KUREPA TREES AND SPECTRA OF $\mathcal{L}_{\omega_1,\omega}$-SENTENCES

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Abstract. We construct a *single* $\mathcal{L}_{\omega_1,\omega}$-sentence ψ that codes Kurepa trees to prove the consistency of the following:

(1) The spectrum of ψ is consistently equal to $[\aleph_0, \aleph_{\omega_1}]$ and also consistently equal to $[\aleph_0, 2^{\aleph_1})$, where $2^{\aleph_1}$ is weakly inaccessible.
(2) The amalgamation spectrum of ψ is consistently equal to $[\aleph_1, \aleph_{\omega_1}]$ and $[\aleph_1, 2^{\aleph_1})$, where again $2^{\aleph_1}$ is weakly inaccessible.
This is the first example of an $\mathcal{L}_{\omega_1,\omega}$-sentence whose spectrum and amalgamation spectrum are consistently both right-open and right-closed. It also provides a positive answer to a question in [14].
(3) Consistently, ψ has maximal models in finite, countable, and uncountable many cardinalities. This complements the examples given in [1] and [2] of sentences with maximal models in countably many cardinalities.
(4) $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence with models in $\aleph_{\omega_1}$, but no models in $2^{\aleph_1}$.
This relates to a conjecture by Shelah that if $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$-sentence with a model of size $\aleph_{\omega_1}$ also has a model of size $2^{\aleph_0}$. Our result proves that $2^{\aleph_0}$ can not be replaced by $2^{\aleph_1}$, even if $2^{\aleph_0} < \aleph_{\omega_1}$.

1. Introduction

Definition 1.1. For an $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ, the *spectrum* of ϕ is the set

$$Spec(\phi) = \{\kappa | \exists M \models \phi \text{ and } |M| = \kappa\}.$$

If $Spec(\phi) = [\aleph_0, \kappa]$, we say that ϕ *characterizes* κ.

The *amalgamation spectrum* of ϕ, AP-$Spec(\phi)$ for short, is the set of all cardinals κ so that the models of ϕ of size κ satisfy the amalgamation property. Similarly define JEP-$Spec(\phi)$ the *joint embedding spectrum* of ϕ.

The *maximal models spectrum* of ϕ is the set

$$\text{MM-}Spec(\phi) = \{\kappa | \exists M \models \phi \text{ and } M \text{ is maximal}\}.$$

Morley and López-Escobar independently established that all cardinals that are characterized by an $\mathcal{L}_{\omega_1,\omega}$- sentence are smaller than $\beth_{\omega_1}$.[1]

Theorem 1.2 (Morley, López-Escobar)**.** *Let Γ be a countable set of sentences of $\mathcal{L}_{\omega_1,\omega}$. If Γ has models of cardinality $\beth_\alpha$ for all $\alpha < \omega_1$, then it has models of all infinite cardinalities.*

In 2002, Hjorth ([5]) proved the following.

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[1]See [9] for more details.

Theorem 1.3 (Hjorth)**.** *For all $\alpha < \omega_1$, $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$-sentence.*

Combining Theorems 1.2 and 1.3 we get under GCH that $\aleph_\alpha$ is characterized by a complete $\mathcal{L}_{\omega_1,\omega}$-sentence if and only if $\alpha < \omega_1$.

Given these results, one might ask if it is consistent under the failure of GCH that $\aleph_{\omega_1}$ is characterizable. The answer is positive because one can force $\aleph_{\omega_1} = 2^{\aleph_0}$ and by [12], $2^{\aleph_0}$ can be characterized by a complete $\mathcal{L}_{\omega_1,\omega}$-sentence. Further, in [14] the question was raised if any cardinal outside the smallest set which contains $\aleph_0$ and which is closed under successors, countable unions, countable products and powerset, can be characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence. Lemma 3.2 proves $\aleph_{\omega_1}$ to be such an example, thus providing a positive answer.

In this paper we work with a single $\mathcal{L}_{\omega_1,\omega}$-sentence ψ that codes a Kurepa tree and we investigate the effects of set-theory on the spectra of ψ. In particular, we prove the consistency of the following.

(1) $2^{\aleph_0} < \aleph_{\omega_1}$ and $Spec(\psi) = [\aleph_0, \aleph_{\omega_1}]$.

(2) $2^{\aleph_0} < 2^{\aleph_1}$, $2^{\aleph_1}$ is weakly inaccessible and $Spec(\psi) = [\aleph_0, 2^{\aleph_1})$.

The only previously known examples of $\mathcal{L}_{\omega_1,\omega}$-sentences with a right-open spectra were of the form $[\aleph_0, \kappa)$, with κ of countable cofinality. If $\kappa = \sup_{n\in\omega} \kappa_n$ and ϕ_n characterizes κ_n, then $\bigvee_n \phi_n$ has spectrum $[\aleph_0, \kappa)$.

Lemma 2.2 proves that our methods can not be used to establish the consistency of $Spec(\psi) = [\aleph_0, \aleph_{\omega_1})$, which remains open. It also open to find a *complete* sentence with a right-open spectrum.

Moreover, ψ is the first example of an $\mathcal{L}_{\omega_1,\omega}$-sentence whose spectrum is consistently both right-open and right-closed. The same observation holds true for the amalgamation spectrum too.

(3) $2^{\aleph_0} < \aleph_{\omega_1}$ and AP-$Spec(\psi)$ =$[\aleph_1, \aleph_{\omega_1}]$.

(4) $2^{\aleph_0} < 2^{\aleph_1}$, $2^{\aleph_1}$ is weakly inaccessible and AP-$Spec(\psi)$ =$[\aleph_1, 2^{\aleph_1})$.

This is the first example of an $\mathcal{L}_{\omega_1,\omega}$-sentence whose amalgamation spectrum is consistently both right-open and right-closed.

Moreover, by manipulating the size of $2^{\aleph_1}$, Corollary 2.15 implies that for κ regular with $\aleph_2 \leq \kappa \leq 2^{\aleph_1}$, κ-amalgamation for $\mathcal{L}_{\omega_1,\omega}$-sentences is non-absolute for models of ZFC.

(5) MM-$Spec(\psi)$ is the set of all cardinals κ such that κ is either equal to $\aleph_1$, or equal to $2^{\aleph_0}$, or there exists a Kurepa tree with exactly κ-many branches. Manipulating the cardinalities on which there are Kurepa trees, we can prove that ψ consistently has maximal models in finite, countable, and uncountable many cardinalities.

Our example complements the examples of $\mathcal{L}_{\omega_1,\omega}$-sentences with maximal models in countably many cardinalities found in [1] and [2].

(6) $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence with models in $\aleph_{\omega_1}$, but no models in $2^{\aleph_1}$.

Shelah conjectured that if $\aleph_{\omega_1} < 2^{\aleph_0}$, then any $\mathcal{L}_{\omega_1,\omega}$-sentence with a model of size $\aleph_{\omega_1}$ also has a model of size $2^{\aleph_0}$. In [13], Shelah proves the consistency of the conjecture. Our result proves that $2^{\aleph_0}$ can not be replaced by $2^{\aleph_1}$, even if $2^{\aleph_0} < \aleph_{\omega_1}$. Our example can not be used to refute Shelah's conjecture.

Section 2 contains the description of the sentence ψ and the results about the model-theoretic properties of ψ. Section 3 contains the consistency results.

2. Kurepa Trees and $\mathcal{L}_{\omega_1,\omega}$

The reader can consult [6] about trees. The following definition summarizes all that we will use in this paper.

Definition 2.1. Assume κ is an infinite cardinal. A κ-tree has height κ and each level has at most $<\kappa$ elements. A κ-Kurepa tree is a κ-tree with at least κ^+ branches of height κ. Kurepa trees, with no κ specified, refer to $\aleph_1$-Kurepa trees. For this paper we will assume that κ-Kurepa trees are *pruned*, i.e. all maximal branches have height κ^+.

If $\lambda \geq \kappa^+$, a (κ, λ)-Kurepa tree is a κ-Kurepa tree with exactly λ branches of height κ. $KH(\kappa, \lambda)$ is the statement that there exists a (κ, λ)-Kurepa tree.

Define $\mathcal{B}(\kappa) = \sup\{\lambda | KH(\kappa, \lambda) \text{ holds}\}$ and $\mathcal{B} = \mathcal{B}(\aleph_1)$.

If κ-Kurepa trees exist, it is immediate that $\mathcal{B}(\kappa)$ is always between κ^+ and 2^κ. We are interested in the case where $\mathcal{B}$ is a supremum but not a maximum. The next lemma proves some restrictions when $\mathcal{B}$ is not a maximum. In Section 3 we prove that it is consistent with ZFC that $\mathcal{B}$ is not a maximum.

Lemma 2.2. *If $\mathcal{B}(\kappa)$ is not a maximum, then $cf(\mathcal{B}(\kappa)) \geq \kappa^+$.*

Proof. Towards contradiction assume that $cf(\mathcal{B}(\kappa)) = \mu \leq \kappa$. Let $\mathcal{B}(\kappa) = sup_{i\in\mu}\kappa_i$ and let $(T_i)_{i\in\mu}$ be a collection of κ-Kurepa trees, where each T_i has exactly κ_i-many cofinal branches. Create new κ-Kurepa trees S_i by induction on $i \leq \mu$: S_0 equals T_0; S_{i+1} equals the disjoint union of S_i together with a copy of T_{i+1}, arranged so that the j^{th} level T_{i+1} coincides with the $(j+i)^{th}$ level of S_i. At limit stages take unions. We leave the verification to the reader that S_μ is a κ-Kurepa tree with exactly $sup_{i\in\mu}\kappa_i = \kappa$ cofinal branches, contradicting the fact that $\mathcal{B}(\kappa)$ is not a maximum. □

Definition 2.3. Let $\kappa \leq \lambda$ be infinite cardinals. A sentence σ in a language with a unary predicate P admits (λ, κ), if σ has a model M such that $|M| = \lambda$ and $|P^M| = \kappa$. In this case, we will say that M is of type (λ, κ).

From [3], theorem 7.2.13, we know

Theorem 2.4. *There is a (first-order) sentence σ such that for all infinite cardinals κ, σ admits (κ^{++}, κ) iff $KH(\kappa^+, \kappa^{++})$.*

We describe here the construction behind Theorem 2.4 in order to use it later.

The vocabulary τ consists of the unary symbols P, L, the binary symbols $V, T, <, \prec, H$, and the ternary symbols F, G. The idea is to build a κ-tree. P, L, V are disjoint and their union is the universe. L is a set that corresponds to the "levels" of the tree. L is linearly ordered by $<$ and it has a minimum and maximum element. Every element $a \in L$ that is not the maximum element has a successor, which we will freely denote by $a+1$. The maximum element is not a successor. For every $a \in L$, $V(a, \cdot)$ is the set of nodes at level a and we assume that $V(a, \cdot)$ is disjoint from L. If $V(a, x)$, we will say that x is at level a. If M is the maximum element of L, $V(M, \cdot)$ is the set of maximal branches through the tree. T is a tree ordering on $V = \bigcup_{a\in L} V(a, \cdot)$. If $T(x, y)$, then x is at some level strictly less than the level of y. If a is a limit, that is neither a successor nor the least element in L, then two distinct elements in $V(a, \cdot)$ can not have the same predecessors. Both "the height of T" and "the height of L" refer to the order type of $(L, <)$. Although it is not necessary for Theorem 2.4, we can stipulate that the Kurepa tree is pruned.

We use the predicate P to bound the size of every initial segment of L of the form $L_{\le a} = \{b \in L | b \le a\}$, where a is not the maximum element of L. We also bound the size of each level $V(a,\cdot)$. For every $a \in L \setminus \{M\}$, where M is the maximum element of L, there is a surjection $F(a,\cdot,\cdot)$ from P to $L_{\le a}$ and another surjection $G(a,\cdot,\cdot)$ from P to $V(a,\cdot)$.

We linearly order the set of maximal branches $V(M,\cdot)$ by $\prec$ so that there is no maximum element. H is a surjection from L to each initial segment of $V(M,\cdot)$ of the form $\{x \in V(M,\cdot) | x \preceq y\}$.

Call σ the (first-order) sentence that stipulates all the above. In all models of σ, if P has size κ, then L has size at most κ^+ and $V(M,\cdot)$ has size at most κ^{++}. In the case that $|V(M,\cdot)| = \kappa^{++}$, then also $|L| = \kappa^+$. So, all models of σ where P is infinite and $|V(M,\cdot)| = |P|^{++}$ code a Kurepa tree. This proves Theorem 2.4.

We want to emphasize here the fact that since well-orderings can not be characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence, it is unavoidable that we will be working with non-well-ordered trees. However, using an $\mathcal{L}_{\omega_1,\omega}$-sentence we can express the fact that P is countably infinite.[2] Let ϕ be the conjunction of σ together with the requirement that P is countably infinite. Then ϕ has models of size $\aleph_2$ iff there exist a Kurepa tree of size $\aleph_2$ iff $KH(\aleph_1,\aleph_2)$.

Fix some $n \ge 2$. Then the above construction of σ (and ϕ) can be modified to produce a first-order sentence σ_n and the corresponding $\mathcal{L}_{\omega_1,\omega}$-sentence ϕ_n so that ϕ_n has a model of size $\aleph_n$ iff there exist a Kurepa tree of size $\aleph_n$ iff $KH(\aleph_1,\aleph_n)$. The argument breaks down at $\aleph_\omega$. Since we will be dealing with Kurepa trees of size potentially larger than $\aleph_\omega$, we must make some modifications.

Let τ' be equal to τ with the symbols $\prec, H$ removed. Let σ' be equal to σ with all requirements that refer to $\prec, H$ removed. Let ψ be the conjunction of σ' and the requirement that P is countably infinite. For any $\lambda \ge \aleph_2$, any $(\aleph_1,\lambda)$-Kurepa tree gives rise to a model of ψ, but unfortunately, there are models of ψ of size $2^{\aleph_0}$, that do not code a Kurepa tree. For instance consider the tree $(\omega^{\le\omega}, \subset)$ which has countable height, but contains $2^{\aleph_0}$ many maximal branches. Notice also that both ϕ and ψ are not complete sentences.

The dividing line for models of ψ to code Kurepa trees is the size of L. By definition L is $\aleph_1$-like, i.e. every initial segment has countable size. If in addition L is uncountable, then we can embed ω_1 cofinally into L.[3] Hence, every model of ψ of size $\ge \aleph_2$ and for which L is uncountable, codes a Kurepa tree. Otherwise, the model does not code a Kurepa tree.

Let $\boldsymbol{K}$ be the collection of all models of ψ, equipped with the substructure relation. I.e. for $M, N \in \boldsymbol{K}$, $M \prec_{\boldsymbol{K}} N$ if $M \subset N$.

Observation 2.5. If $M \prec_{\boldsymbol{K}} N$, the following follow from the definition:

(1) L^M is an initial segment of L^N. Towards contradiction, assume that L^N contains some point x and there exists some $y \in L^M$ such that $x < y$. Then the function $F^M(y,\cdot,\cdot)$ defined on M disagrees with the function $F^N(y,\cdot,\cdot)$ defined on N. Contradiction.
(2) for every non-maximal $a \in L^M$, $V^M(a,\cdot)$ equals $V^N(a,\cdot)$. The argument is similar to the argument for (1), using the functions $G^M(y,\cdot,\cdot)$ and $G^N(y,\cdot,\cdot)$ this time.

[2] There are many ways to do this. The simplest way is to introduce countably many new constants $(c_n)_{n\in\omega}$ and require that $\forall x,\ P(x) \to \bigvee_n x = c_n$.
[3] The embedding is not necessarily continuous, i.e. it may not respect limits.

(3) the tree ordering is preserved.

We will express (1) – (3) by saying that "(the tree defined by) M is an *initial segment* of (the tree defined by) N".

Corollary 2.6. *Assume M is an initial segment of N. Then:*

- *If $L^N = L^M$, then N differs from M only in the maximal branches it contains.*
- *If L^N is a strict end-extension of L^M, then M must be countable.*

Proof. Part (1) is immediate from Observation 2.5 (2). Part (2) follows from the requirement that all models of ψ have countable levels. □

Convention 2.7. For the rest of the paper, when we talk about "the models of ψ" we will mean $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$.

The next theorem characterizes the spectrum and the maximal models spectrum of ψ.

Theorem 2.8. *The spectrum of ψ is characterized by the following properties:*

(1) $[\aleph_0, 2^{\aleph_0}] \subset Spec(\psi)$;
(2) if there exists a Kurepa tree with κ many branches, then $[\aleph_0, \kappa] \subset Spec(\psi)$;
(3) no cardinal belongs to $Spec(\psi)$ except those required by (1) and (2). I.e. if ψ has a model of size κ, then either $\kappa \leq 2^{\aleph_0}$, or there exists a Kurepa tree which κ many branches.

The maximal models spectrum of ψ is characterized by the following:

(4) ψ has maximal models in cardinalities $2^{\aleph_0}$ and $\aleph_1$;
(5) if there exists a Kurepa tree with exactly κ many branches, then ψ has a maximal model in κ;
(6) ψ has maximal models only in those cardinalities required by (4) and (5).

Proof. For (1) and (2), we observed already that $(\omega^{\leq\omega}, \subset)$ is a model of ψ and that every Kurepa tree gives rise to a model of ψ.

For (3), let N be a model of ψ of size κ. If L^N is countable, then N has size $\leq 2^{\aleph_0}$. If L^N is uncountable, then N codes a κ-tree with $|N|$-many branches. By the proof of theorem 2.4, this is a Kurepa tree, assuming that $|N| \geq \aleph_2$. Otherwise, $|N| \leq \aleph_1 \leq 2^{\aleph_0}$.

To prove (4), notice that $(\omega^{\leq\omega}, \subset)$ is a maximal model of ψ, and it is easy to construct trees with height ω_1 and $\leq \aleph_1$ many branches.

Now, let N code a Kurepa tree with exactly κ many branches. By Corollary 2.6, N is maximal. This proves (5).

For (6), let N be a maximal model. If N was countable, we could end-extend it and it would not be maximal. So, N must be uncountable. We split into two cases depending on the size of L^N. If L^N is countable, assume without loss of generality, that it has height ω (otherwise consider a cofinal subset of order type ω). So, (V^N, T^N) is a pruned tree which is a subtree of $\omega^{\leq\omega}$. The set of maximal branches through (V^N, T^N) is a closed subset of the Baire space (cf. [8], Proposition 2.4). Since closed subsets of ω^ω have size either $\aleph_0$ or $2^{\aleph_0}$, we conclude that N has size $2^{\aleph_0}$. The second case is when L^N is uncountable. If N has size $\aleph_1$, we are done. If $|N| \geq \aleph_2$, by Corollary 2.6 and maximality of N, the tree defined by N contains exactly $|N|$ many maximal branches. Therefore, N defines a Kurepa tree. □

Recall that $\mathcal{B} = \mathcal{B}(\aleph_1)$ is the supremum of the size of Kurepa trees.

Corollary 2.9. *(1) If there are no Kurepa trees, then Spec(ψ) equals* $[\aleph_0, 2^{\aleph_0}]$ *and MM-Spec(ψ) equals* $\{\aleph_1, 2^{\aleph_0}\}$.

(2) If $\mathcal{B}$ is a maximum, i.e. there is a Kurepa tree of size $\mathcal{B}$, then ψ characterizes $\max\{2^{\aleph_0}, \mathcal{B}\}$.

(3) If $\mathcal{B}$ is not a maximum, then Spec(ψ) equals either $[\aleph_0, 2^{\aleph_0}]$ or $[\aleph_0, \mathcal{B})$, whichever is greater. Moreover ψ has maximal models in $\aleph_1$, $2^{\aleph_0}$ and in cofinally many cardinalities below $\mathcal{B}$.

Proof. (1) and (2) follow immediately from Theorem 2.8. We only establish (3). If $\mathcal{B}$ is not a maximum, then $[\aleph_0, \mathcal{B}) \subset Spec(\psi)$ and $Spec(\psi)$ equals either $[\aleph_0, 2^{\aleph_0}]$ or $[\aleph_0, \mathcal{B})$, whichever is greater. For the the last assertion, assume $\mathcal{B} = \sup_i \kappa_i$ and for each i, there is a Kurepa tree with κ_i many branches. Then each κ_i is in the MM-*Spec*(ψ) by Theorem 2.8 (5). □

In Section 3 we prove the following consistency results.

Theorem 2.10. *The following are consistent with ZFC:*

(i) ZFC+ ($2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}$) + "$\mathcal{B}$ is a maximum", i.e. there exists a Kurepa tree of size $\aleph_{\omega_1}$".

(ii) ZFC+ ($\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}$)+ "$\mathcal{B}$ is a maximum".

(iii) ZFC+ ($2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}$) + "$2^{\aleph_1}$ is weakly inaccessible + "for every $\kappa < 2^{\aleph_1}$ there is a Kurepa tree with exactly κ-many maximal branches, but no Kurepa tree has exactly $2^{\aleph_1}$-many branches."

Moreover, in (i) and (ii) we can replace $\aleph_{\omega_1}$ by most cardinals below or equal to $2^{\aleph_1}$ and $2^{\aleph_0}$ respectively.

From [7] we know the consistency of the following:

Theorem 2.11 (R. Jin)**.** *Assume the existence of two strongly inaccessible cardinals. It is consistent with CH (or $\neg$CH) plus $2^{\aleph_1} > \aleph_2$ that there exists a Kurepa tree with $2^{\aleph_1}$ many branches and no ω_1-trees have λ-many branches for some λ strictly between $\aleph_1$ and $2^{\aleph_1}$. In particular, no Kurepa trees have less than $2^{\aleph_1}$ many branches.*

It follows from the proof of Theorem 2.11 that if λ is regular cardinal above $\aleph_2$, we can force the size of $2^{\aleph_1}$ to equal λ.

Corollary 2.12. *There exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ such that it is consistent with ZFC that*

(1) ψ characterizes $2^{\aleph_0}$;

(2) CH (or $\neg$CH), $2^{\aleph_1}$ is a regular cardinal greater than $\aleph_2$ and ψ characterizes $2^{\aleph_1}$;

(3) $2^{\aleph_0} < \aleph_{\omega_1}$ and ψ characterizes $\aleph_{\omega_1}$; and

(4) $2^{\aleph_0} < 2^{\aleph_1}$, $2^{\aleph_1}$ is weakly inaccessible and Spec(ψ) $= [\aleph_0, 2^{\aleph_1})$.

For the same ψ it is consistent with ZFC that

(5) MM-Spec(ψ) $=\{\aleph_1, 2^{\aleph_0}\}$

(6) MM-Spec(ψ) $=\{\aleph_1, 2^{\aleph_0}, 2^{\aleph_1}\}$

(7) $2^{\aleph_0} < 2^{\aleph_1}$, $2^{\aleph_1}$ is weakly inaccessible and MM-Spec(ψ) $=[\aleph_1, 2^{\aleph_1})$.

Proof. The result follows from Theorem 2.8 and Corollary 2.9 using the appropriate model of ZFC each time. For (1) consider a model with no Kurepa trees, or a model where $\mathcal{B} < 2^{\aleph_0}$, e.g. case (ii) of Theorem 2.10. For (2) and (6) use Theorem 2.11. For (3),(4), use Theorem 2.10 cases (i),(iii) respectively. For (5) consider a model of ZFC with no Kurepa trees. For (7) use Theorem 2.10 case (iii) again. □

Corollary 2.13. *It is consistent with ZFC that $2^{\aleph_0} < \aleph_{\omega_1} < 2^{\aleph_1}$ and there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence with models in $\aleph_{\omega_1}$, but no models in $2^{\aleph_1}$.*

2.1. Amalgamation and Joint Embedding Spectra.

In this section we provide the amalgamation and joint embedding spectrum of models of ψ.

The following characterizes JEP-*Spec*(ψ) and AP-*Spec*(ψ).

Theorem 2.14. *(1) $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ fails JEP in all cardinals;*
(2) $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ satisfies AP for all uncountable cardinals that belong to Spec(ψ), but fails AP in $\aleph_0$.

Proof. The first observation is that in all cardinalities there exists two linear orders L^M, L^N none of which is an initial segment of the other. By Observation 2.5(1), M, N can not be be jointly embedded to some larger structure in $\boldsymbol{K}$. So, JEP fails in all cardinals.

A similar argument to JEP proves that there exist three countable linear orders L^{M_0}, L^{M_1}, L^{M_2} such that L^{M_0} is an initial segment of both L^{M_1} and L^{M_2}, and the triple $(L^{M_0}, L^{M_1}, L^{M_2})$ can not be amalgamated. This proves that amalgamation fails in $\aleph_0$.

Now, assume that M, N are uncountable models of ψ and $M \prec_{\boldsymbol{K}} N$. By Corollary 2.6, $L^M = L^N$ and M, N agree on all levels, except N may contain more maximal branches. We use this observation to prove amalgamation.

Let M_0, M_1, M_2 be uncountable models in $\boldsymbol{K}$ with $M_0 \prec_{\boldsymbol{K}} M_1, M_2$. Since $L^{M_0} = L^{M_1} = L^{M_2}$ and M_0, M_1, M_2 agree on all levels, except possible the maximal level, define the amalgam N of (M_0, M_1, M_2) to be the union of M_0 together with all maximal branches in M_1 and M_2. If two maximal branches have exactly the same predecessors, we identify them. It follows that N is a structure in $\boldsymbol{K}$ and $M_i \prec_{\boldsymbol{K}} N$, $i = 1, 2, 3$. □

Notice that, in general, the amalgamation is not disjoint, since both M_1 and M_2 may contain the same maximal branch.

Corollary 2.15. *The following are consistent:*

1. *(1) AP-Spec(ψ) $= [\aleph_1, 2^{\aleph_0}]$;*
2. *(2) AP-Spec(ψ) $= [\aleph_1, 2^{\aleph_1}]$;*
3. *(3) $2^{\aleph_0} < \aleph_{\omega_1}$ and AP-Spec(ψ) $= [\aleph_1, \aleph_{\omega_1}]$; and*
4. *(4) $2^{\aleph_0} < 2^{\aleph_1}$, $2^{\aleph_1}$ is weakly inaccessible and AP-Spec(ψ) $= [\aleph_1, 2^{\aleph_1})$.*

Proof. The result follows from Theorem 2.14 and Corollary 2.12. □

It follows from Corollary 2.15 that if $\aleph_2 \le \kappa \le 2^{\aleph_1}$ is a regular cardinal, then the κ-amalgamation property for $\mathcal{L}_{\omega_1,\omega}$-sentences is not absolute for models of ZFC. The result is useful especially under the failure of GCH, since GCH implies that $2^{\aleph_1} = \aleph_2$. Assuming GCH the absoluteness question remains open for $\kappa \ge \aleph_3$.

In addition, it is an easy application of Shoenfield's absoluteness that $\aleph_0$-amalgamation is an absolute property for models of ZFC. The question for $\aleph_1$-amalgamation remains open. Recall that by [4], model-existence in $\aleph_1$ for $\mathcal{L}_{\omega_1,\omega}$-sentences is an absolute property for models of ZFC.

Open Questions 2.16.

(1) Is $\aleph_1$-amalgamation for $\mathcal{L}_{\omega_1,\omega}$-sentences absolute for models of ZFC?
(2) Let $3 \leq \alpha < \omega_1$. Does the non-absoluteness of $\aleph_\alpha$-amalgamation hold if we assume GCH?

3. Consistency Results

In this section we prove the consistency results announced by Theorem 2.10. Recall that $\mathcal{B}$ is the supremum of $\{\lambda |$there exists a Kurepa tree with λ-many branches$\}$.

Theorem 3.1. *It is consistent with ZFC that:*

(1) $2^{\aleph_0} < \aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_1}$ *and there exists a Kurepa tree of size* $\aleph_{\omega_1}$*.*
(2) $\aleph_{\omega_1} = \mathcal{B} < 2^{\aleph_0}$ *and there exists a Kurepa tree of size* $\aleph_{\omega_1}$*.*

We start with a model V_0 of ZFC+GCH. Let $\mathbb{P}$ be the standard σ-closed poset for adding a Kurepa tree K with $\aleph_{\omega_1}$-many ω_1-branches. More precisely, conditions in $\mathbb{P}$ are of the form (t, f), where:

- t is a tree of height $\beta + 1$ for some $\beta < \omega_1$ and countable levels;
- f is a function with $\text{dom}(f) \subset \aleph_{\omega_1}$, $|\text{dom}(f)| = \omega$, and $\text{ran}(f) = t_\beta$, where t_β is the β-th level of t.

Intuitively, t is an initial segment of the generically added tree, and each $f(\delta)$ determines where the δ-th branch intersects the tree at level β. The order is defined as follows: $(u, g) \leq (t, f)$ if,

- t is an initial segment of u, $\text{dom}(f) \subset \text{dom}(g)$,
- for every $\delta \in \text{dom}(f)$, either $f(\delta) = g(\delta)$ (if t and u have the same height) or $f(\delta) <_u g(\delta)$.

We have that $\mathbb{P}$ is countably closed and has the $\aleph_2$-chain condition. Suppose that H is $\mathbb{P}$-generic over V_0. Then $\bigcup_{(t,f)\in H} t$ is a Kurepa tree with $\aleph_{\omega_1}$-many branches, where for $\delta < \aleph_{\omega_1}$, the δ-th branch is given by $\bigcup_{(t,f)\in H, \delta\in\text{dom}(f)} f(\delta)$. These branches are distinct by standards density arguments. Also, note that since $|B|$ cannot exceed $2^{\aleph_1}$ and in this model $\aleph_{\omega_1} = |B|$, we have that $\aleph_{\omega_1} < 2^{\aleph_1}$.

The model $V_0[H]$ proves part (1) of Theorem 3.1. The same model also answers positively a question raised in [14]. The question was whether any cardinal outside the smallest set which contains $\aleph_0$ and which is closed under successors, countable unions, countable products and powerset, can be characterized by an $\mathcal{L}_{\omega_1,\omega}$-sentence. $\aleph_{\omega_1}$ is consistently such an example.

Lemma 3.2. *Let C be the smallest set of cardinals that contains $\aleph_0$ and is closed under successors, countable unions, countable products and powerset. In $V_0[H]$, the set C does not contain $\aleph_{\omega_1}$.*

Proof. Since $2^\omega < \aleph_{\omega_1} < 2^{\omega_1}$, it is enough to show that $\aleph_{\omega_1}$ is not the countable product of countable cardinals. Suppose $\langle \alpha_n \mid n < \omega \rangle$ is an increasing sequence of countable ordinals and let $\alpha = \sup_n \alpha_n + 1$. Then $\prod_n \aleph_{\alpha_n} = (\aleph_\alpha)^\omega = \aleph_{\alpha+1}$. □

Let $\mathbb{C} = Add(\omega, \aleph_{\omega_1+1})$ denote the standard poset for adding $\aleph_{\omega_1+1}$-many Cohen reals. Suppose G is $\mathbb{C}$-generic over $V := V_0[H]$. (Note that $\mathbb{C}$ is interpreted the same in V_0 and in V and that actually genericity over V_0 implies genericity over V by the ccc.)

We claim that the forcing extension $V_0[H \times G] = V[G]$ satisfies $2^{\aleph_0} > \aleph_{\omega_1} = \mathcal{B}$. Let T be a Kurepa tree in $V[G]$. Denote $\mathbb{C}_{\omega_1} := Add(\omega, \omega_1)$, i.e. the Cohen poset for adding ω_1 many reals. The following fact is standard and can be found in [6], but we give the proof for completeness.

Lemma 3.3. *There is a generic $\bar{G}$ for $\mathbb{C}_{\omega_1}$, such that $V[\bar{G}] \subset V[G]$ and $T \in V[\bar{G}]$.*

Proof. Since T is a tree of height ω_1 and countable levels, we can index the nodes of T by $\langle \alpha, n\rangle$, for $\alpha < \omega_1$ and $n < \omega$, where the first coordinate denotes the level of the node. In particular each level $T_\alpha = \{\alpha\} \times \omega$. Note that this is in the ground model (although of course the relation $<_T$ may not be). Working in $V[G]$, for every $\alpha < \beta < \omega_1$ and n, m, let $p_{\alpha,\beta,m,n} \in G$ decide the statement $\langle \alpha, m\rangle <_{\dot{T}} \langle \beta, n\rangle$. Let $d_{\alpha,\beta,m,n} = \mathrm{dom}(p_{\alpha,\beta,m,n})$; this is a finite subset of $\aleph_{\omega_1+1} \times \omega$. Now let $d^* = \bigcup_{\alpha,\beta,m,n} d_{\alpha,\beta,m,n}$ and $d = \{i < \aleph_{\omega_1+1} \mid (\exists k)\langle i, k\rangle \in d^*\}$. Then d has size at most ω_1. By increasing d if necessary, assume that $|d| = \omega_1$.

Write G as $\prod_{i \in \aleph_{\omega_1+1}} G_i$, where every G_i is $Add(\omega, 1)$-generic, and let $\bar{G} = \prod_{i \in d} G_i$. Then $\bar{G}$ is $\mathbb{C}_{\omega_1}$-generic, containing every $p_{\alpha,\beta,m,n}$. And so $T \in V[\bar{G}] \subset V[G]$.

□

Fix $\bar{G}$ as in the above lemma.

Lemma 3.4. *All ω_1-branches of T in $V[G]$ are already in the forcing extension $V[\bar{G}]$.*

Proof. This is because $\mathbb{C}/\bar{G}$, i.e. the forcing to get from $V[\bar{G}]$ to $V[G]$ is Knaster:

Suppose $\dot{b}$ is forced to be an ω_1-branch of T. For $\alpha < \omega$ let $p_\alpha \Vdash_{\mathbb{C}/\bar{G}} u_\alpha \in \dot{b} \cap T_\alpha$, where T_α is the α-th level. Then there is an unbounded $I \subset \omega_1$ such that for all $\alpha, \beta \in I$, p_α and p_β are compatible, and so $\{u_\alpha \mid \alpha \in I\}$ generate the branch in $V[\bar{G}]$.

□

For every $\alpha < \aleph_{\omega_1}$, let $\mathbb{P}_\alpha := \{(t, f \restriction \aleph_\alpha) \mid (t, f) \in \mathbb{P}\}$. Then clearly the poset $\mathbb{P}$ is the union of the sequence $\langle \mathbb{P}_\alpha \mid \alpha < \omega_1\rangle$ and each $\mathbb{P}_\alpha$ is a regular $\aleph_2$-cc subordering of $\mathbb{P}$ that adds $\aleph_\alpha$ many branches to the generic Kurepa tree. Let H_α be the generic filter for $\mathbb{P}_\alpha$ obtained from H. Also, for a condition $p \in \mathbb{P}$, we use the notation $p = (t^p, f^p)$.

Claim 3.5. For every ω_1 branch b of T, there is some $\alpha < \omega_1$, such that $b \in V_0[H_\alpha \times \bar{G}]$.

Proof. As before, we index the nodes of T by $\langle \alpha, n\rangle$, for $\alpha < \omega_1$ and $n < \omega$, where the first coordinate denotes the level of the node. Similarly to the arguments in Lemma 3.3, we can find a set $d \subset \aleph_{\omega_1}$ of size ω_1, such that T is in $V_0[\bar{G} \times \bar{H}]$, where $\bar{H} = \{(t, f \restriction d) \mid (t, f) \in H\}$. Then $\bar{H}$ is actually a generic filter for $\mathbb{P}_1$, and we view $V_0[\bar{G} \times H] = V_0[\bar{G} \times \bar{H}][H']$, where H' is $\mathbb{P}/\bar{H} := \{(t, f) \in \mathbb{P} \mid (t, f \restriction \aleph_1) \in \bar{H}\}$ -generic.

Suppose $\dot{b}$ is a $\mathbb{P}$-name for a cofinal branch through T, which is not in $V_0[\bar{G} \times \bar{H}]$. We say $p \Vdash \dot{b}(\alpha) = n$ to mean that p forces that $\dot{b} \cap \dot{T}_\alpha = \{\langle \alpha, n\rangle\}$. In $V_0[\bar{G} \times \bar{H}]$ let $\mathbb{Q} := \{(t, f \restriction \aleph_1 + 1) \mid (t, f) \in \mathbb{P}/\bar{H}\}$. Define a $\mathbb{Q}$-name τ, by setting

$$\tau = \{(\langle \alpha, n\rangle, q) \mid \alpha < \omega_1, n < \omega, f^q(\omega_1) = n, \exists p \in \mathbb{P}/\bar{H}, t^p \restriction (\alpha+1) = t^q, p \Vdash \dot{b}(\alpha) = n\}.$$

Also let $K = \{(t, f) \mid \exists \alpha < \omega_1, \exists p \in H, t^p \restriction \alpha + 1 = t, \omega_1 \in \mathrm{dom}(f), p \Vdash \dot{b}(\alpha) = f(\omega_1)\}$. Since b is not in $V_0[\bar{G} \times \bar{H}]$, it is straightforward to check that K is $\mathbb{Q}$ -generic, and also that $\dot{b}_H = \tau_K$.

Finally, since $\bar{H} * K \in V_0[\bar{G} \times H]$ is generic for the suborder $\{(t, f \restriction \aleph_1 + 1) \mid (t,f) \in \mathbb{P}\}$, K must be in $V_0[\bar{G} \times H_\alpha]$, for some $\alpha < \omega_1$. Then $b \in V_0[\bar{G} \times H_\alpha]$. □

Lemma 3.6. *If $\mathbb{P} \times \mathbb{C}_{\omega_1}$ adds more than $\aleph_{\omega_1}$-many ω_1-branches to T then there is some $\alpha < \omega_1$, so that $\mathbb{P}_\alpha \times \mathbb{C}_{\omega_1}$ adds more than $\aleph_{\omega_1}$ many ω_1-branches to T.*

Proof. For every $\alpha < \omega_1$, let H_α be $\mathbb{P}_\alpha$-generic over V_0, induced by H. Then $V_0[H_\alpha \times \bar{G}] \subset V_0[H \times \bar{G}] = V[\bar{G}]$.

Suppose that for some $\lambda > \aleph_{\omega_1}$, T has λ-many branches, enumerate them by $\langle b_i \mid i < \lambda \rangle$. For every $i < \lambda$, let $\alpha_i < \omega_1$ be such that $b_i \in V_0[H_{\alpha_i} \times \bar{G}]$ given by Claim 3.5. Then for some $\alpha < \omega_1$ there is an unbounded $I \subset \lambda$, such that for all $i \in I$, $\alpha_i = \alpha$.

But that implies that in $V_0[H_\alpha \times \bar{G}]$, $2^{\omega_1} > \aleph_{\omega_1}$, which is a contradiction since we started with $V_0 \models GCH$. So the forcing extension of $\mathbb{P} \times \mathbb{C}$ has at most $\aleph_{\omega_1}$ many ω_1-branches of T, i.e $\mathcal{B} = \aleph_{\omega_1}$ in this forcing extension. □

$V[G] = V_0[H][G]$ proves part (2) of Theorem 3.1.

Theorem 3.7. *It is consistent with ZFC that $2^{\aleph_0} < \mathcal{B} = 2^{\aleph_1}$, for every $\kappa < 2^{\aleph_1}$ there is a Kurepa tree with exactly κ-many maximal branches, but no Kurepa tree has $2^{\aleph_1}$-many maximal branches.*

Proof. The proof uses the forcing axiom principle GMA defined by Shelah. GMA_κ states that for every κ-closed, stationary κ^+-linked, well met poset $\mathbb{P}$ with greatest lower bound if κ is regular and for every collection of less than 2^κ many dense sets there is a filter for $\mathbb{P}$ meeting them. For an exact definition of *stationary κ^+-linked*, see section 4 of [10].

We take a model constructed in [10], section 4. More precisely, following the arguments in that section, from some fairly mild large cardinals (a Mahlo cardinal will suffice), we get a model V, where the following holds:

(1) GMA_{ω_1},
(2) CH,
(3) 2^{ω_1} is weakly inaccessible,
(4) every Σ^1_1-subset of $\omega_1^{\omega_1}$ of cardinality 2^{ω_1} contains a perfect set.

We claim that this is the desired model. Let $\omega_1 < \kappa < 2^{\omega_1}$. To see that there is a Kurepa tree with exactly κ-many maximal branches, let $\mathbb{P}$ be the standard poset to add such a tree (i.e. we take the poset from earlier but with κ in place of $\aleph_{\omega_1}$). Then $\mathbb{P}$ satisfies the hypothesis of GMA_{ω_1}, and there are only κ-many dense sets to meet in order to get a Kurepa tree with κ-many branches.

Also, the last item of the properties listed above implies that there are no Kurepa trees with 2^{ω_1}-many branches; for details see the discussion of page 22 of [11]. □

Although we will not give the details here, the results presented here can be extended to κ-Kurepa trees with $\kappa \geq \aleph_2$.

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