) but with \( F^{0r_n} := F \). We claim that \( r \) is nothing but \( p + t \).

Clearly, \( r \leq p \). Also, by the way \( r \) is defined,

\[
\psi_n(r) = \langle \mathcal{w}_n(r), c \rangle = \langle \mathcal{w}_n(q), c \rangle \leq_{S_n \times \text{Col}((\kappa_n-1)^+, \kappa_n)} \langle s, c \rangle = t,
\]

where the rightmost inequality follows from the fact that \( q = p + s \).

Let \( u \in P_n \) be such that \( u \leq p \) and \( \psi_n(u) \leq_{S_n \times \text{Col}((\kappa_n-1)^+, \kappa_n)} t \). In particular, \( \mathcal{w}_n(u) \leq_n s \) and \( j_n(F^{0u_n})(\alpha_{u_n}) \supseteq c = j_n(F)(\alpha_{q_n}) \), and so

\[
F^{0u_n}(\nu) \supseteq F(\pi_{\alpha_{u_n}, \alpha_{q_n}}(\nu)) = F^{0r_n}(\pi_{\alpha_{u_n}, \alpha_{q_n}}(\nu)),
\]

for each \( \nu \in \pi_{\text{mc}(\alpha_{u_n}), \alpha_{u_n}}[A^{u_n}] \). Clearly, \( u \leq p + s \) and so, by the above expression, \( u \leq r \), as wanted.

Altogether, \( r = p + t \). Also, by (*) and exactness of \( \mathcal{w}_n \), \( \psi_n(r) = \langle \mathcal{w}_n(q), c \rangle = \langle s, c \rangle = t \). Thus, \( \psi_n \) is an exact nice projection.

(3) Let \( D \subseteq \mathbb{Q}_{\kappa_n}^\psi \) be a directed set of size \( < \kappa_n \). In particular, \( \psi_n[D] = \{ \langle s, c \rangle \} \) for some \( \langle s, c \rangle \in S_n \times \text{Col}((\kappa_n-1)^+, \kappa_n) \). In particular, for each condition \( p \in D \), \( j_n(F^{0p_n})(\alpha_{p_n}) = c \). Arguing as usual, let \( a \in [\lambda]^{<\kappa_n} \) be such that both \( a \cap \kappa^+ \) and \( a \) have \( \leq_{E_{\kappa_n}} \) order-preservation, \( \alpha \) and \( \beta \) respectively, and \( a \supseteq \bigcup_{p \in D} a^{p_n} \). Thus, for each \( p, q \in D \), \( c = j_n(F^{0p_n} \circ \pi_{\alpha, \alpha_{p_n}})(\alpha) \) and

\[
B_{p, q} := \{ \nu < \alpha \mid F^{0p_n}(\pi_{\alpha, \alpha_{p_n}}(\nu)) = F^{0q_n}(\pi_{\alpha, \alpha_{q_n}}(\nu)) \} \in E_{\kappa_n, \alpha}.
\]

Set \( A := \pi_{\beta, \alpha}^{-1}[B] \). By shrinking \( A \) if necessary, we may further assume \( \pi_{\beta, \alpha}(\alpha_{p_n})[A] \subseteq A^{p_n} \), for each \( p \in D \). Since \( \psi_n \upharpoonright D \) is constant the map \( \mathcal{w}_n : p \mapsto \langle (\rho_{p_i}^{h_i}, h_{p_i}^{p_i}, h_{1}^{p_i}, h_{2}^{p_i}) \mid i < n \rangle \) is so. Let \( \langle (\rho_k, h_k^1, h_k^2) \mid k < n \rangle \) be such constant value. For each \( k < \omega \), set \( f_k := \bigcup_{p \in D} f^{p_n} \) and \( F^0 \) be such that \( \text{dom}(F^0) = B \) and \( F^0(\nu) := F^{0p_n}(\pi_{\alpha, \alpha_{p_n}}(\nu)) \), for some \( p \in D \).

Observe that \( \{ F^{1p_n}, F^{2p_n} \mid p \in D \} \) forms a directed subset of \( \mathbb{C}_n \) of size \( < \kappa_n \) (cf. Lemma 4.34). Using the \( \kappa_n \)-directed-closedness of \( \mathbb{C}_n \) we may let \( \langle F^1, F^2 \rangle \in C_n \) be a \( \subseteq \)-lower bound. Moreover, we may assume that \( \text{dom}(F^1) \subseteq \bigcup_{n, \alpha} E_{n, \alpha} \) and so, by shrinking if necessary, \( \text{dom}(F^1) = \bigcup_{n, \alpha} E_{n, \alpha} \).

\[18\] See the proof of Claim 4.34.3.

\[19\] Notice that here we do not need that \( \mathcal{w}_n \) is exact.
Define \( p^* := \langle p_k^* \mid k < \omega \rangle \) as follows:

\[
p_k^* := \begin{cases} 
(f_k, p_k, h_k^0, h_k^1, h_k^2), & \text{if } k < n; \\
(a, A, f_k, F_k^0, F_k^1, F_k^2), & \text{if } k = n; \\
(a_k, A_k, f_k, F_k^0, F_k^1, F_k^2), & \text{if } k > n,
\end{cases}
\]

where \((a_k, A_k, f_k, F_k^0, F_k^1, F_k^2)\) is constructed as described in Lemma 4.20. Clearly, \( p^* \in \mathcal{P}_{\omega_1} \) and gives a lower bound for \( D \).

(4) By Lemma 2.6(2), \((T_n \downarrow \psi_n(p)) \times ((\mathcal{P}^{\psi_n})_n \downarrow p)\) projects onto \( \mathcal{P}_n \downarrow p \). In this particular case, the above projection map is moreover an isomorphism.

\[ \Box \]

Lemma 4.40. Assume GCH. Let \( n < \omega \).

1. \( \mathcal{P}_n \) is \( \mu^+ \)-Linked;
2. \( \mathcal{P}_n \) forces \( \text{CH}_\theta \) for any cardinal \( \theta \geq \sigma_n \);
3. \( \mathcal{P}^{\mu^+_n} \) preserves the GCH.

Proof.

1. By Definition 4.12, Lemma 4.21 and the fact that \( |H_\mu| = \mu \).

2. As \( \mathcal{P}_n \) has size \( \leq \mu^+ \), Clause (1) together with a counting-of-nice-names argument implies that \( 2^{\theta} = \theta^+ \) for any cardinal \( \theta \geq \mu^+ \). By Lemma 4.36, in any generic extension by \( \mathcal{P}_n \), \( |\mu| = cf(\mu) = \kappa_n = (\sigma_n)^+ \). It thus left to verify that \( \mathcal{P}_n \) forces \( 2^{\theta} = \theta^+ \) for \( \theta \in \{\sigma_n, \kappa_n\} \).
   - The number of nice names for subsets of \( \kappa_n \) is \( (\mu^+)\kappa_n = \mu^+ \), and hence \( \text{CH}_{\kappa_n} \) is forced by \( \mathcal{P}_n \).
   - By Clauses (1), (3) and (4) of Lemma 4.39, together with Easton’s lemma, \( \mathcal{P}_n \) forces \( \text{CH}_{\sigma_n} \) if and only if \( \mathcal{T}_n \) forces \( \text{CH}_{\sigma_n} \). By Clause (1) of Lemma 4.39, \( \mathcal{T}_n \) is a \( \kappa_n \)-cc poset of size \( \kappa_n \), so, the number of \( \mathcal{T}_n \)-nice names for subsets of \( \sigma_n \) is at most \( \kappa_n^{<\kappa_n} = \kappa_n = \sigma_n^+ \), as wanted.

3. By Lemma 4.29, \( \mathcal{P}^{\mu^+_n} \) preserves GCH below \( \sigma_n \). By Remark 4.23 and the the fact that \( \mathcal{S}_n \) has size \( < \sigma_n \), we infer from Clause (2) that GCH holds at cardinals \( \geq \sigma_n \), as well.

\[ \Box \]

5. Exact forking projections

In this short section we introduce the notion of exact forking projection, a strengthening of the following key concept from Part I of this series:

Definition 5.1 ([PRS19, §4]). Suppose that \((\mathcal{P}, \ell_\mathcal{P}, c_\mathcal{P})\) is a \( \Sigma \)-Prikry triple, \( \mathcal{A} = (A, \leq) \) is a notion of forcing, and \( \ell_\mathcal{A} \) and \( c_\mathcal{A} \) are functions with \( \text{dom}(\ell_\mathcal{A}) = \text{dom}(c_\mathcal{A}) = A \).

A pair of functions \((\cdot, \pi)\) is said to be a forking projection from \((\mathcal{A}, \ell_\mathcal{A})\) to \((\mathcal{P}, \ell_\mathcal{P})\) iff all of the following hold:

1. \( \pi \) is a projection from \( \mathcal{A} \) onto \( \mathcal{P} \), and \( \ell_\mathcal{A} = \ell_\mathcal{P} \circ \pi \);
2. for all \( a \in A \), \( \ell(a) \) is an order-preserving function from \((\mathcal{P} \downarrow \pi(a), \leq)\) to \((\mathcal{A} \downarrow a, \leq)\);
3. for all \( p \in \mathcal{P} \), \( \{a \in A \mid \pi(a) = p\} \) admits a greatest element, which we denote by \([p]_\mathcal{A}^\mathcal{A} \).
(4) for all $n, m < \omega$ and $b \leq_a^{n+m} a$, $m(a, b)$ exists and satisfies:

$$m(a, b) = \exists\pi(a)(m(\pi(a), \pi(b)))$$

(5) for all $a \in A$ and $r \leq \pi(a)$, $
\exists\pi(a)(\pi(\exists\pi(a))(r)) = r$;

(6) for all $a \in A$ and $r \leq \pi(a)$, $a = [\pi(a)]^\exists\pi(a)$ iff $\exists\pi(a)(r) = [r]^\exists\pi(a)$;

(7) for all $a \in A$, $a' \leq_\exists\pi(a)$ and $r \leq_\exists\pi(a')$ $\exists\pi(a')(r) \leq \exists\pi(a)(r)$.

The pair $(\exists\pi, \pi)$ is said to be a forking projection from $(A, \ell_A, c_A)$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P})$ iff, in addition to all of the above, the following holds:

(8) for all $a, a' \in A$, if $c_A(a) = c_A(a')$, then $c_\mathbb{P}(\pi(a)) = c_\mathbb{P}(\pi(a'))$ and, for all $r \in P_0^{\pi(a)} \cap P_0^{\pi(a')}$, $\exists\pi(a)(r) = \exists\pi(a')(r)$.

In [PRS20, §2], we drew a map of connections between $\Sigma$-Prikry forcings and forking projection, demonstrating that this notion is crucial to define a viable iteration scheme for $\Sigma$-Prikry posets. We now turn to quickly collect analogous results for exact forking projection and $(\Sigma, S)$-Prikry forcings.

**Setup 5.** Throughout the rest of this section, we suppose that:

- $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $1_\mathbb{P}$;
- $\mathbb{A} = (A, \subseteq)$ is a notion of forcing with a greatest element $1_\mathbb{A}$;
- $\Sigma = \langle \sigma_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal $\kappa$, and $\mu$ is a cardinal such that $1_\mathbb{P} \Vdash \mu = \kappa^+$;
- $S = \langle S_n \mid n < \omega \rangle$ is a sequence of notions of forcing, $S_n = (S_n, \leq_n)$, with $|S_n| < \sigma_n$;
- $\ell_\mathbb{P}$, $c_\mathbb{P}$ and $\bar{\omega} = \langle \omega_n \mid n < \omega \rangle$ are such that $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P}, \bar{\omega})$ is $(\Sigma, S)$-Prikry;
- $\ell_A$ and $c_A$ are functions with $\text{dom}(\ell_A) = \text{dom}(c_A) = A$, and $\bar{\omega} = \langle \omega_n \mid n < \omega \rangle$ is a sequence of functions.

**Definition 5.2.** A pair of functions $(\exists\pi, \pi)$ is said to be an exact forking projection from $(A, \ell_A, c_A)$ to $(\mathbb{P}, \ell_\mathbb{P}, \bar{\omega})$ iff all of the following hold:

(a) Clauses (1)–(7) of Definition 5.1 with respect to $(A, \ell_A)$ and $(\mathbb{P}, \ell_\mathbb{P})$;
(b) $\pi$ is an exact nice projection from $A$ onto $\mathbb{P}$. Furthermore, for all $a \in A$ and $p \leq_\pi a$, $a + p = \exists\pi(a)(p)$;
(c) $\bar{\omega} = \bar{\pi} \cdot \pi$, that is, $\omega_n = \omega_n \circ \pi$ for all $n < \omega$.

The pair $(\exists\pi, \pi)$ is said to be an exact forking projection from $(A, \ell_A, c_A, \bar{\omega})$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P}, \bar{\omega})$ if, in addition, Clause (8) of Definition 5.1 is satisfied.

**Remark 5.3.** If $(\exists\pi, \pi)$ is an exact forking projection as above, then each $\omega_n$ is a nice exact projection from $A \geq n$ to $S_n$. Moreover, for each $k \geq n$, $\omega_n \mid A_k$ is again an exact nice projection.

We now turn to collect sufficient conditions — assuming the existence of an exact forking projection from $(A, \ell_A, c_A, \bar{\omega})$ to $(\mathbb{P}, \ell_\mathbb{P}, c_\mathbb{P}, \bar{\omega})$ — for $(A, \ell_A, c_A, \bar{\omega})$ to be $(\Sigma, S)$-Prikry on its own, and then address the problem of ensuring that the $A_n$’s be suitable for reflection. This study will be
needed in Section 6, most notably, in the proof of Theorem 6.9. A reader proficient with Parts I and II of this series is encouraged to skip directly to Lemma 5.14 below.

**Lemma 5.4** (Canonical form). Suppose that $(\mathbb{P}, \ell_F, c_F, \mathfrak{F})$ and $(A, \ell_A, c_A, \mathfrak{F})$ are $(\Sigma, \mathfrak{F})$-Prikry notions of forcing. Denote $\mathbb{P} = (P, \leq)$ and $\mathbb{A} = (A, \leq)$.

If $(A, \ell_A, c_A, \mathfrak{F})$ admits an exact forking projection to $(\mathbb{P}, \ell_F, c_F, \mathfrak{F})$ as witnessed by a pair $(\hat{\phi}, \pi)$, then we may assume that all of the following hold true:

1. each element of $A$ is a pair $(x, y)$ with $\pi(x, y) = x$;
2. for all $a \in A$, $[\pi(a)]^A = (\pi(a), \emptyset)$;
3. for all $p, q \in P$, if $c_F(p) = c_F(q)$, then $c_A([p]^A) = c_A([q]^A)$.

**Proof.** This is proved as in [PRS20, Lemma 2.16]. \hfill \Box

**Lemma 5.5.** Suppose that $(\hat{\phi}, \pi)$ is a forking projection from $(A, \ell_A)$ to $(\mathbb{P}, \ell_F)$, and that Clauses (1), (5), (7), (8) and (9) of Definition 3.3. are valid for $(A, \ell_A)$. If $\mathbb{1}_P \Vdash_P "\hat{\kappa} is singular",$ then $\mathbb{1}_A \Vdash_A \hat{\mu} = \hat{\kappa}^+$.

**Proof.** This is proved as in [PRS19, Corollary 4.13], this time appealing to Lemma 3.12(3), rather than to [PRS19, Lemma 2.10(2)]. \hfill \Box

We now go over the clauses of Definition 3.3, starting with Clause (1).

**Fact 5.6** ([PRS19, Lemma 4.3 and 4.5]). Suppose that $(\hat{\phi}, \pi)$ is a forking projection from $(A, \ell_A)$ to $(\mathbb{P}, \ell_F)$, or, just a pair of maps satisfying Clauses (1), (2) and (4) of Definition 5.1. For each $a \in A$, the following holds:

1. $(\hat{\phi}(a) \upharpoonright W(\pi(a)))$ forms a bijection from $W(\pi(a))$ to $W(a)$;
2. for all $n < \omega$ and $r \in F_n^\pi(a)$, $(\hat{\phi}(a)(r)) \in A_n^a$.

In particular, $(A, \ell_A)$ is a graded poset.

As for Clause (2) of Definition 3.3, we provide the following sufficient condition.

**Lemma 5.7.** Suppose that $(\hat{\phi}, \pi)$ is a forking projection from $(A, \ell_A)$ to $(\mathbb{P}, \ell_F)$, or, just a pair of maps satisfying Clauses (1), (2), (5) and (7) of Definition 5.1. For every $n < \omega$, if $A_n^a$ is $\aleph_1$-closed, then so is $A_n^a$.

**Proof.** This is proved exactly as in [PRS19, Lemma 4.6]. \hfill \Box

For Clause (3), we have the following.

**Fact 5.8** ([PRS19, Lemma 4.7]). Suppose that $(\hat{\phi}, \pi)$ is a forking projection from $(A, \ell_A, c_A)$ to $(\mathbb{P}, \ell_F, c_F)$, or, just a pair of maps satisfying Clauses (1), (2), (4), (7) and (8) of Definition 5.1. For all $a, a' \in A$, if $c_A(a) = c_A(a')$, then $A_0^a \cap A_0^{a'}$ is non-empty.

Clause (4) of Definition 3.3 is secured by Definition 5.1(4), so we move on to Clause (5).
Corollary 5.9. Suppose that \((\mathcal{R}, \pi)\) is a forking projection from \((A, \ell_A)\) to \((P, \ell_P)\), or, just a pair of maps satisfying Clauses (1), (2) and (4) of Definition 5.1. Then, for all \(a \in A\), \(|W(a)| < \mu\).

Proof. This follows immediately from Fact 5.6.

Remark 5.10. By Fact 5.6(1), if \(|W(p)| \leq \kappa\) for every \(p \in P\), then in the above Corollary we may moreover infer that \(|W(a)| \leq \kappa\) for every \(a \in A\).

The following takes care of Clause (6).

Fact 5.11 ([PRS19, Lemma 4.10]). Suppose that \((\mathcal{R}, \pi)\) is a forking projection from \((A, \ell_A)\) to \((P, \ell_P)\), or, just a pair of maps satisfying Clauses (1), (4) and (7). For all \(a' \leq a\) in \(A\), \(b \mapsto w(a, b)\) forms an order-preserving map from \(W(a')\) to \(W(a)\).

As a sufficient condition for Clause (7), we have the following key concept, an abstraction of the usual diagonalization results in the theory of Prikry-type forcings.

Definition 5.12 ([PRS19, Definition 4.11]). A forking projection \((\mathcal{R}, \pi)\) is said to have the mixing property iff for all \(a \in A\), \(n < \omega, q \leq^0 \pi(a)\), and a function \(g : W_n(q) \rightarrow A \downarrow a\) such that \(\pi \circ g\) is the identity map, there exists \(b \leq^0 a\) with \(\pi(b) = q\) such that \(\mathcal{R}(b)(r) \leq^0 g(r)\) for every \(r \in W_n(q)\).

Fact 5.13 ([PRS19, Lemma 4.12]). If \((\mathcal{R}, \pi)\) has the mixing property, then \((A, \ell_A)\) has the CPP.

Clause (8) of Definition 3.3 is taken care of by Remark 5.3. Thus, we are left with verifying Clause (9). Here is where Definition 5.2(c) comes into play.

Lemma 5.14. Suppose that \((\mathcal{R}, \pi)\) is an exact forking projection from \((A, \ell_A, \zeta)\) to \((P, \ell_P, \bar{\zeta})\). Let \(n < \omega\). Suppose that \(\theta\) is a cardinal such that:

(i) \(P^n_n\) is \(\theta\)-closed (resp. \(\theta\)-directed-closed), and
(ii) \(A^n_n\) is \(\theta\)-closed (resp. \(\theta\)-directed-closed).

Then \(A^n_n\) is \(\theta\)-closed (resp. \(\theta\)-directed-closed), as well.

Proof. Let \(D \subseteq A^n_n\) be a set of conditions of size \(< \theta_n\). For concreteness, we assume that \(D\) is directed and prove that it admits a \(\leq^\omega\)-lower bound; the proof of the case that \(D\) is well-ordered is similar.

Let \(s \in S_n\) denote the sole member of \(\varsigma_n[D]\). Since \(\zeta = \bar{\zeta} \cdot \pi\), \(\pi[D]\) is a directed set of conditions in \(P^n_n\) of size \(< \theta_n\), so by hypothesis (i), we may find a \(\leq^\omega\)-lower bound for it, say, \(p\). Set \(D^h := \{\mathcal{R}(a)(p) \mid a \in D\}\). By Clauses (5) and (7) of Definition 5.1, \(D^h\) is a directed set of conditions in \(A^n_n\) with \(|D^h| < \theta_n\) such that \(|\pi[D^h]| = |\{p\}| = 1\). So by hypothesis (ii), we may find a \(\leq^\omega\)-lower bound for \(D^h\), say \(a\).

Note that \(s_n(a) = \mathcal{R}_n(\pi(a)) = \mathcal{R}_n(p) = s\). Together with Clause (2) of Definition 5.1, we infer that \(a\) is a \(\leq^\omega\)-lower bound for \(D\), as wanted. \(\square\)
We conclude this section by providing a sufficient condition for the \( A_n \)'s to be suitable for reflection.

**Lemma 5.15.** Let \( n \) be a positive integer. Assume:

1. \( \kappa_n-1, \kappa_n \) are regular uncountable cardinals with \( \kappa_n-1 \leq \sigma_n < \kappa_n \);
2. \( (A_n, \ell_n, c_n, \zeta_n) \) is \((\Sigma, \bar{S})\)-Prikry;
3. \( (\mathfrak{h}, \pi) \) is an exact forking projection from \( (A_n, \ell_n, \zeta_n) \) to \((\mathbb{P}, \ell, \bar{\omega})\);
4. \( (\mathbb{P}_n, S_n, w_n) \) is suitable for reflection with respect to \( \langle \sigma_{n-1}, \kappa_{n-1}, \kappa_n, \mu \rangle \);
5. \( S_n \times A_n^\mathfrak{c} \) forces \( "|\mu| = \text{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}" \).

Then \( (A_n, S_n, s_n) \) is suitable for reflection with respect to \( \langle \sigma_{n-1}, \kappa_{n-1}, \kappa_n, \mu \rangle \).

**Proof.** Clauses (1), (2) and (4) of Definition 2.11 hold by virtue of hypotheses, (iv), (ii) and (iv) respectively.

Now let us address Clause (3). Given hypothesis (v), we are left with verifying that \( A_n \) forces \( "|\mu| = \text{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}" \). By Lemma 2.6, for every \( a \in A_n \), \( (S_n \downarrow s_n(a)) \times (A_n^\mathfrak{c} \downarrow a) \) projects onto \( A_n \downarrow a \). In addition, by hypothesis (iii), \( A_n \) projects onto \( \mathbb{P}_n \). Since both ends force \( "|\mu| = \text{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}" \), the same is true for \( A_n \).

\[\square\]

6. **Stationary Reflection and Killing a Fragile Stationary Set**

In this section, we isolate a natural notion of a fragile set and study two aspects of it. In the first subsection, we prove that, given a \((\Sigma, \bar{S})\)-Prikry poset \( \mathbb{P} \) and an \( r^* \)-fragile stationary set \( \hat{T} \), Sharon’s functor \( A(\cdot, \cdot) \) from [PRS19, §6] yields a \((\Sigma, \bar{S})\)-Prikry poset \( A(\mathbb{P}, \hat{T}) \) admitting an exact forking projection to \( \mathbb{P} \) and killing the stationarity of \( \hat{T} \). In the second subsection, we make the connection between fragile stationary sets, suitability for reflection and non-reflecting stationary sets. The two subsections can be read independently of each other.

**Setup 6.** As a setup for the whole section, we assume that \((\mathbb{P}, \ell, c, \bar{\omega})\) is a given \((\Sigma, \bar{S})\)-Prikry notion of forcing. Denote \( \mathbb{P} = (P, \leq), \Sigma = \langle \sigma_n \mid n < \omega \rangle, \bar{\omega} = \langle \omega_n \mid n < \omega \rangle, \bar{S} = \langle S_n \mid n < \omega \rangle \). Also, define \( \kappa \) and \( \mu \) as in Definition 3.3, and assume that \( 1 \mathbb{P} \vdash \kappa \) is singular” and that \( \text{cf}(\kappa)^+ = \mu \).

The following concept is implicit in the proof of [CFM01, Theorem 11.1]:

**Definition 6.1.** Suppose \( r^* \in P \) forces that \( \hat{T} \) is a \( \mathbb{P} \)-name for a stationary subset \( T \) of \( \mu \). We say that \( \hat{T} \) is \( r^*-\text{fragile} \) if, looking for each \( n < \omega \) at \( \hat{T}_n := \{ (\check{\alpha}, p) \mid (\alpha, p) \in \mu \times P_n \text{ & } p \Vdash \check{\alpha} \in \check{T} \} \), then, for every \( q \leq r^* \), \( q \Vdash_{\mathbb{P}_n} \hat{T}_{\ell(q)} \) is nonstationary”.

**6.1. Killing one fragile set.** Let \( r^* \in P \) and \( \hat{T} \) be a \( \mathbb{P} \)-name for an \( r^*- \)fragile stationary subset of \( \mu \). Let \( I := \omega \setminus \ell(r^*) \). By Definition 6.1, for all \( q \leq r^* \) with \( \ell(q) \in I \), \( q \Vdash_{\mathbb{P}_n} \hat{T}_{\ell(q)} \) is nonstationary”. Thus, for each \( n \in I \), we may pick a \( \mathbb{P}_n \)-name \( \check{C}_n \) for a club subset of \( \mu \) such that, for all \( q \leq r^* \) with \( \ell(q) = n \),

\[ q \Vdash_{\mathbb{P}_n} \hat{T}_n \cap \check{C}_n = \emptyset. \]
Consider the following binary relation:
\[ R := \{ (\alpha, q) \in \mu \times P \mid q \leq r^* \land \forall r \leq q[\ell(r) \in I \rightarrow r \parallel_{\Psi_{\ell(r)}} \check{\alpha} \in \check{C}_{\ell(r)}] \}. \]

Note that, for all \((\alpha, q) \in R, q \parallel_{\Psi} \check{\alpha} \notin \check{T} \).

Now, we spell out (only) the necessary details of the definition of \(\text{A}(\mathbb{P}, \check{T})\) from [PRS19, §6].

**Definition 6.2.** Suppose \(p \in P\). A labeled \(p\)-tree is a function \(S : W(p) \rightarrow [\mu]^{<\mu}\) such that for all \(q \in W(p)\):

1. \(S(q)\) is a closed bounded subset of \(\mu\);
2. \(S(q') \supseteq S(q)\) whenever \(q' \leq q\);
3. \(q \parallel_{\mathbb{P}} S(q) \cap \check{T} = \emptyset\);
4. for all \(q' \leq q\) in \(W(p)\), either \(S(q') = \emptyset\) or \((\max(S(q')) \equiv q) \in R\).

**Definition 6.3.** For \(p \in P\), we say that \(\vec{S} = (S_i \mid i \leq \alpha)\) is a \(p\)-strategy iff all of the following hold:

1. \(\alpha < \mu\);
2. \(S_i\) is a labeled \(p\)-tree for all \(i \leq \alpha\);
3. for every \(i < \alpha\) and \(q \in W(p)\), \(S_i(q) \subseteq S_{i+1}(q)\);
4. for every \(i < \alpha\) and a pair \(q' \leq q\) in \(W(p)\), \((S_{i+1}(q) \setminus S_i(q)) \subseteq (S_{i+1}(q') \setminus S_i(q'))\);
5. for every limit \(i \leq \alpha\) and \(q \in W(p)\), \(S_i(q)\) is the ordinal closure of \(\bigcup_{j<i} S_j(q)\). In particular, \(S_0(q) = \emptyset\) for all \(q \in W(p)\).

**Definition 6.4.** Let \(\text{A}(\mathbb{P}, \check{T})\) be the notion of forcing \(\text{A} := (A, \leq)\), where:

1. \((p, \vec{S}) \in A\) iff \(p \in P\), and \(\vec{S}\) is either the empty sequence, or a \(p\)-strategy;
2. \((p', \vec{S}') \leq (p, \vec{S})\) iff:
   a. \(p' \leq p\);
   b. \(\text{dom}(\vec{S}') \supseteq \text{dom}(\vec{S})\);
   c. \(S'_i(q) = S_i(w(p, q))\) for all \(i \in \text{dom}(\vec{S})\) and \(q \in W(p')\).

For all \(p \in P\), denote \([p]^\text{A} := (p, \emptyset)\).

**Definition 6.5.**

1. Let \(\ell_A := \ell \circ \pi\), where \(\pi : A \to \mathbb{P}\) is defined via \(\pi(p, \vec{S}) := p\);
2. Define \(c_A : A \to H_\mu\) via \(c_A(p, \vec{S}) := (c(p), \{(i, c(q), S_i(q)) \mid i \in \text{dom}(\vec{S}), q \in W(p)\})\).
3. Given \(a = (p, \vec{S})\) in \(A\), define \(\check{\ell}(a) : \mathbb{P} \downarrow p \to A\) by letting for each \(p' \leq p\), \(\check{\ell}(a)(p') := (p', \vec{S}')\), where \(\vec{S}'\) is the sequence \(\langle S'_i : W(p') \to [\mu]^{<\mu} \mid i < \text{dom}(\vec{S}) \rangle\) to satisfy:
   \[ S'_i(q) = S_i(w(p, q)) \text{ for all } i \in \text{dom}(\vec{S}') \text{ and } q \in W(p'). \]
4. Define \(\zeta = \langle \zeta_n \mid n < \omega \rangle\) by letting \(\zeta_n := \omega_n \circ \pi\) for every \(n < \omega\).
Comparing the seven clauses of Definition 2.3 from Part I (that is, [PRS19]) with the corresponding ones from this paper’s Definition 3.3, we see that the only difference is in Clause (2). That is, in our context, the $P_n$’s are merely $\aleph_1$-closed. In effect, all of the following results from Part I are still valid:

- Lemma 2.6 with $\theta \leq 2^{\aleph_0}$;
- Lemma 2.8;
- Claims 5.6.1 and 5.6.2(1);
- Lemma 6.6., Lemma 6.7 and Theorem 6.8.
- Lemmas 6.12, 6.13 and Lemma 6.15;
- Lemma 6.16.

**Corollary 6.6.**

1. For every cardinal $\nu \geq \mu$, if $\mathbb{P}$ is a subset of $H_\nu$, then so is $A$;
2. $(\hat{r}^\ast, \emptyset) \Vdash A \text{ "} T \text{ is non-stationary";}$
3. For each $n < \omega$, $A_n^\pi$ is $\mu$-directed-closed;
4. $(\hat{\chi}, \pi)$ is a forking projection from $(A, \ell_A, c_A)$ to $(\mathbb{P}, \ell, c)$ having the mixing property.

**Proof.**

1. This is Lemma 6.6 of Part I.
2. This is Theorem 6.8 of Part I.
3. This is Lemma 6.15 of Part I.
4. These are Lemmas 6.13 and 6.16 of Part I. □

**Lemma 6.7.** $(\hat{\chi}, \pi)$ is an exact forking projection from $(A, \ell_A, c_A, \vec{\chi})$ to $(\mathbb{P}, \ell, c, \vec{\omega})$.

**Proof.** Recalling Corollary 6.6(4), we are left with verifying Clauses (b) and (c) of Definition 5.2.

(b) Given $a \in A$ and $p \leq \pi(a)$, Definitions 6.4 and 6.5 make clear that $\hat{\chi}(a)(p)$ is the greatest element of the set $\{b \in A \mid b \leq a \wedge \pi(b) \leq p\}$.

(c) This is guaranteed by the last clause of by Definition 6.5. □

**Lemma 6.8.** For each $n < \omega$, $A_n$ is $\aleph_1$-closed, and $A_n^\pi$ is $\sigma_n$-directed-closed.

**Proof.** By Lemma 6.7, $(\hat{\chi}, \pi)$ is an exact forking projection. Thus, we get the conclusion by Lemma 5.7 and Lemma 5.14, respectively, together with Corollary 6.6(3).

**Theorem 6.9.** $(A, \ell_A, c_A, \vec{\chi})$ is $(\Sigma, \vec{S})$-Prikry, and $1_A \Vdash \mu = \check{\kappa}^+$.  

**Proof.** We go over the clauses of Definition 3.3:

1. By Fact 5.6.
2. By Lemma 6.8.
3. By $|H_\mu| = \mu$ and Fact 5.8.
4. By Definition 3.3(4).
5. By Corollary 5.9.
6. By Fact 5.11.
7. By Fact 5.13 together with Corollary 6.6(4).
(8) Remark 5.3
(9) By Lemma 6.8.

Finally, by Fact 5.5, $1 \Vdash \check{\mu} = \check{\kappa}^+$. 

To sum up the content of this subsection:

**Corollary 6.10.** Suppose that $(\Sigma, \check{S})$-Prikry quadruple $(P, \ell, c, \check{\gamma})$ such that, $P = (P, \leq)$ is a subset of $H_{\check{\mu}^+}$, $1 \Vdash P \check{\mu} = \check{\kappa}^+$ and $1 \Vdash \kappa$ is singular.

For every $r^* \in P$ and a $P$-name $z$ for an $r^*$-fragile stationary subset of $\check{\mu}$, there are a $(\Sigma, \check{S})$-Prikry quadruple $(A, \ell_A, c_A, \check{\gamma})$ and a pair of maps $(\check{\gamma}, \pi)$ such that all the following hold:

(a) $(\check{\gamma}, \pi)$ is an exact forking projection from $(A, \ell_A, c_A, \check{\gamma})$ to $(P, \ell, c, \check{\gamma})$ that has the mixing property;

(b) $1 \Vdash A \check{\mu} = \check{\kappa}^+$;

(c) $A = (A, \leq)$ is a subset of $H_{\check{\mu}^+}$;

(d) For every $n < \omega$, $A_n$ is a $\check{\mu}$-directed-closed;

(e) $[r^*]_A$ forces that $z$ is nonstationary.

**Proof.** Since all the assumptions of Setup 6 are valid we obtain from Definitions 6.4 and 6.5, a notion of forcing $A = (A, \leq)$ together with maps $\ell_A$ and $c_A$, and a sequence $\check{\gamma}$ such that, by Theorem 6.9, $(A, \ell_A, c_A, \check{\gamma})$ is $(\Sigma, \check{S})$-Prikry and $1 \Vdash A \check{\mu} = \check{\kappa}^+$. By Lemma 6.7 and Corollary 6.6(4), the pair of maps $(\check{\gamma}, \pi)$ from Definition 6.5 is an exact forking projection from $(A, \ell_A, c_A, \check{\gamma})$ to $(P, \ell, c, \check{\gamma})$ having the mixing property. The last three clauses are taken care of by Corollary 6.6. 

6.2. Fragile sets vs non-reflecting stationary sets. For every $n < \omega$, denote $\Gamma_n := \{ \alpha < \check{\mu} \mid \text{cf} V(\alpha) < \sigma_{n-1} \}$, where, by convention, we define $\sigma_{-2}$ and $\sigma_{-1}$ to be $\aleph_0$.

The next lemma is an analogue of [PRS19, Lemma 6.1] and will be crucial for the proof of reflection in the model of the Main Theorem.

**Lemma 6.11.** Suppose that:

(i) for every $n < \omega$, $V^\mathbb{P}_n \models \text{Refl}(E_{<\sigma_{n-2}}^\mu, E_{<\sigma_{n}}^\mu)$;

(ii) $r^*$ is a condition in $\mathbb{P}$;

(iii) $\hat{T}$ is a nice $\mathbb{P}$-name for a subset of $\Gamma_{\ell(r^*)}$;

(iv) $r^* \mathbb{P}$-forces that $\hat{T}$ is a non-reflecting stationary set.

Then $\hat{T}$ is $r^*$-fragile.

**Proof.** Suppose not, and let $q$ be an extension of $r^*$ witnessing that. Set $n := \ell(q)$, so that $q \Vdash \mathbb{P}_n \text{ "}\hat{T}_n \text{ is stationary".}$

Since $\hat{T}$ is a nice $\mathbb{P}$-name for a subset of $\Gamma_{\ell(r^*)}$, it altogether follows that $q \mathbb{P}_n$-forces that $\hat{T}_n$ is a stationary subset of $E_{<\sigma_{n-2}}^\mu$.

Let $G_n$ be $\mathbb{P}_n$-generic containing $q$. By Clause (i), we have that $T_n := (\hat{T}_n)_{G_n}$ reflects at some ordinal $\gamma$ of cofinality $< \sigma_n$. Since $|S_n| < \sigma_n$ and...
\(\mathbb{P}^{\sigma_n}\) is \(\sigma_n\)-directed-closed, it follows from Lemma 2.6(2) that \(\theta := \text{cf}^V(\gamma)\) is \(< \sigma_n\). Fix in \(V\) a club \(C\) in \(\gamma\) of order-type \(\theta\).

Work in \(V[G_n]\). Set \(A := T_n \cap C\), so that \(A\) is a stationary subset of \(\gamma\) of size \(< \theta\). Let \(H_n\) be the \(S_n\)-generic filter induced from \(G_n\) by \(\varpi_n\). Since \(\mathbb{P}^{\sigma_n}\) is \(\sigma_n\)-directed-closed, it follows from Lemma 2.6(2) that \(A \in V[H_n]\).

Let \((\alpha_i | i < \theta)\) be some enumeration (possibly with repetitions) of \(A\), and let \(\langle \dot{\alpha}_i | i < \theta \rangle\) be an \(S_n\)-name for it. Pick a condition \(r\) in \(\mathbb{P}_n/H_n\) such that \(r \Vdash P_n \dot{A} \subseteq T_n \cap \gamma\) and such that \(\varpi_n(r) \Vdash S_n \dot{A} = \{ \dot{\alpha}_i | i < \theta \}\). Denote \(s := \varpi_n(r)\). We now go back and work in \(V\).

**Claim 6.11.1.** Let \(i < \theta\) and \(\alpha < \gamma\). For all \(r' \leq^\varpi_n r\) and \(s' \leq_n s\), if \(s' \Vdash S_n \dot{\alpha}_i = \dot{\alpha}\), then there are \(r'' \leq^\varpi_n r'\) and \(s'' \leq_n s'\) such that \(r'' + s'' \Vdash \exists \alpha \in T\).

**Proof.** Suppose \(r', s'\) are as above. As \(r'\) extends \(r\) and \(s'\) extends \(s\), it follows that \(r' + s' \Vdash \exists \dot{\alpha} \in T_n\) and \(s' \Vdash \exists \dot{\alpha} \in \dot{A}\). So, by the definition of the name \(T_n\), there is some \(p \leq^0 \dot{r}' + s'\) such that \(p \Vdash \dot{\alpha} \in \dot{T}_n\). By Lemma 2.6, we may then find \(s'' \leq_n s'\) and \(r'' \leq^\varpi_n r'\) such that \(r'' + s'' = p\). So \(r'' + s'' \Vdash \exists \alpha \in T\), as desired. \(\square\)

Fix an injective enumeration \(\langle (i_\xi, s_\xi) | \xi < \chi \rangle\) of \(\theta \times (S_n \downarrow s)\). Note that \(\chi < \sigma_n\). So, as \(\mathbb{P}^{\sigma_n}\) is \(\sigma_n\)-directed-closed, we may build a \(\leq^\varpi_n\)-decreasing sequence of conditions \((r_\xi | \xi \leq \chi)\), such that, for every \(\xi < \chi\), \(r_\xi \leq^\varpi_n r\), and, for any \(\alpha < \gamma\), if \(s_\xi \Vdash S_n \dot{\alpha}_i = \dot{\alpha}\), then there is \(s'_\xi \leq_n s_\xi\) such that \(r_\xi + s'_\xi \Vdash \dot{\alpha} \in \dot{T}\). Finally, let \(r^* := r_\chi\). Note that \(\varpi_n(r^*) = \varpi_n(r) = s\), and hence \(\varpi_n(r^*) \in H_n\).

**Claim 6.11.2.** \(r^* \Vdash \mathbb{P}/H_n A \subseteq \dot{T} \cap \dot{\gamma}\).

**Proof.** For each \(i < \theta\), by density, there is some \(s' \leq_n s\) in \(H\) such that \(s'\) decides \(\dot{\alpha}_i\) to be some ordinal \(\alpha < \gamma\). Fix \(\xi < \chi\) such that \((i_\xi, s_\xi) = (i, s')\). Then, by the construction, \(r_\xi + s'_\xi \Vdash \dot{\alpha} \in \dot{T}\). In particular, \(r^* + s'_\xi \Vdash \exists \alpha \in \dot{T} \cap \dot{\gamma}\). \(\square\)

Finally, since \((\mathbb{P}, \ell, c, \varpi)\) is \((\Sigma, \dot{S})\)-Prikry, Lemma 3.12(1) implies that \(\mathbb{P}/H_n\) does not add any new subsets of \(\theta\) nor any new subsets of \(C\), hence \(\mathbb{P}/H_n\) preserves the stationarity of \(A\), hence the stationarity of \(T \cap \gamma\). This contradicts hypothesis \((iv)\). \(\square\)

### 7. Iteration Scheme

In this section, we define an iteration scheme for \((\Sigma, \dot{S})\)-Prikry forcings, following closely and expanding the work from [PRS20, §4].

**Setup 7.** The blanket assumptions for this section are as follows:

- \(\mu\) is some cardinal satisfying \(\mu^{<\mu} = \mu\), so that \(|H_\mu| = \mu\);
- \(\langle (\sigma_n, \sigma^*_n) | n < \omega \rangle\) is a sequence of pairs of regular uncountable cardinals, such that, for every \(n < \omega\), \(\sigma_n \leq \sigma^*_n \leq \mu\) and \(\sigma_n \leq \sigma_{n+1}\).
For a pair of ordinals \( \gamma \leq \alpha \leq \mu^+ \):

1. \( \emptyset_\alpha := \alpha \times \{0\} \) denotes the \( \alpha \)-sequence with constant value \( 0 \); 
2. For a \( \gamma \)-sequence \( p \) and an \( \alpha \)-sequence \( q \), \( p \ast q \) denotes the unique \( \alpha \)-sequence satisfying that for all \( \beta < \alpha \):
   
   \[
   (p \ast q)(\beta) = \begin{cases} 
   q(\beta), & \text{if } \gamma \leq \beta < \alpha; \\
   p(\beta), & \text{otherwise}.
   \end{cases}
   \]

3. Let \( P_\alpha := (P_\alpha, \leq_\alpha) \) and \( P_\gamma := (P_\gamma, \leq_\gamma) \) be forcing posets such that \( P_\alpha \subseteq \alpha H_{\mu^+} \) and \( P_\gamma \subseteq \gamma H_{\mu^+} \). Also, assume \( p \rightarrow p \upharpoonright \gamma \) defines a projection between \( P_\alpha \) and \( P_\gamma \). We denote by \( i_\gamma^\alpha : V^{P_\gamma} \rightarrow V^{P_\alpha} \) the map defined by recursion over the rank of each \( P_\gamma \)-name \( \sigma \) as follows:
   
   \[
   i_\gamma^\alpha(\sigma) := \{(i_\gamma^\alpha(\tau), p \ast \emptyset_\alpha) | (\tau, p) \in \sigma \}.
   \]

Our iteration scheme requires three building blocks:

**Building Block I.** We are given a \( (\Sigma, \mathcal{S}) \)-Prikry forcing \( (\mathcal{Q}, \ell, c, \vec{s}) \). We moreover assume that \( \mathcal{Q} = (\mathcal{Q}, \leq_\mathcal{Q}) \) is a subset of \( H_{\mu^+} \), \( I_\mathcal{Q} \models \mathcal{Q} \) and \( I_\mathcal{Q} \models \kappa = \mathcal{K}^+ \). To streamline the matter, we also require that \( I_\mathcal{Q} \) be equal to \( 0 \).

**Building Block II.** For every \( (\Sigma, \mathcal{S}) \)-Prikry quadruple \( (\mathcal{P}, \ell_\mathcal{P}, c_\mathcal{P}, \vec{s}) \) such that, \( \mathcal{P} = (P, \leq) \) is a subset of \( H_{\mu^+} \), \( I_\mathcal{P} \models \mathcal{P} \) and \( I_\mathcal{P} \models \kappa = \mathcal{K}^+ \), every \( r^* \in P \), and every \( \mathcal{P} \)-name \( z \in H_{\mu^+} \), we are given a corresponding \( (\Sigma, \mathcal{S}) \)-Prikry quadruple \( (A, \ell_A, c_A, \vec{s}) \) and a pair of maps \( (\pi, \pi) \), such that the following hold true:

   a. \( (\pi, \pi) \) is an exact forking projection from \( (A, \ell_A, c_A, \vec{s}) \) to \( (\mathcal{P}, \ell_\mathcal{P}, c_\mathcal{P}, \vec{s}) \) that has the mixing property;
   
   b. \( I_A \models \mathcal{P} \) and \( \mu = \mathcal{K}^+ \);
   
   c. \( A = (A, \leq) \) is a subset of \( H_{\mu^+} \);
   
   d. for every \( n < \omega \), \( A_n^\pi \) is \( \sigma_n^* \)-directed-closed.

By Lemma 5.4, we may streamline the matter and also require that:

   e. each element of \( A \) is a pair \( (x, y) \) with \( \pi(x, y) = x \);
   
   f. for every \( a \in A \), \( [\pi(a)]^A = (\pi(a), \emptyset) \);
   
   g. for every \( p, q \in P \), if \( c_\mathcal{P}(p) = c_\mathcal{P}(q) \), then \( c_A([p]^A) = c_A([q]^A) \).

**Building Block III.** We are given a function \( \psi : \mu^+ \rightarrow H_{\mu^+} \).

**Goal 7.2.** Our goal is to define a system \( \langle (P_\alpha, \leq_\alpha), P_\alpha \subseteq ^\alpha H_{\mu^+}, \text{ for all } \alpha \in \alpha \rangle \) such that for all \( \gamma \leq \alpha \leq \mu^+ \):

   i. \( P_\alpha \) is a poset \( (P_\alpha, \leq_\alpha) \), \( P_\alpha \subseteq ^\alpha H_{\mu^+} \) and, for all \( p \in P_\alpha \), \( \|B_p\| < \mu \), where \( B_p := \{ \beta + 1 | \beta \in dom(p) \& p(\beta) \neq \emptyset \} \);
The map $\pi_{\alpha,\gamma} : P_\alpha \to P_\gamma$ defined by $\pi_{\alpha,\gamma}(p) := p \upharpoonright \gamma$ forms an exact nice projection from $P_\alpha$ to $P_\gamma$ and $\ell_\alpha = \ell_1 \circ \pi_{\alpha,\gamma}$.

(ii) If $\alpha > 0$, then $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$ is a $(\Sigma, \vec{\omega})$-Prikry notion of forcing whose greatest element is $\phi_\alpha$, $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$ and $\phi_\alpha \Vdash P_\alpha \mu = \kappa^+$. Moreover, $\vec{\omega}_\alpha = \vec{\omega}_\gamma \bullet \pi_{\alpha,\gamma}$ for every $\gamma \leq \alpha$.

(iii) $P_0$ is a trivial forcing, $P_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I, and $P_{\alpha+1}$ is isomorphic to $\mathbb{A}$ given by Building Block II when invoked with respect to $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$ and a pair $(r^*, z)$ which is decoded from $\psi(\alpha)$.

(iv) If $\alpha > 0$, then $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$ is a $(\Sigma, \vec{\omega})$-Prikry notion of forcing whose greatest element is $\phi_\alpha$, $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$ and $\phi_\alpha \Vdash P_\alpha \mu = \kappa^+$. Moreover, $\vec{\omega}_\alpha = \vec{\omega}_\gamma \bullet \pi_{\alpha,\gamma}$ for every $\gamma \leq \alpha$.

(v) If $0 < \gamma < \alpha \leq \mu^+$, then $(\varpi_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is an exact forking projection from $(P_\alpha, \ell_\alpha, \vec{\omega}_\alpha)$ to $(P_\gamma, \ell_\gamma, \vec{\omega}_\gamma)$; in case $\alpha < \mu^+$, $(\varpi_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is furthermore an exact forking projection from $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$ to $(P_\gamma, \ell_\gamma, c_\gamma, \vec{\omega}_\gamma)$.

(vi) If $0 < \gamma \leq \beta \leq \alpha$, then, for all $p \in P_\alpha$ and $r \leq \gamma$, $\varpi_{\alpha,\gamma}(p \upharpoonright \beta)(r) = (\varpi_{\alpha,\gamma}(p)(r)) \upharpoonright \beta$.

7.1. Defining the iteration. For every $\alpha < \mu^+$, fix an injection $\phi_\alpha : \alpha \to \mu$. As $|H_\mu| = \mu$, we may also fix a sequence $\langle e^i \mid i < \mu \rangle$ of functions from $\mu^+$ to $H_\mu$ such that for every function $e : C \to H_\mu$ with $C \in [\mu^+]^{<\mu}$, there is $i < \mu$ such that $e \subseteq e^i$.

The upcoming definition is by recursion on $\alpha \leq \mu^+$, and we continue as long as we are successful. We shall later verify that the described process is indeed successful.

- Let $P_0 := \langle \langle \emptyset, \leq \rangle_0 \rangle$ be the trivial forcing. Let $\ell_0$ and $c_0$ be the constant function $\langle \emptyset, 0 \rangle$ and $\vec{\omega}_0 = \langle \langle \emptyset, 0 \rangle \mid n < \omega \rangle$. Finally, let $\varpi_{0,0}$ be the constant function $\langle \emptyset, \langle \emptyset, 0 \rangle \rangle$, so that $\varpi_{0,0}(\emptyset)$ is the identity map.

- Let $P_1 := \langle P_1, \leq_1 \rangle$, where $P_1 := \langle 1, Q \rangle$ and $p \leq_1 p'$ iff $p(0) \leq_Q p'(0)$. Evidently, $p \mapsto p(0)$ form an isomorphism between $P_1$ and $\mathbb{Q}$, so we naturally define $\ell_1 := \ell \circ \chi$, $c_1 := c \circ \chi$ and $\vec{\omega}_1 := \vec{\omega} \bullet \chi$. Hereafter, the sequence $\vec{\omega}_1$ is denoted by $\langle \omega^1_n \mid n < \omega \rangle$.

For all $p \in P_1$, let $\varpi_{1,0}(p) : \langle \emptyset \rangle \to \{p\}$ be the constant function, and let $\varpi_{1,1}(p)$ be the identity map.

- Suppose $\alpha < \mu^+$ and that $\langle \langle P_\beta, \ell_\beta, c_\beta, \vec{\omega}_\beta, \langle \varpi_{\beta,\gamma} \mid \gamma \leq \beta \rangle \rangle \mid \beta \leq \alpha \rangle$ has already been defined. We now define $(P_{\alpha+1}, \ell_{\alpha+1}, c_{\alpha+1}, \vec{\omega}_{\alpha+1})$ and $\langle \varpi_{\alpha+1,\gamma} \rangle \mid \gamma \leq \alpha + 1 \rangle$.

- If $\psi(\alpha)$ happens to be a triple $(\beta, r, \sigma)$, where $\beta < \alpha$, $r \in P_\beta$ and $\sigma$ is a $P_\beta$-name, then we appeal to Building Block II with $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$, $r^* := r \circ \theta_\alpha$ and $z := r^*_\beta(\sigma)$ to get a corresponding $(\Sigma, \vec{\omega})$-Prikry quadruple $(\mathbb{A}, \ell_\mathbb{A}, \chi_\mathbb{A}, \vec{\omega}_\mathbb{A})$.

- Otherwise, we obtain $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A}, \vec{\omega}_\mathbb{A})$ by appealing to Building Block II with $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$, $r^* := \emptyset_\alpha$ and $z := \emptyset$.

In both cases, we also obtain an exact forking projection $(\varpi, \pi)$ from $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A}, \vec{\omega}_\mathbb{A})$ to $(P_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)$. Furthermore, each condition in $\mathbb{A} = (A, \leq)$ is a pair $(x, y)$ with $\pi(x, y) = x$, and, for every $p \in P_\alpha$, $[p]^\mathbb{A} = (p, \emptyset)$. Now, define $P_{\alpha+1} := (P_{\alpha+1}, \leq_{\alpha+1})$ by letting $P_{\alpha+1} := \{x^\gamma(y) \mid (x, y) \in A\}$, and
then letting $p \leq A_{\alpha+1} p'$ iff $(p \restriction \alpha, p(\alpha)) \preceq (p' \restriction \alpha, p'(\alpha))$. Put $\ell_{\alpha+1} := \ell_1 \circ \pi_{\alpha+1,1}$ and define $c_{\alpha+1} : P_{\alpha+1} \to H_\mu$ via $c_{\alpha+1}(p) := c_\alpha(p \restriction \alpha, p(\alpha))$.

Let $\tilde{\pi}_\alpha = (\pi_\alpha^n \mid n < \omega)$ be defined in the natural way, i.e., for each $n < \omega$ and $x^n(y) \in (P_\alpha)^{n \geq n}$, we set $\pi_\alpha^n(x^n(y)) := s_n(x, y)$.

Next, let $p \in P_{\alpha+1}$, $\gamma \leq \alpha + 1$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\hat{\pi}_{\alpha+1, \gamma}(p)(r)$. For $\gamma = \alpha + 1$, let $\hat{\pi}_{\alpha+1, \gamma}(p)(r) := r$, and for $\gamma \leq \alpha$, let

$$\hat{\pi}_{\alpha+1, \gamma}(p)(r) := x^n(y) \iff \hat{\pi}(p \restriction \alpha, p(\alpha))((\hat{\pi}_{\alpha, \gamma}(p \restriction \alpha))(r)) = (x, y).$$

Suppose $\alpha \leq \mu^+$ is a nonzero limit ordinal, and that the sequence $\langle (P_\beta, \ell_\beta, c_\beta, \tilde{\pi}_\beta, (\hat{\pi}_{\beta, \gamma} \mid \gamma \leq \beta) \rangle \mid \beta < \alpha \rangle$ has already been defined according to Goal 7.2.

Define $P_\alpha := (P_\alpha, \leq_\alpha)$ by letting $P_\alpha$ be all $\alpha$-sequences $p$ such that $|B_\gamma| < \mu$ and $\forall \beta < \alpha(p \restriction \beta \in P_\beta)$. Let $p \leq_\alpha q$ iff $\forall \beta < \alpha(p \restriction \beta \leq_\beta q \restriction \beta)$. Let $\ell_\alpha := \ell_1 \circ \pi_{\alpha,1}$. Next, we define $c_\alpha : P_\alpha \to H_\mu$, as follows.

If $\alpha < \mu^+$, then, for every $p \in P_\alpha$, let

$$c_\alpha(p) := \{(\phi_\gamma(C \cap \gamma), c_\gamma(p \restriction \gamma)) \mid \gamma \in B_p\}.$$

If $\alpha = \mu^+$, then, given $p \in P_\alpha$, first let $C := \text{cl}(B_p)$, then define a function $e : C \to H_\mu$ by stipulating:

$$e(\gamma) := (\phi_\gamma(C \cap \gamma), c_\gamma(p \restriction \gamma)),$$

and then let $c_\alpha(p) := i$ for the least $i < \mu$ such that $e \subseteq e^i$. Set $\tilde{\pi}_\alpha := \tilde{\pi}_1 \cdot \pi_{\alpha,1}.

Finally, let $p \in P_\alpha$, $\gamma \leq \alpha$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\hat{\pi}_{\alpha, \gamma}(p)(r)$. For $\gamma = \alpha$, let $\hat{\pi}_{\alpha, \gamma}(p)(r) := r$, and for $\gamma < \alpha$, let $\hat{\pi}_{\alpha, \gamma}(p)(r) := \bigcup\{\hat{\pi}_{\beta, \gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha\}$.

**Convention 7.3.** Even though $(P_0, \ell_0)$ is not a graded poset, in order to smooth the matter, we define $\leq_0$ to be $\leq_0$, and likewise, for every $p \in P_0$, we interpret $(P_0)^p$ as $\{q \in P_0 \mid q \leq_0 p\}$.  

**7.2. Verification.** We now verify that for all $\alpha \leq \mu^+$, the tuple $(P_\alpha, \ell_\alpha, c_\alpha,\tilde{\pi}_\alpha, (\hat{\pi}_{\alpha, \gamma} \mid \gamma \leq \alpha))$ fulfills requirements (i)–(vi) of Goal 7.2. By the recursive definition given so far, it is obvious that Clauses (i) and (iii) hold, so we focus on the rest. The next lemma deals with an expanded version of Clause (vi).

**Lemma 7.4.** For all $\gamma \leq \alpha \leq \mu^+$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \restriction \gamma$, if we let $q := \hat{\pi}_{\alpha, \gamma}(p)(r)$, then:

1. $q \restriction \beta = \hat{\pi}_{\beta, \gamma}(p \restriction \beta)(r)$ for all $\beta \in [\gamma, \alpha]$;
2. $B_q = B_p \cup B_r$;
3. $q \restriction \gamma = r$;
4. If $\gamma = 0$, then $q = p$;
5. $p = (p \restriction \gamma) * \emptyset_\alpha$ iff $q = r * \emptyset_\alpha$;
6. For all $p' \leq_n^* p$, if $r \leq_\gamma p' \restriction \gamma$, then $\hat{\pi}_{\alpha, \gamma}(p')(r) \leq_\alpha \hat{\pi}_{\alpha, \gamma}(p)(r)$.

**Proof.** The proof is the same as in [PRS20, Lemma 4.6].

We move on to Clause (ii):
Lemma 7.5. Let $\gamma \leq \alpha \leq \mu^+$. 

(1) $\pi_{\alpha,\gamma}$ is an exact nice projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$. Furthermore, for all $p \in P_\alpha$ and $r \leq \gamma \pi_{\alpha,\gamma}(p)$, $p + r = \mathfrak{n}_{\alpha,\gamma}(p)(r)$;

(2) $\ell_\alpha = \ell_\gamma \circ \pi_{\alpha,\gamma}$.

Proof. (1) By induction on $\alpha \leq \mu^+$. The case $\gamma = \alpha$ is trivial, since $\pi_{\gamma,\gamma}$ and $\mathfrak{n}_{\gamma,\gamma}(p)$ are the identity maps. Also, it is clear that $\pi_{\alpha,\gamma}$ is order-preserving and that $\pi_{\alpha,\gamma}(\emptyset_\alpha) = \emptyset_\gamma$.

- Assume $\alpha = \alpha' + 1$ and that the claim holds for $\alpha'$. Let $\gamma < \alpha$, $p \in P_\alpha$ and $r \leq \gamma \pi_{\alpha,\gamma}(p)$. Recall that $P_\alpha = P_{\alpha' + 1}$ was defined by feeding $(P_{\alpha'}, \ell_{\alpha'}, c_{\alpha'}, \pi_{\alpha'})$ into Building Block II, thus obtaining a $(\Sigma, \tilde{S})$-Prikry quadruple $(\tilde{A}, \ell_{\tilde{A}}, c_{\tilde{A}}, \tilde{\gamma})$ along with an exact forking projection $(\tilde{\eta}, \pi)$, and the following holds

$$\mathfrak{n}_{\alpha,\gamma}(p)(r) = x \prec \langle y \rangle$$

if $\mathfrak{n}(p \upharpoonright \alpha', p(\alpha'))(\mathfrak{n}_{\alpha',\gamma}(p \upharpoonright \alpha')(r)) = (x, y)$.

We shall show that $\mathfrak{n}_{\alpha,\gamma}(p)(r)$ is the greatest element of the following set:

$$\hat{Q} := \{ q \in P_\alpha \mid q \leq \alpha p \wedge \pi_{\alpha,\gamma}(q) = r \}.$$

To show that $\mathfrak{n}_{\alpha,\gamma}(p)(r)$ is in $\hat{Q}$, set $(x, y) := \mathfrak{n}(p \upharpoonright \alpha', p(\alpha'))(\mathfrak{n}_{\alpha',\gamma}(p \upharpoonright \alpha')(r))$. By Definition 5.1(1), $(x, y) \leq (p \upharpoonright \alpha', p(\alpha'))$. So, by the definition of $\leq \alpha' + 1$, we altogether infer that $\mathfrak{n}_{\alpha',\gamma}(p)(r) = x \prec \langle y \rangle \leq \alpha' + 1$. By Definition 5.1(5), $x = \mathfrak{n}_{\alpha',\gamma}(p \upharpoonright \alpha')(r)$. By the inductive hypothesis on $\alpha'$:

- $\mathfrak{n}_{\alpha',\gamma}(p \upharpoonright \alpha')(r) = (p \upharpoonright \alpha') + r$, and
- $r = \pi_{\alpha',\gamma}(p \upharpoonright \alpha' + r)$.

In particular,

$$\pi_{\alpha' + 1,\gamma}(\mathfrak{n}_{\alpha' + 1,\gamma}(p)(r)) = \pi_{\alpha',\gamma}(x) = r,$$

so that $\mathfrak{n}_{\alpha,\gamma}(p)(r)$ is in $\hat{Q}$.

To show that $\mathfrak{n}_{\alpha,\gamma}(p)(r)$ is the greatest element of $\hat{Q}$, let $q \in \hat{Q}$ be arbitrary. From $q \leq \alpha$, we infer that $(q \upharpoonright \alpha', q(\alpha')) \leq (p \upharpoonright \alpha', p(\alpha'))$ and $(q \upharpoonright \alpha') \leq \mathfrak{n}_{\alpha,\gamma}(q) = r$. So, from $\pi_{\alpha,\gamma}(q) = r$, we moreover infer that $(q \upharpoonright \alpha') \leq (p \upharpoonright \alpha' + r)$. By Clause (a) of Building Block II, then

$$(q \upharpoonright \alpha', q(\alpha')) \leq (p \upharpoonright \alpha', p(\alpha')) + ((p \upharpoonright \alpha') + r) = (p \upharpoonright \alpha', p(\alpha') + \mathfrak{n}_{\alpha',\gamma}(p \upharpoonright \alpha')(r)).$$

Finally, from Clause (b) of Definition 5.2, the rightmost condition is nothing but $(x, y)$. Altogether, $(q \upharpoonright \alpha', q(\alpha')) \leq (x, y)$, so that $q \leq \alpha' + 1 x \prec \langle y \rangle = \mathfrak{n}_{\alpha,\gamma}(p)(r)$.

- Assume $\alpha \in \text{acc}(\mu^+ + 1)$ is such that the claim holds for all $\beta < \alpha$. Let $\gamma < \alpha$, $p \in P_\alpha$ and $r \leq \gamma \pi_{\alpha,\gamma}(p)$. Set $q := \mathfrak{n}_{\alpha,\gamma}(p)(r)$. By Lemma 7.4(1) and the inductive hypothesis, for all $\beta \in [\gamma, \alpha]$,

$$q \upharpoonright \beta = \mathfrak{n}_{\beta,\gamma}(p \upharpoonright \beta)(r) = ((p \upharpoonright \beta) + r).$$

It thus follows from the definition of $\leq_\alpha$ that $q = p + r$.

(2) Recall that, for all $\beta \leq \mu^+$, $\ell_\beta := \ell_1 \circ \pi_{\beta,1}$. In effect, $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1} = \ell_1 \circ (\pi_{\gamma,1} \circ \pi_{\alpha,\gamma}) = (\ell_1 \circ \pi_{\gamma,1}) \circ \pi_{\alpha,\gamma} = \ell_\gamma \circ \pi_{\alpha,\gamma}$. \hfill \square

Our next task is to verify Clause (v) of Goal 7.2.
Lemma 7.6. Suppose that \( \alpha \leq \mu^+ \) is such that for all nonzero \( \beta \leq \gamma < \alpha \), \((P_\gamma,c_\gamma,\ell_\gamma,\vec{w}_\gamma)\) is a \((\Sigma,\vec{S})\)-Prikry forcing with \( \vec{w}_\gamma = \vec{w}_\beta \bullet \pi_{\gamma,\beta} \).

Then, for all nonzero \( \gamma \leq \alpha \), \((\bar{t}_{\alpha,\gamma},\pi_{\alpha,\gamma})\) is an exact forking projection from \((P_\alpha,\ell_\alpha,c_\alpha,\vec{w}_\alpha)\) to \((P_\gamma,\ell_\gamma,\vec{w}_\gamma)\). If \( \alpha < \mu^+ \), then \((\bar{t}_{\alpha,\gamma},\pi_{\alpha,\gamma})\) is furthermore an exact forking projection from \((P_\alpha,\ell_\alpha,c_\alpha,\vec{w}_\alpha)\) to \((P_\gamma,\ell_\gamma,c_\gamma,\vec{w}_\gamma)\).

Proof. By [PRS20, Lemma 4.7], the above maps are forking projections. To show exactness, we need the following claim:

Claim 7.6.1. For all nonzero \( \gamma \leq \alpha \) and \( p \in P_\alpha \), \( \vec{w}_\alpha = \vec{w}_\gamma \bullet \pi_{\alpha,\gamma} \). Moreover, \((\bar{t}_{\alpha,\gamma},\pi_{\alpha,\gamma})\) is exact.

Proof. By induction on \( \alpha \leq \mu^+ \):

- The case \( \alpha = 1 \) is trivial, since then, \( \gamma = \alpha \).
- Suppose \( \alpha = \alpha' + 1 \) and the claim holds for \( \alpha' \). Recall that \( P_\alpha = P_{\alpha'+1} \) was defined by feeding \((P_{\alpha'},\ell_{\alpha'},c_{\alpha'},\vec{w}_{\alpha'})\) into Building Block II, thus obtaining a \((\Sigma,\vec{S})\)-Prikry forcing \((\bar{a},\ell_{\bar{a}},c_{\bar{a}},\vec{c}_{\bar{a}},\vec{s}_{\bar{a}})\) along with the pair \((\bar{t},\pi)\). Also, we have that \( (x,y) \in A \) iff \( x^\gamma(y) \in P_\alpha \).

By exactness of \((\bar{t},\pi)\) and our recursive definition,

\[
\vec{w}_s^\alpha(x^\gamma(y)) = s(x,y) = s_s^\alpha_\alpha(\pi(x,y)) = \vec{w}_s^{\alpha'}(x) = \vec{w}_s^\alpha(\pi_{\alpha,\alpha'}(x^\gamma(y))),
\]

for all \( n < \omega \) and \( x^\gamma(y) \in (P_\alpha)_{\geq n} \). Hence, \( \vec{w}_\alpha = \vec{w}_\alpha' \bullet \pi_{\alpha,\alpha'} \). Using the induction hypothesis for \( \vec{w}_\alpha' \) we arrive at \( \vec{w}_\alpha = \vec{w}_\gamma \bullet \pi_{\alpha,\gamma} \), as wanted. The exactness of \((\bar{t}_{\alpha,\gamma},\pi_{\alpha,\gamma})\) follows from this equality and Lemma 7.5.

- For \( \alpha \in \text{acc}(\mu^+ + 1) \), the conclusion follows by combining the induction hypotheses with the definition of \( \vec{w}_\alpha \) at limit stages.

This completes the proof.

By now, we have verified all clauses of Goal 7.2 with the exception of Clause (iv). Before we are in conditions to do that, let us point out that the pair \((\bar{t}_{\alpha,1},\pi_{\alpha,1})\) has mixing property for every \( \alpha \geq 1 \).

Lemma 7.7. Let \( 1 \leq \alpha \leq \mu^+ \), and suppose that, for all nonzero \( \beta \leq \gamma < \alpha \), \((P_\gamma,\ell_\gamma,c_\gamma,\vec{w}_\gamma)\) is a \((\Sigma,\vec{S})\)-Prikry quadruple, and \( \vec{w}_\gamma = \vec{w}_\beta \bullet \pi_{\gamma,\beta} \). Then \((\bar{t}_{\alpha,1},\pi_{\alpha,1})\) has the mixing property.

Proof. See the proof of [PRS20, Lemma 4.8].

The following result will be key in the proof of Lemma 7.9, where we verify Clause (iv) of Goal 7.2.

Lemma 7.8. Let \( n < \omega \). For each \( \alpha \in [2,\mu^+] \), \( (P_\alpha)^{\bar{t}_{\alpha,1}} \) is \( \sigma_n^* \)-directed-closed.

Proof. We argue by induction on \( \alpha \). The proof of the base case \( \alpha = 2 \) follows by an argument similar (and actually simpler) to the successor case below. Thus, we will suppose that the base case have been covered and that we are given an \( \alpha \in (2,\mu^+] \) such that the conclusion of the lemma is true for each \( \bar{\alpha} \) with \( 1 < \bar{\alpha} < \alpha \).
Case $\alpha = \beta + 1$: Fix a directed set $D \subseteq (P_\alpha)_n^{\pi_\alpha,1}$ of size less than $\sigma_n^*$. Since $\pi_{\alpha,\beta}$ is a projection and $\pi_{\alpha,1} = \pi_{\alpha,\beta} \circ \pi_{\beta,1}$, $D_\beta := \pi_{\alpha,\beta}[D]$ is a directed family of $(P_\beta)_n^{\pi_\beta,1}$ with $\pi_{\beta,1}[D_\beta] = \pi_{\alpha,1}[D]$. By the inductive assumption, let $p_\beta$ be a $\leq_{\pi_{\beta,1},1}$-lower bound for $D_\beta$. In particular, $\pi_{\alpha,1}[D] = \{\pi_{\beta,1}(p_\beta)\}$. Now consider

$$D := \{\tilde{\pi}_{\alpha,\beta}(p)(p_\beta) \mid p \in D\}.$$ 

Since $(\tilde{\phi}_{\alpha,\beta}, \pi_{\alpha,\beta})$ is a forking projection and $D$ is directed, Definition 5.1(7), $\tilde{D}$ is directed in $(P_\alpha)_n^{\pi_\alpha,1}$. Also, by Definition 5.1(5), $\pi_{\alpha,1}[\tilde{D}] = \{\pi_{\beta,1}(p_\beta)\}$ and $\pi_{\alpha,\beta}[\tilde{D}] = \{p_\beta\}$. Clearly, $|\tilde{D}| < \mu$.

Recall that $P_\alpha$ was defined by feeding $(P_\beta, \ell_\beta, c_\beta, \vec{\beta})$ into Building Block II, thus obtaining a $(\Sigma, \vec{S})$-Prikry quadruple $(\delta, \ell_\delta, c_\delta, \vec{\gamma})$ along with an exact forking projection $(\delta, \pi)$. Thus, $\tilde{D}$ can be identified with a directed subset of $(\delta)^n$. By Clause (d) of Building Block II, $(\delta)^n$ is $\sigma_n$-directed-closed and so there is a $\leq_{\pi_{\alpha,\beta}}$-lower bound $p$ for $\tilde{D}$. Then,

$$\pi_{\alpha,1}(p) = \pi_{\beta,1}(\pi_{\alpha,\beta}(p)) = \pi_{\beta,1}(p_\beta),$$

which is the sole condition in $\pi_{\alpha,1}[\tilde{D}]$. Thus, $p$ yields the desired $\leq_{\pi_{\alpha,\beta}}$-lower bound for $\tilde{D}$.

Case $\alpha$ is limit: Fix a directed family $D \subseteq (P_\alpha)_n^{\pi_\alpha,1}$ of size less than $\sigma_n^*$. Let $C := \text{cl}(\bigcup_{p \in D} B_p) \cup \{1, \alpha\}$. As $|D| < \sigma_n^* \leq \mu$, the regularity of $\mu$ implies that $|C| < \mu$. We will define a $\leq$-increasing sequence $\langle p_\beta \mid \beta \in C \rangle \in \prod_{\beta \in C}(P_\beta)_n$ such, for all $\beta \in C$ and $\tilde{\beta} \in C \cap \beta$, $p_\beta$ is a lower bound for $\{p \mid \beta \mid p \in D\}$ and $\pi_{\beta,1}(p_\beta) = \pi_{\tilde{\beta},1}(p_\tilde{\beta})$. The definition is by recursion on $\beta \in C$:

- For $\beta = 1$, $\{p \mid 1 \mid p \in D\}$ is a singleton, so let $p_1$ be this value.
- Suppose $\beta > 1$ is a non-accumulation point of $C \cap \alpha$. Let $\delta := \delta + 1$ and $\gamma := \text{sup}(C \cap \beta)$. Clearly, $\gamma \leq \delta$. For each $p \in D$, $p|\delta = p|\gamma * \theta_\delta$, and by Lemma 7.4(5), $\tilde{\pi}_{\delta,\gamma}(p | \delta)(p_\gamma) = p_{\gamma} * \theta_\delta$. Set $q := p_{\gamma} * \theta_\delta$ and consider

$$\tilde{D} := \{\tilde{\pi}_{\beta,\delta}(p | \beta) | p \in D\}.$$ 

Then $\tilde{D}$ is a directed subset of $(P_\beta)_n^{\pi_{\beta,1},1}$ of size less than $\sigma_n^*$. As $\beta = \delta + 1$, by the successor case above, let $p_\beta$ be a $\leq_{\pi_{\beta,1},1}$-lower bound for $\tilde{D}$. Then $p_\beta$ is also a lower bound for $D$, such that $\pi_{\beta,1}(p_\beta) = \pi_{\gamma,1}(p_\gamma) = \pi_{\tilde{\beta},1}(p_{\tilde{\beta}})$, for each $\tilde{\beta} \in C \cap \beta$.

- Suppose $\beta \in \text{acc}(C)$. Define $p_\beta := \bigcup_{\tilde{\beta} \in (C \cap \beta)} P_{\tilde{\beta}}$. By regularity of $\mu$ and since $|C| < \mu$, we have $|B_{p_\beta}| < \mu$, and so $p_\beta \in P_{\beta}$. Now, for all $p \in D$ and $\tilde{\beta} \in C \cap \beta$, we have $p_\beta | \tilde{\beta} = p_{\tilde{\beta}} \leq_{\tilde{\beta}} p | \tilde{\beta}$. So, $p_\beta$ is a lower bound for $\{p \mid \beta \mid p \in D\}$. Moreover, by definition, $\pi_{\beta,1}(p_\beta) = \pi_{\tilde{\beta},1}(p_{\tilde{\beta}})$, and so $p_\beta$ is as desired.

- Suppose $\beta = \alpha$, but $\beta \notin \text{acc}(C)$. Set $\tilde{\alpha} := \text{sup}(C \cap \alpha)$, and $p_\alpha := \alpha * \theta_\alpha$. As the interval $(\tilde{\alpha}, \alpha]$ is disjoint from $\bigcup_{p \in D} B_p$, ...
for every \( p \in D \),
\[
p_\alpha = (p_\alpha \restriction \bar{\alpha}) \ast \emptyset_\alpha \leq_\alpha (p \restriction \bar{\alpha}) \ast \emptyset_\alpha = p.
\]

Also, \( \pi_{\alpha,1}(p_\alpha) = \pi_{\bar{\alpha},1}(p_\bar{\alpha}) = \pi_{\beta,1}(p_\beta) \), for each \( \beta \in C \cap \alpha \).

Clearly, \( p_\alpha \) is a \( \leq_{\pi_{\alpha,1}} \)-lower bound for \( D \), as desired. \( \square \)

**Lemma 7.9.** For all nonzero \( \alpha \leq \mu^+ \), \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)\) is \((\Sigma, \vec{S})\)-Prikry with a greatest element \( \emptyset_\alpha \), \( \ell_\alpha = \ell_1 \circ \pi_{\alpha,1} \) and \( 1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \hat{\mu} = \kappa^+ \). Moreover, \( \vec{\omega}_\alpha = \vec{\omega}_\gamma \bullet \pi_{\alpha,\gamma} \) for every \( \gamma \leq \alpha \).

**Proof.** We argue by induction on \( \alpha \leq \mu^+ \). The base case \( \alpha = 1 \) follows from the fact that \( \mathbb{P}_1 \) is isomorphic to \( \mathbb{Q} \) given by Building Block I. The successor step \( \alpha = \beta + 1 \) follows from the fact that \( \mathbb{P}_{\beta+1} \) was obtained by invoking Building Block II.

Next, suppose that \( \alpha \in \text{acc}(\mu^+ + 1) \) is such that the conclusion of the lemma holds below \( \alpha \). In particular, the hypothesis of Lemma 7.6 is satisfied, so that, for all nonzero \( \beta \leq \gamma \leq \alpha \), \((\mathbb{P}_\gamma, \ell_\gamma, \vec{\omega}_\gamma)\) is a forking projection from \((\mathbb{P}_\gamma, \ell_\gamma, \vec{\omega}_\gamma)\) to \((\mathbb{P}_\beta, \ell_\beta, \vec{\omega}_\beta)\). We now go over the clauses of Definition 3.3.

By the very same proof of [PRS20, Lemma 4.9], we have that Clauses (1)–(7) of the definition of \((\Sigma, \vec{S})\)-Prikry hold for \((\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\omega}_\alpha)\), and that \( \ell_\alpha = \ell_1 \circ \pi_{\alpha,1} \). Thus, it remains to show Clauses (8) and (9).

(8) As \((\ell_{\alpha,1}, \pi_{\alpha,1})\) is an exact forking projection from \((\mathbb{P}_\alpha, \ell_\alpha, \vec{\omega}_\alpha)\) to \((\mathbb{P}_1, \ell_1, \vec{\omega}_1)\), \( \vec{\omega}_\alpha = \vec{\omega}_1 \bullet \pi_{\alpha,1} \). Thus, the result follows using the fact that exact forking projections is preserved under composition and the fact that \( \vec{\omega}_1 \) witnesses Definition 3.3(8).

(9) Let \( n < \omega \). Since \((\ell_{\alpha,1}, \pi_{\alpha,1})\) is an exact forking projection from \((\mathbb{P}_\alpha, \ell_\alpha, \vec{\omega}_\alpha)\) to \((\mathbb{P}_1, \ell_1, \vec{\omega}_1)\), Lemma 5.14 reduces our task to showing that \( \mathbb{P}_1 \upharpoonright_{\pi_n}^{\pi_1} \) and \( \mathbb{P}_\alpha \upharpoonright_{\pi_{\alpha,1}}^{\pi_1} \) are \( \sigma_n \)-directed-closed. The former follows from the fact that \((\mathbb{P}_1, \ell_1, c_1, \vec{\omega}_1)\) is \((\Sigma, \vec{S})\)-Prikry, and the latter follows from Lemma 7.8.

Finally, by the same proof as in [PRS20, Corollary 4.9.2], we have that, for each \( 1 \leq \alpha \leq \mu^+ \), \( 1_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \hat{\mu} = \kappa^+ \). \( \square \)

### 8. A Proof of the Main Theorem

In this section, we arrive at the primary application of the framework developed thus far. We will be constructing a model where GCH holds below \( \aleph_\omega \), \( 2^{\aleph_\omega} = \aleph_{\omega+2} \) and every stationary subset of \( \aleph_{\omega+1} \) reflects.

#### 8.1. Setting up the ground model

We want to obtain a ground model with GCH and \( \omega \)-many supercompact cardinals, which are Laver indestructible under GCH-preserving forcing. The first lemma must be well-known, but we could not find it in the literature, so we give an outline of the proof.

**Lemma 8.1.** Suppose \( \vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle \) is an increasing sequence of supercompact cardinals. Then there is a generic extension where GCH holds and \( \vec{\kappa} \) remains an increasing sequence of supercompact cardinals.
Proof. By preparing the ground model ala Laver [Lav78], we may assume that, for each \( n < \omega \), the supercompactness of \( \kappa_n \) is indestructible under \( \kappa_n \)-directed-closed forcing.

**Claim 8.1.1.** There is a generic extension in which \( \kappa \) remains an increasing sequence of supercompact cardinals, and \( \Theta := \{ \theta \in \text{CARD} \mid \theta < \theta = \theta \} \) forms a proper class. \( \square \)

For ease of notation, we denote it by \( V \).

Now, working in \( V \), let \( \mathbb{J} \) be Jensen’s iteration to force the GCH. Namely, \( \mathbb{J} \) is the inverse limit of the Easton-support iteration \( \langle \mathbb{J}_\alpha : Q_\beta \mid \beta \leq \alpha \in \text{Ord} \rangle \) such that, if \( 1 \models_{\mathbb{J}_\alpha} \) “\( \alpha \) is a cardinal”, then \( 1 \models_{\mathbb{J}_\alpha} \) “\( Q_\alpha = \text{Add}(\alpha^+, 1) \)” and \( 1 \models_{\mathbb{J}_\alpha} \) “\( Q_\alpha \) trivial”, otherwise. Let \( G \) be a \( \mathbb{J} \)-generic filter over \( V \).

Let \( n < \omega \). We claim that \( \mathbb{J} \) preserves the supercompactness of \( \kappa_n \). To this end, let \( \theta \) be an arbitrary cardinal. By possibly enlarging \( \theta \), we may assume that \( \theta \in \Theta \). Let \( j : V \rightarrow M \) be an elementary embedding induced by a \( \theta \)-supercompact measure over \( P_{\kappa_n}(\theta) \). In particular, we are taking \( j \) such that \( \text{crit}(j) = \kappa_n \), \( j(\kappa_n) > \theta \), \( (\theta \cap M) \cap \mathbb{V} \subseteq \mathbb{M} \) and

\[
M = \{ j(f)(j^*\theta) \mid f : P_{\kappa_n}(\theta) \rightarrow \mathbb{V} \}.
\]

Observe that \( \mathbb{J} \) can be factored into three forcings: the iteration up to \( \kappa_n \), the iteration in the interval \( [\kappa_n, \theta) \) and, finally, the iteration in the interval \( \theta, \text{Ord} \). For an interval of ordinals \( \mathcal{I} \), let \( G_\mathcal{I} \) denote the \( \mathbb{J}_\mathcal{I} \)-generic filter induced by \( G \). Similarly, we define \( G_\mathcal{I}^\ast := G_\mathcal{I} \cap j(\mathbb{J})_\mathcal{I} \).

**Claim 8.1.2.** In \( V[G] \), there is a lifting \( j_1 : V[G_{\kappa_n}] \rightarrow M[G_{\kappa_n}^\ast \mathbb{I}_{j(\kappa_n)}] \) of \( j \) such that

\[
\left( \theta M[G_{\kappa_n}^\ast \mathbb{I}_{j(\kappa_n)}] \right) \cap V[G_{\kappa_n}^\ast \mathbb{I}_{j(\kappa_n)}] \subseteq M[G_{\kappa_n}^\ast \mathbb{I}_{j(\kappa_n)}].
\]

Moreover, \( H_\theta^M[G_{\kappa_n}^\ast \mathbb{I}_{j(\kappa_n)}] \subseteq V[G_\theta] \). \( \square \)

**Claim 8.1.3.** In \( V[G] \), there is a lifting \( j_2 : V[G_\theta] \rightarrow M[G_{\kappa_n}^\ast \mathbb{I}_{j(\theta)}] \) of \( j_1 \). \( \square \)

**Claim 8.1.4.** In \( V[G] \), there is a lifting \( j_3 : V[G] \rightarrow M[G_{\kappa_n}^\ast \mathbb{I}_{j(\theta)} \ast \mathbb{K}] \) of \( j_2 \).

Finally, define

\[
\mathcal{U} := \{ X \in P_{\kappa_n}(\theta) \mid j^\ast \theta \in j_3(X) \}.
\]

As \( j^\ast \theta \in M \subseteq M[G_{j(\theta)}^\ast \mathbb{I} \ast \mathbb{K}] \), standard arguments now show that \( \mathcal{U} \) is a \( \theta \)-supercompact measure over \( P_{\kappa_n}(\theta) \). In particular, \( \kappa_n \) is \( \theta \)-supercompact in \( V[G] \), as wanted. \( \square \)

Note that in the model of the conclusion the above lemma, the \( \kappa_n \)'s are no longer indestructible. Our next task is to remedy that, while maintaining GCH. For this we need the following slight variation of the usual Laver preparation [Lav78].
Lemma 8.2. Suppose that GCH holds, $\chi < \kappa$ are infinite regular cardinals, and $\kappa$ is supercompact. Then there exists a $\chi$-directed-closed notion of forcing $L^\kappa_\chi$ that preserves GCH and makes the supercompactness of $\kappa$ indestructible under $\kappa$-directed-closed forcings that preserve GCH.

Proof. Let $f$ be a Laver function on $\kappa$, as in [Cum10, Theorem 24.1]. Let $L^\kappa_\chi$ be the direct limit of the Laver-style forcing iteration $\langle \mathbb{R}_\alpha; \check{Q}_\beta \mid \chi \leq \beta < \alpha < \kappa \rangle$ where, if $\alpha$ is inaccessible, $\mathbb{R}_\alpha \Vdash \text{GCH}$, and $f(\alpha)$ encodes an $\mathbb{R}_\alpha$-name $\tau \in H_{\alpha^+}$ for some $\alpha$-directed-closed forcing that preserves the GCH of $V^{\mathbb{R}_\alpha}$, then $\check{Q}_\alpha$ is chosen to be such $\mathbb{R}_\alpha$-name. Otherwise, $\check{Q}_\alpha$ is chosen to be the trivial forcing.

As in the proof of [Cum10, Theorem 24.12], we have that after forcing with $L^\kappa_\chi$, the supercompactness of $\kappa$ becomes indestructible under $\kappa$-directed-closed forcings that preserve GCH.

We claim that GCH holds in $V^{L^\kappa_\chi}$. This is clear for cardinals $\geq \kappa$, since the iteration has size $\kappa$. Now, let $\lambda < \kappa$ and inductively assume GCH$_{<\lambda}$. Observe that $L^\kappa_\chi \cong \mathbb{R}_{\lambda+1} * \check{Q}$, where $\check{Q}$ is an $\mathbb{R}_{\lambda+1}$-name for a $\lambda^+$-directed-closed forcing. In particular, $P(\lambda)^{V^{L^\kappa_\chi}} = P(\lambda)^{V^{\mathbb{R}_{\lambda+1}}}$, and so it is enough to show that $V^{\mathbb{R}_{\lambda+1}} \Vdash \text{CH}_\lambda$. There are two cases.

If $\lambda$ is singular, then $|\mathbb{R}_\lambda| = \lambda^+$, and $\check{Q}_\lambda$ is trivial, so $V^{\mathbb{R}_{\lambda+1}} \Vdash \text{CH}_\lambda$.

Otherwise, let $\alpha$ be the largest inaccessible, such that $\alpha < \lambda$. Then $\mathbb{R}_{\lambda+1}$ is just $\mathbb{R}_\alpha * \check{Q}_\alpha$ followed by trivial forcing. Since $|\mathbb{R}_\alpha| = \alpha$ and by construction $\check{Q}_\alpha$ preserves GCH, the result follows. \hfill \Box

Corollary 8.3. Suppose that $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then, in some forcing extension, all of the following hold:

1. GCH holds;

2. $\vec{\kappa}$ is an increasing sequence of supercompact cardinals;

3. For every $n < \omega$, the supercompactness of $\kappa_n$ is indestructible under notions of forcing that are $\kappa_n$-directed-closed and preserves the GCH.

Proof. By Lemma 8.1, we may assume that we are working in a model in which Clauses (1) and (2) already hold. Next, let $L$ be the direct limit of the iteration $\langle \mathbb{L}_n * \check{Q}_n \mid n < \omega \rangle$, where $\mathbb{L}_0$ is the trivial forcing and, for each $n$, if $1 \Vdash_{\mathbb{L}_n} \text{“} \kappa_n \text{ is supercompact} \text{“}$, then $1 \Vdash_{\mathbb{L}_n} \check{Q}_n$ is the $(\kappa_{n-1})$-directed-closed, GCH-preserving forcing making the supercompactness of $\kappa_n$ indestructible under GCH-preserving $\kappa_n$-directed-closed notions of forcing. (More precisely, in the notation of the previous lemma, $\check{Q}_n$ is $\mathbb{L}_{\kappa_{n-1}} \check{\kappa}_n$, where, by convention, $\kappa_{-1}$ is $\kappa_0$).

Note that, by induction on $n < \omega$, and Lemma 8.2, we maintain that $1 \Vdash_{\mathbb{L}_n} \text{“} \kappa_n \text{ is supercompact and GCH holds} \text{“}$. And then when we force with $\check{Q}_n$ over that model, we make this supercompactness indestructible under GCH-preserving forcing.
Then, after forcing with $\mathbb{L}$, GCH holds, and each $\kappa_n$ remains supercompact, indestructible under $\kappa_n$-directed-closed forcings that preserve GCH.

### 8.2. Connecting the dots.

**Setup 8.** For the rest of this section, we make the following assumptions:

- $\vec{k} = \langle \kappa_n | n < \omega \rangle$ is an increasing sequence of supercompact cardinals. By convention, we set $\kappa_0 := \kappa_0$;
- For every $n < \omega$, the supercompactness of $\kappa_n$ is indestructible under notions of forcing that are $\kappa_n$-directed-closed and preserve the GCH;
- $\kappa := \sup_{n < \omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- GCH holds below $\lambda$. In particular, $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$;
- $\Sigma := \langle \sigma_n | n < \omega \rangle$, where $\sigma_0 := \aleph_1$ and $\sigma_{n+1} := (\kappa_n)^+$ for all $n < \omega$.

By convention, we set $\sigma_{-2}$ and $\sigma_{-1}$ to be $\aleph_0$;
- $\vec{S}$ is as in Definition 4.13.

We now want to appeal to the iteration scheme of the previous section. First, observe that $\mu$, $\langle (\sigma_n, \mu) | n < \omega \rangle$, $\vec{S}$ and $\Sigma$ respectively fulfill all the blanket assumptions of Setup 7. We now introduce our three building blocks of choice:

**Building Block I.** We let $(Q, \ell, c, \vec{v})$ be EBPFC as defined in Section 4. By Corollary 4.31, this is $(\Sigma, \vec{S})$-Prikry. Also, $Q$ is a subset of $H_{\mu^+}$ and, by Lemma 4.30, $1_Q \Vdash Q \mu = \kappa^+$. In addition, $\kappa$ is singular, so that $1_Q \Vdash Q \kappa$ is singular”.

**Building Block II.** For every $(\Sigma, \vec{S})$-Prikry quadruple $(P, \ell_P, c_P, \vec{v})$ such that $P = (P, \leq)$ is a subset of $H_{\mu^+}$, $1_P \Vdash_P \mu = \kappa^+$, and $1_P \Vdash_P \kappa$ is singular”, every $r^* \in P$, and every $P$-name $z \in H_{\mu^+}$, we are given a corresponding $(\Sigma, \vec{S})$-Prikry quadruple $(A, \ell_A, c_A, \vec{z})$ such that the following hold true:

(a) $(\hat{\mathcal{H}}, \pi)$ is an exact forking projection from $(A, \ell_A, c_A, \vec{z})$ to $(P, \ell_P, c_P, \vec{v})$ that has the mixing property;
(b) $1_A \Vdash_A \hat{\mu} = \hat{\kappa}^+$;
(c) $A = (A, \leq)$ is a subset of $H_{\mu^+}$;
(d) for each $n < \omega$, $A^n$ is $\mu$-directed-closed;
(e) if $r \in P$ and $z$ is a $P$-name for an $r^*$-fragile stationary subset of $\mu$ then 

\[ [r^*]A \Vdash_A \text{"}z\text{" is nonstationary".} \]

**Remark 8.4.** The above block is obtained as follows.

- If $r^* \in P$ and $z$ is a $P$-name for an $r^*$-fragile stationary subset of $\mu$, then we invoke Corollary 6.10.
- Otherwise, let $A := (A, \leq)$, where $A := P \times \{\emptyset\}$ and $(p, q) \leq (p', q')$ iff $p \leq p'$. Define $\pi : A \rightarrow P$ via $\pi(x, y) := x$. Define $\hat{\mathcal{H}}$ via $\hat{\mathcal{H}}(a)(p) := (p, \emptyset)$ and let $\ell_A := \ell_P \circ \pi$, $c_A := c_P \circ \pi$ and $\vec{z} = \vec{v} \circ \pi$. It is straightforward to verify that $(A, \ell_A, c_A, \vec{z})$ and $(\hat{\mathcal{H}}, \pi)$ satisfy all the above requirements.
Building Block III. As $2^\mu = \mu^+$, we fix a surjection $\psi : \mu^+ \to H_{\mu^+}$ such that the preimage of any singleton is cofinal in $\mu^+$.

Now, we appeal to the iteration scheme of Section 7 with these building blocks, and obtain, in return, a sequence $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \varpi_\alpha) \mid 1 \leq \alpha \leq \mu^+ \rangle$ of $(\Sigma, \bar{S})$-Prikry quadruples. By Lemmas 7.8 and 7.9, for all nonzero $\alpha \leq \mu^+$, $(\mathbb{P}_\alpha)_{n-1}^{\alpha-1}$ is $\mu$-directed-closed and $\mathbb{I}_{\mathbb{P}_\alpha} \Vdash \bar{\mu} = \kappa^+$. Note that by the first clause of Goal 7.2, $|P_\alpha| \leq \mu^+$ for every $\alpha \leq \mu^+$.

Lemma 8.5. Let $n \in \omega \setminus 2$ and $\alpha \in [2, \mu^+)$. Then $((\mathbb{P}_\alpha)_n, \mathbb{S}_n, \varpi_n^\alpha)$ is suitable for reflection with respect to $\langle \sigma_{n-2}, \kappa_{n-1}, \kappa_n, \mu \rangle$.

**Proof.** We go over the clauses of Lemma 5.15 with $\mathbb{P}_\alpha$ playing the role of $\mathbb{A}$, $\varpi_n^\alpha$ playing the role of $\mathbb{c}_n$, and $\mathbb{P}_1$ playing the role of $\mathbb{P}$.

As $\mathbb{P}_1$ is isomorphic to the notion of forcing given by Building Block I, which is given by Section 4, we simplify the notation here, and — for the scope of this proof — we let $\mathbb{P}$ denote the forcing $\mathbb{P}$ from Section 4.

Clause (i) is part of the assumptions of Setup 8. Clauses (ii) and (iii) are given by our iteration theorem. Clause (iv) is due to Corollary 4.37, and the fact that $\mathbb{P}_1$ is isomorphic to Gitik’s EBPFC. Now, we turn to address Clause (v).

That is, we need to prove that in any generic extension by $\mathbb{S}_n \times (\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$,

$$|\mu| = \text{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}.$$  

The upcoming discussion assumes the notation of Section 4. By Lemma 4.39, we have:

1. $T_n$ has the $\kappa_n$-cc and size $\kappa_n$;
2. $\psi_n$ defines an exact nice projection;
3. $(\mathbb{P}^\psi_n)_n$ is $\kappa_n$-directed-closed;
4. For each $p \in P_n$, $\mathbb{P}_n \downarrow p$ and $(T_n \downarrow \psi_n(p)) \times ((\mathbb{P}^\psi_n)_n \downarrow p)$ are forcing equivalent.

By Lemma 4.36, $\mathbb{P}_n$ forces $|\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^+ = (\kappa_{n-1})^{++}$, and by our remark before the statement of this lemma, $(\mathbb{P}_\alpha)_{n-1}^{\alpha-1}$ is $\mu$-directed-closed, hence $\kappa_n$-directed-closed. Combining Clauses (1), (3) and (4) above with Easton’s Lemma, $(\mathbb{P}^\psi_n)_n \times (\mathbb{P}_\alpha)_{n-1}^{\alpha-1}$ is $\kappa_n$-distributive over $V^T_n$, and so $\mathbb{P}_n \times (\mathbb{P}_\alpha)_{n}^{\varpi_n^\alpha}$ forces $\kappa_n = (\kappa_{n-1})^{++}$. Moreover, as $\mathbb{P}_n \times (\mathbb{P}_\alpha)_{n}^{\varpi_n^\alpha}$ projects to $\mathbb{P}_n$ and the former preserves $\kappa_n$, it also forces $|\mu| = \text{cf}(\mu)$. Altogether, $\mathbb{P}_n \times (\mathbb{P}_\alpha)_{n}^{\varpi_n^\alpha}$ forces $|\mu| = \text{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}$. To establish that the same configuration is being forced by $\mathbb{S}_n \times (\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$, we give a sandwich argument, as follows:

- $\mathbb{P}_n \times (\mathbb{P}_\alpha)_{n}^{\varpi_n^\alpha}$ projects to $\mathbb{S}_n \times (\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$, as witnessed by $(p, q) \mapsto (\varpi_n(p), q)$;
- For any condition $p$ in $(\mathbb{P}_\alpha)_n$, $(\mathbb{S}_n \downarrow \varpi_n^\alpha(p)) \times ((\mathbb{P}_\alpha)_n^{\varpi_n^\alpha} \downarrow p)$ projects to $(\mathbb{P}_\alpha)_n \downarrow p$, by Lemma 2.6.
- $(\mathbb{P}_\alpha)_n$ projects to $\mathbb{P}_n$ via $\pi_n^{\alpha,1}$.
This completes the proof. □

**Lemma 8.6.** Let \( n < \omega \) and \( 0 < \alpha < \mu^+ \). Then \( (P_\alpha)^{\omega_\alpha}_n \) preserves GCH.

**Proof.** The case \( \alpha = 1 \) is taken care of by Lemma 4.40.

Now, let \( \alpha \geq 2 \). Since \( (P_\alpha)^{\omega_\alpha}_n \) is \( \sigma_n \)-directed-closed, it preserves GCH below \( \sigma_n \). By the sandwich analysis from the proof of Lemma 8.5, in any generic extension by \( (P_\alpha)^{\omega_\alpha}_n \), \( |\mu| = \text{cf}(\mu) = \kappa_n = (\sigma_n)^+ \). So, as \( (P_\alpha)^{\omega_\alpha}_n \) is a notion of forcing of size \( \leq \mu^+ \), collapsing \( \mu \) to \( \kappa_n \), it forces \( 2^\theta = \theta^+ \) for any cardinal \( \theta > \kappa_n \).

It thus left to verify that \( (P_\alpha)^{\omega_\alpha}_n \) forces \( 2^\theta = \theta^+ \) for \( \theta \in \{ \sigma_n, \kappa_n \} \).

Arguing as in Lemma 8.5, for any condition \( p \in (P_\alpha)_n \), \( (\mathbb{T}_n \downarrow \psi_n(p)) \times ((\mathbb{P}^\alpha_\alpha)_n \downarrow p) \times (P_\alpha)^{\omega_\alpha}_n \) projects onto \( (P_\alpha)^{\omega_\alpha}_n \). Recall that the first factor of the product is a \( \kappa_n \)-cc forcing of size \( \leq \kappa_n \). By Lemma 7.8, the second factor is altogether a \( \kappa_n \)-directed-closed forcing. Thus, by Easton’s lemma, this product preserves CH\( \sigma_n \) if and only if \( \mathbb{T}_n \downarrow \psi_n(p) \) does. And this is indeed the case, as the number of \( \mathbb{T}_n \)-nice names for subsets of \( \sigma_n \) is at most \( \kappa_n \).

Again, arguing as in Lemma 8.5, \( (P_1)_n \times (P_\alpha)^{\omega_\alpha}_n \) projects onto \( S_n \times (P_\alpha)^{\omega_\alpha}_n \), which projects onto \( (P_\alpha)^{\omega_\alpha}_n \). Since \( (P_\alpha)^{\omega_\alpha}_n \) is \( \mu \)-directed-closed, it preserves CH\( \sigma_n \). Also, it preserves \( \mu \) and so, by Lemma 4.40(1) and the absoluteness of the \( \mu^+ \)-Linked property, \( (P_1)_n \) is also \( \mu^+ \)-Linked in \( V^{(P_\alpha)^{\omega_\alpha}_n} \).

Once again, counting-of-nice-names arguments implies that this latter forcing forces \( 2^{\kappa_n} \leq \mu^+ = (\kappa_n)^+ \). Thus, \( (P_1)_n \times (P_\alpha)^{\omega_\alpha}_n \) preserves CH\( \kappa_n \) and so does \( (P_\alpha)^{\omega_\alpha}_n \). □

**Theorem 8.7.** In \( V^{P_{\mu^+}} \), all of the following hold true:

1. All cardinals \( \geq \kappa \) are preserved;
2. \( \kappa = \aleph_\omega \), \( \mu = \aleph_{\omega + 1} \) and \( \lambda = \aleph_{\omega + 2} \);
3. \( 2^{\aleph_n} = \aleph_{n + 1} \) for all \( n < \omega \);
4. \( 2^{\aleph_\omega} = \aleph_{\omega + 2} \);
5. Every stationary subset of \( \aleph_{\omega + 1} \) reflects.

**Proof.** (1) We already know that \( \mathbb{P}^P_{\mu^+} \models \mu = \kappa^+ \). By Lemma 3.12(2), \( \kappa \) remains strong limit cardinal in \( V^{P_{\mu^+}} \). Finally, as Clause (3) of Definition 3.3 holds for \( (P_{\mu^+}, \ell_{\mu^+}, c_{\mu^+}, \mathbb{S}_{\mu^+}) \), \( P_{\mu^+} \) has the \( \mu^+ \)-chain-condition, so that all cardinals \( \geq \kappa^+ \) are preserved.

(2) Let \( G \subseteq P_{\mu^+} \) be an arbitrary generic over \( V \). By virtue of Clause (1) and Setup 8, it suffices to prove that \( V[G] \models \kappa = \aleph_\omega \). Let \( G_1 \) the \( P_1 \)-generic filter generated by \( G \) and \( \pi_{\mu^+} \). By Corollary 4.32, \( V[G_1] \models \kappa = \aleph_\omega \). Thus, let us prove that \( V[G] \) and \( V[G_1] \) have the same cardinals \( \leq \kappa \).

Of course, \( V[G_1] \subseteq V[G] \), and so any \( V[G] \)-cardinal is also a \( V[G_1] \)-cardinal. Towards a contradiction, suppose that there is a \( V[G_1] \)-cardinal \( \theta < \kappa \) that ceases to be so in \( V[G] \). Any surjection witnessing this can be encoded as a bounded subset of \( \kappa \), hence as a bounded subset of some
\[ \sigma_n \] for some \( n < \omega \). Thus, Lemma 3.12(1) implies that \( \theta \) is not a cardinal in \( V[H_n] \), where \( H_n \) is the \( S_m \)-generic filter generated by \( G_1 \) and \( \varpi_1^n \). As \( V[H_n] \subseteq V[G_1] \), \( \theta \) is not a cardinal in \( V[G_1] \), which is a contradiction.

(3) On one hand, by Lemma 3.12(1), \( \mathcal{P}(\aleph_n)^{V_{\mu^+}} = \mathcal{P}(\aleph_n)^{V_{\mu_1}} \) for some \( m < \omega \). On the other hand, as \( GCH_{\aleph_1} \) holds (cf. Setup 8), Remark 4.14 shows that \( S_m \) preserves \( \mathbf{CH} \). Altogether, \( V^{2_{\mu^+}} \models \mathbf{CH} \).

(4) By Setup 8, \( V \models 2^\kappa = \kappa^+ \). In addition, \( \mathbb{P}_{\mu^+} \) is isomorphic to a notion of forcing lying in \( H_{\mu^+} \) (see [PRS20, Remark 4.3(1)]) and \( |H_{\mu^+}| = \lambda \). Thus, \( V_{\mu^+}^{2_{\mu^+}} \models 2^\kappa \leq \lambda \). In addition, \( \mathbb{P}_{\mu^+} \) projects to \( \mathbb{P}_1 \), which is isomorphic to \( Q_1 \), being a poset blowing up \( 2^\kappa \) to \( \lambda \), as seen in Proposition 4.32, so that \( V_{\mu^+}^{2_{\mu^+}} \models 2^\kappa \geq \lambda \). So, \( V_{\mu^+}^{2_{\mu^+}} \models 2^\kappa = \lambda \). Thus, together with Clause (2), \( V_{\mu^+}^{2_{\mu^+}} \models 2^{\aleph_0} = \aleph_{\omega+2} \).

(5) Let \( G \) be \( \mathbb{P}_{\mu^+} \)-generic over \( V \) and hereafter work in \( V[G] \). Towards a contradiction, suppose that there exists a stationary set \( T \subseteq \mu \) that does not reflect. By shrinking, we may assume the existence of some regular cardinal \( \theta < \mu \) such that \( T \subseteq \mathcal{P}_\theta^\mu \). Fix \( r^* \in G \) and a \( \mathbb{P}_{\mu^+} \)-name \( \tau \) such that \( \tau_G \) is equal to such a \( T \) and such that \( r^* \) forces \( \tau \) to be a stationary subset of \( \mu \) that does not reflect. Since \( \mu = \kappa^+ \) and \( \kappa \) is singular in \( V \), by possibly enlarging \( r^* \), we may assume that \( r^* \) forces \( \tau \) to be a subset of \( \Gamma_{\ell(r^*)} \) (see page 38).

Furthermore, we may require that \( \tau \) be a nice name, i.e., each element of \( \tau \) is a pair \( (\xi, p) \) where \( (\xi, p) \in \Gamma_{\ell(r^*)} \times \mathcal{P}_{\mu^+} \), and, for each ordinal \( \xi \in \Gamma_{\ell(r^*)} \), the set \( \{ p \in \mathcal{P}_{\mu^+} \mid (\xi, p) \in \tau \} \) is a maximal antichain.

As \( \mathbb{P}_{\mu^+} \) satisfies Clause (3) of Definition 3.3, \( \mathbb{P}_{\mu^+} \) has in particular the \( \mu^+ \)-cc. Consequently, there exists a large enough \( \beta < \mu^+ \) such that \( B_{r^*} \cup \bigcup \{ B_p \mid (\xi, p) \in \tau \} \subseteq \beta \).

Let \( r := r^* \upharpoonright \beta \) and set
\[
\sigma := \{ (\xi, p \upharpoonright \beta) \mid (\xi, p) \in \tau \}.
\]
From the choice of Building Block III, we may find a large enough \( \alpha < \mu^+ \) with \( \alpha > \beta \) such that \( \psi(\alpha) = (\beta, r, \sigma) \). As \( \beta < \alpha, r \in \mathbb{P}_\beta \) and \( \sigma \) is a \( \mathbb{P}_\alpha \)-name, the definition of our iteration at step \( \alpha + 1 \) involves appealing to Building Block II with \( (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \check{c}_\alpha) \), \( r^* := r \upharpoonright \theta_\alpha \) and \( z := \check{i}_\beta(\sigma) \).\(^{20}\) For each ordinal \( \eta < \mu^+ \), denote \( G_\eta := \pi_{\mu^+, \eta}[G] \). By our choice of \( \beta \) and since \( \alpha > \beta \), we have
\[
(\xi, p \upharpoonright \eta) \in \sigma \} = \{ (\xi, p \upharpoonright \theta_{\mu^+}) \mid (\xi, p) \in z \},
\]
so that, in \( V[G] \),
\[
T = \tau_G = \sigma_{G_\beta} = zG_\alpha.
\]
In addition, \( r^* = r^* \upharpoonright \theta_{\mu^+} \) and so \( \ell(r^*) = \ell(r^*) \).

As \( r^* \) forces that \( \tau \) is a non-reflecting stationary subset of \( \Gamma_{\ell(r^*)} \), it follows that \( r^* \) \( \mathbb{P}_\alpha \)-forces the same about \( z \).

Claim 8.7.1. \( z \) is \( r^* \)-fragile.

\(^{20}\)Recall Convention 7.1.
Proof. Recalling Lemma 6.11, it suffices to prove that for every \( n < \omega \),

\[
V^{(P_\alpha)_n} \models \text{Refl}(E^\mu_{\leq \sigma_{n-2}}, E^\mu_{\leq \sigma_n}).
\]

This is trivially the case for \( n \leq 1 \). So, let us fix an arbitrary \( n \geq 2 \).

By Lemma 8.5, \((P_\alpha, S_n, \pi_{\alpha n})\) is suitable for reflection with respect to \( \langle \sigma_{n-2}, \kappa_{n-1}, \kappa_n, \mu \rangle \). Since \((P_\alpha)_{\pi_{\alpha n}}\) is \( \sigma_n \)-directed-closed and (by Lemma 8.6) it preserves GCH, \( \kappa_{n-1} \) is a supercompact cardinal indestructible under forcing with \((P_\alpha)_{\pi_{\alpha n}}\). So, recalling Setup 8, \((P_\alpha)_{\pi_{\alpha n}}\) preserves the supercompactness of \( \kappa_{n-1} \). Thus, by Lemma 2.12, \( V^{(P_\alpha)_n} \models \text{Refl}(E^\mu_{\leq \sigma_{n-2}}, E^\mu_{\leq \sigma_n}) \). \qed

As \( z \) is \( r^* \)-fragile and \( \pi_{\mu+\alpha+1}(r^*) = r^* \circ \emptyset_{\alpha+1} = [r^*]_{\pi_{\alpha+1}} \in G_{\alpha+1} \), Clause (e) of Building Block II implies that there exists (in \( V[G_{\alpha+1}] \)) a club subset of \( \mu \) disjoint from \( T \). In particular, \( T \) is nonstationary in \( V[G_{\alpha+1}] \) and thus nonstationary in \( V[G] \). This contradicts the very choice of \( T \). The result follows from the above discussion and the previous claim. \qed

We are now ready to prove the Main Theorem.

**Theorem 8.8.** Suppose that \( \langle \kappa_n \mid n < \omega \rangle \) is an increasing sequence of supercompact cardinals. Then there exists a forcing extension where all of the following hold:

1. \( 2^{\kappa_n} = \kappa_{n+1} \) for all \( n < \omega \);
2. \( 2^{\kappa_\omega} = \kappa_{\omega+2} \);
3. every stationary subset of \( \kappa_{\omega+1} \) reflects.

**Proof.** Using Corollary 8.3, we may assume that all the blanket assumptions of Setup 8 are met. Specifically:

- \( \bar{\kappa} = \langle \kappa_n \mid n < \omega \rangle \) is an increasing sequence of supercompact cardinals that are indestructible under \( \kappa_n \)-directed-closed notions of forcing that preserve the GCH;
- \( \kappa := \sup_{n<\omega} \kappa_n, \mu := \kappa^+ \) and \( \lambda := \kappa^{++} \);
- GCH holds.

Now, appeal to Theorem 8.7. \qed

9. Acknowledgments

Poveda was partially supported by the Spanish Government under grant MTM2017-86777-P, by Generalitat de Catalunya (Catalan Government) under grant SGR 270-2017 and by MECD Grant FPU15/00026. Rinot was partially supported by the European Research Council (grant agreement ERC-2018-StG 802756) and by the Israel Science Foundation (grant agreement 2066/18). Sinapova was partially supported by the National Science Foundation, Career-1454945.
References


---

**Departament de Matemàtiques i Informàtica, Universitat de Barcelona. Gran Via de les Corts Catalanes, 585, 08007 Barcelona, Catalonia.**

**Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.**

**Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045, USA**

**URL:** http://www.assafrinot.com

**URL:** https://homepages.math.uic.edu/~sinapova/