SINGULAR CARDINALS AND SQUARE PROPERTIES

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Abstract. We analyze the effect of singularizing cardinals on square properties. An old theorem of Dzamonja-Shelah/Gitik says that if you singularize an inaccessible cardinal while preserving its successor, then □κ,ω holds in the bigger model. We extend this to the situation where a finite interval of cardinals above κ is collapsed. More precisely, we show that if V ⊂ W, κ is inaccessible in V, cfW(κi+1) = ω for all 0 ≤ i ≤ n, and κn+1 = κW, then W |= □κ,ω.

1. Introduction

The square principle was isolated by Jensen in his fine structure analysis of L. Square at κ, □κ, states that there exists a sequence ⟨Cα | α < κ+⟩ such that each Cα is a club subset of α, o.t.(Cα) ≤ κ, and if δ ∈ lim Cα, then Cα ∩ δ = Cδ. There are various weakenings allowing multiple guesses for the clubs at each point. More precisely, the principle □κ,λ states that there is a sequence ⟨Cα | α < κ+⟩, such that:

1. 1 ≤ |Cα| ≤ λ,
2. every C ∈ Cα is a club in α, o.t.(C) ≤ κ,
3. if C ∈ Cα and δ ∈ lim(C), then C ∩ δ ∈ Cδ.

Square principles hold in models that sufficiently resemble L. On the other hand, they are at odds with reflection properties and fail above large cardinals. For example, there is tension between failure of SCH and failure of the weaker square properties. One reason is that it is difficult to avoid weaker square principles at singular cardinals. The standard way of singularizing cardinals is by Prikry forcing, and Cummings and Schimmerling showed in [1] that after Prikry forcing at κ, □κ,ω holds in the generic extension. More generally, Gitik [3] and Dzamonja-Shelah [2] showed independently, that if V ⊂ W are transitive class models of ZFC such that κ is an inaccessible cardinal in V, singular in W, and (κ+)V = (κ+)W, then W |= □κ,ω. On the other hand, in Gitik-Sharon [4], a forcing was constructed where κ is singularized, (κω+1)V becomes the successor of κ, and weak square of κ fails in the generic extension.

The natural question is what happens if κ is singularized and only a finite gap of cardinals above κ is collapsed. We show that if V ⊂ W are such that κ is inaccessible in V, κ,κ,κ+,...,κ+n have cofinality ω in W, and (κn+1)V = (κ+)W, then □κ,ω holds in the outer model. Similarly to what was done in [2] and [3], we show the existence of a “pseudo Prikry sequence” witnessing...
that $\kappa^{+n}$ is singular in the bigger model. Moreover, we generalize these arguments to obtain a pseudo Prikry sequence in $W$ through $\mathcal{P}_\kappa((\kappa^{+n})^V)$. We use the term “pseudo Prikry” because the properties of these sequence will be the best approximation of the genericity of a Prikry sequence, without referring to any measure.

We will use the following notation: $\text{cof}(\tau)$ and $\text{cof}(<\tau)$ denote points of cofinality $\tau$ and less than $\tau$, respectively. For a set $C$, $\text{lim}(C)$ is the set of all limit points of $C$ and $\text{nacc}(C)$ is the non-limit points of $C$.

2. Pseudo Prikry sequences

**Proposition 2.1.** (Shelah) Let $n < \omega$ and $\kappa > \aleph_1$ be a regular cardinal. There is a sequence $\langle S_\delta \mid \delta < \kappa^{+n} \rangle$ of stationary subsets of $\kappa^{+n+1}$, such that $\bigcup_{\delta < \kappa^{+n}} S_\delta = \kappa^{+n+1} \cap \text{cof}(<\kappa)$ and each $S_\delta$ carries a partial square sequence $\langle C^\delta_\alpha \mid \alpha \in S_\delta \cap \text{Lim} \rangle$, where:

- each $C^\delta_\alpha$ is a club subset of $\alpha$ with $\text{o.t.}(C^\delta_\alpha) < \kappa^{+n}$, and
- if $\beta \in \text{lim}(C^\delta_\alpha)$, then $\beta \in S_\delta$ and $C^\delta_\alpha \cap \beta = C^\beta_\beta$.

**Theorem 2.2.** Suppose $V \subset W$ are such that $\kappa$ is an inaccessible cardinal in $V$, a singular cardinal in $W$, $\text{cf}^V((\kappa+k)^V) = \omega$ for every $k \leq m$, and $(\kappa^{m+1})^V = (\kappa^+)^W$. Suppose that in $V$, $\langle D^\alpha_\alpha \mid \alpha < \kappa^{+m+1} \rangle$ are club subsets of $\kappa^{+m}$. Then in $W$ there is a sequence $\langle \delta_n \mid n < \omega \rangle$ cofinal in $\kappa^{+m}$, such that for every $\alpha < \kappa^{+m+1}$, for all large $n$, $\delta_n \in D^\alpha_\alpha$. Furthermore, for any $\lambda < \kappa$, we may assume that for all $n$, $\text{cf}(\delta_n) \geq \lambda$.

**Proof.** Denote $\mu := \kappa^{+m+1}$, and let $\langle D^\alpha_\alpha \mid \alpha < \mu \rangle \in V$ be club subsets of $\kappa^{+m}$. First we will construct a sequence $\langle \delta_n \mid n < \omega \rangle$ meeting each of these sets on a tail end. Then we will amend the argument to ensure that the cofinality is as desired.

In $V$, let $\langle S_\delta \mid \delta < \kappa^{+m} \rangle$ be as in the above proposition. In particular, each $S_\delta$ carries a partial square sequence of clubs of order type less than $\kappa^{+m}$. Since $\bigcup_{\delta < \kappa^{+m}} S_\delta = \mu \cap \text{cof}^V(<\kappa) \supset \mu \cap \text{cof}^W(\omega_1)$, let $\delta < \kappa^{+m}$ be such that in $W$, $S_\delta \cap \text{cof}(\omega)$ is stationary, and denote $S := S_\delta$. Let $\langle C^\alpha_\alpha \mid \alpha \in S \rangle$ be the corresponding partial square sequence.

**Claim 2.3.** In $V$ there is a sequence $\langle D^\alpha_\alpha \mid \alpha < \mu \rangle$ of club subsets of $\kappa^{+m}$, such that

1. For each $\alpha < \mu$, $D^\alpha_\alpha \subset D^\alpha_\alpha$.
2. For every $\alpha < \beta < \mu$, $|D^\beta_\beta \setminus D^\alpha_\alpha| < \kappa^{+m}$
3. For every $\alpha < \beta < \mu$, $\beta \in S$, if $\alpha \in C^\beta_\beta$, then $D^\beta_\beta \subset D^\alpha_\alpha$.

**Proof.** Work in $V$. Let $D^\beta_0 = D^\beta_0$. Suppose we have defined $\langle D^\alpha_\alpha \mid \alpha < \beta \rangle$, $\beta < \mu$. If $\beta \in S$, set

$$D^\beta_\beta = D^\beta_\beta \cap \Delta^\alpha_\beta \cap \bigcap_{\alpha \in C^\beta_\beta} D^\alpha_\alpha.$$ 

If $\beta \notin S$, let $D^\beta_\beta = D^\beta_\beta \cap \Delta^\alpha_\beta D^\beta_\beta$. 


In $W$, let $\langle \lambda_n \mid n < \omega \rangle$ be an increasing cofinal sequence in $(\kappa^+)^W$. Let $I_n$ be the interval $(\lambda_n, \lambda_{n+1})$. For every $\alpha < \mu$, let $d_\alpha = \{ n \mid I_n \cap D_\alpha^* \neq \emptyset \}$, and define a sequence $\langle \lambda_n^\alpha \mid n \in d_\alpha \rangle$, by setting
\[
\lambda_n^\alpha = \sup(D_\alpha^* \cap I_n).
\]
Note that for $\alpha < \beta$, $d_\beta \setminus d_\alpha$ is finite, and if $\beta \in S, \alpha \in C_\beta$, then $d_\beta \subset d_\alpha$. Denote $E_\alpha := (\lambda_n^\alpha \mid n \in d_\alpha)$. For $\alpha < \beta$, we say that there is a major change between $\alpha$ and $\beta$ if $E_\alpha \triangle E_\beta$ is infinite. Note that for $\alpha < \beta < \gamma$, if there is a major change between $\alpha$ and $\beta$, then there is a major change between $\alpha$ and $\gamma$.

**Claim 2.4.** There is some $\alpha$, such that for every $\alpha < \beta < \mu$, there is no major change between $\alpha$ and $\beta$.

**Proof.** Otherwise, for every $\alpha$, let $f(\alpha) > \alpha$ be such that there is a major change between $\alpha$ and $f(\alpha)$. The set $C := \{ \alpha < \mu \mid (\exists \beta < \alpha)(f(\beta) < \alpha) \}$ is a club, so pick $\alpha \in \lim(C) \cap S \cap cof(W)(\omega_1)$. Let $\langle \alpha_i \mid i < \omega_1 \rangle$ be a cofinal sequence through $\lim(C_\alpha) \cap C$. Then if $i < j < \omega_1$, $D_{\alpha_j} \subset D_{\alpha_i}$, and so $d_{\alpha_j} \subset d_{\alpha_i}$. Then $\langle d_{\alpha_i} \mid i < \omega_1 \rangle$ must be eventually constant. So, let $d$ be such that for all large $i$, $d = d_{\alpha_i}$.

Now for every $n \in d$, the sequence $\langle \lambda_n^\alpha_i \mid i < \omega_1 \rangle$ is weakly decreasing, so for some $i_n < \omega_1$ and $\lambda_n$, we have that for all $i \geq i_n$, $\lambda_n^\alpha_i = \lambda_n$. Set $i = \sup_{n \in d} i_n < \omega_1$. Let $E := \{ \lambda_n \mid n \in d \}$. Then for all $\xi \geq i$, $E_{\alpha_\xi} = E$.

But for $i < \xi < \xi'$, since $\alpha_\xi \in C$, we have that $f(\alpha_\xi) < \alpha_{\xi'}$, so there is a major change between $\alpha_\xi$ and $\alpha_{\xi'}$. But $E_{\alpha_\xi} = E = E_{\alpha_{\xi'}}$. Contradiction.

Now let $\alpha$ be as in the claim and set $E := E_\alpha$. Then $E$ is the desired sequence, i.e. for all $\beta < \mu$, on a tail end $E$ is contained in $D_\beta^*$.

Next, we show how to get a sequence $\langle \delta_n \mid n < \omega \rangle$ as obtained above, but with the additional requirement that for all $n$, $cf(\delta_n) \geq \lambda$ (in $W$). We follow the argument in Dzamonja-Shelah [2]. First note that for any club $D \subset (\kappa^+)^W$, the set
\[
D^* := \{ \delta \in D \mid o.t.(D \cap \delta) \text{ is divisible by } \lambda \}
\]
is a club, such that if $\delta \in nacc(D^*)$, then $cf(W)(\delta) \geq \lambda$.

For every $\delta \in (\kappa^+)^W$, fix a club $a_\delta \subset D$ in $V$ with order type $cf(V)(\delta)$. We define a sequence of clubs $\langle D_\alpha^* \mid \alpha < \mu \rangle$ by induction on $\alpha$. Set $D_0^* = \{ \delta \in D_0 \mid o.t.(D_0 \cap \delta) \text{ is divisible by } \lambda \}$. Suppose we have already defined $\langle D_\alpha^* \mid \alpha < \beta \rangle$. If $\beta \in S$, let $D_\beta^* = D_\beta \cap \triangle_{\alpha<\beta} D_\alpha^* \cap \bigcap_{\alpha \in C_\beta} D_\alpha^{**}$. If $\beta \notin S$, let $D_\beta^* = D_\beta \cap \triangle_{\alpha<\beta} D_\alpha^{**}$. Then set $D_\beta^{**} := \{ \delta \in D_\beta^* \mid o.t.(D_\beta^* \cap \delta) \text{ is divisible by } \lambda \}$.

Then $\langle D_\alpha^{**} \mid \alpha < \mu \rangle$ satisfy (1), (2), and (3) in Claim 2.3, with the new key property that if $\delta \in nacc(D_\alpha^{**})$, then $cf(W)(\delta) \geq \lambda$.

Now, for $\alpha < \mu$, let $T_\alpha^0 := E_\alpha$, where $E_\alpha$ is defined above, but with respect to $D_\alpha^{**}$, and let
Proof. We prove the theorem for \( \delta \) between there is some \( \alpha \), \( \kappa \) bounded in \( (\mu, \in V) \). Suppose that \( T \) is \( \kappa \)-club in \( nacc(T) \), \( \kappa \) are club subsets of \( D \). Then let \( \kappa \) have uncountable cofinality. It is enough to show that \( \sup(a \cap D^{**}) = \delta \). So \( \delta \in nacc(D^{**}) \), and so \( \sup(a) \geq \delta \).

We say that there is a major change between \( \alpha \) and \( \beta \) if \( T^\alpha \Delta T^\beta \) is unbounded in \((\kappa + m)^N\). By running the same argument as above we get that there is some \( \alpha \), such that for all \( \alpha < \beta < \mu \), there is no major change between \( \alpha, \beta \). Then setting \( T := T^\alpha \), we have that for all \( \beta < \mu \), for some \( \delta, T \setminus \delta \subset D^{**} \).

Then let \( \langle \delta_n \mid n < \omega \rangle \) be cofinal in \( nacc(T) \). This sequence is as desired.

Next we generalize to club subsets of \( P_\kappa(\kappa + m) \).

**Theorem 2.6.** Suppose \( V \subset W \) are such that \( \kappa \) is an inaccessible cardinal in \( V \), a singular cardinal in \( W \), \( \text{cf}^W((\kappa + k)^V) = \omega \) for every \( k \leq m \), and \((\kappa + m + 1)^V = (\kappa)^W \). Suppose that in \( V \), \( \langle D_\alpha \mid \alpha < \kappa^{+m+1} \rangle \) are club subsets of \( P_\kappa(\kappa + m) \). Then in \( W \) there is a sequence \( \langle x_i \mid i < \omega \rangle \), such that for every \( \alpha < \kappa^{+m+1} \), for all large \( i \), \( x_i \in D_\alpha \).

**Proof.** We prove the theorem for \( m = 1 \). The general case is similar, using induction. Denote \( \mu := \kappa^{+} \).

**Claim 2.7.** There is a club \( D \) in \( P_\kappa(\kappa^{+}) \), such that for all \( x, y \in D \) such that \( \kappa \cap x, \kappa \cap y \), sup\((x), sup(y)\) are ordinals with uncountable cofinality,

- if \( \kappa \cap x = \kappa \cap y \) and sup\((x) = sup(y)\), then \( x = y \),
- if \( \kappa \cap x < \kappa \cap y \) and sup\((x) = sup(y)\), then \( x \subset y \).

**Proof.** Let \( \gamma \) be some big enough cardinal, and \( <_\gamma \) be a well-ordering of \( H_\gamma \).

Let \( D' = \{ M \cap \kappa^+ \mid M \prec (H_\gamma, \in, <_\gamma ...), |M| < \kappa, \kappa \in M, M \cap \kappa \in \kappa \} \). Then set \( D = D' \cap \{ x \mid \text{cf}((\kappa \cap x) > \omega \) and \( \text{cf}(\sup(x)) > \omega \} \rightarrow x \) is \( \omega \)-closed.

Suppose \( x, y \in D \) are such that sup\((x) = sup(y) = \alpha, \kappa \cap x \leq \kappa \cap y \), and \( \kappa \cap x, \kappa \cap y, \alpha \) have uncountable cofinality. It is enough to show that \( x \leq y \).

Since \( x, y \) are both \( \omega \)-club in \( \alpha \), \( x \cap y \) is also \( \omega \)-club in \( \alpha \). Now let \( M, N \) witness that \( x, y \in D \). For every \( \eta \in x \cap y \), by elementarity \( M \) and \( N \) have the same functions from \( \kappa \) to \( \eta \). Since \( M \cap \kappa \leq N \cap \kappa \), we get that \( M \cap \eta \subseteq N \cap \eta \). By doing this for all \( \eta \in x \cap y \), we get \( x \subseteq y \).

Suppose that \( \langle D_\alpha \mid \alpha < \mu \rangle \) are club subsets of \( P_\kappa(\kappa^{+}) \), each \( D_\alpha \subset D \). As before let \( S \cap \kappa \cap \text{cof}^W(\omega_1) \subset \text{stationary} \) (in \( W \)), such that there is a partial square sequence \( \langle C_\alpha \mid \alpha \in S \cap \text{Lim} \rangle \in V \), where for each \( \alpha, o.t. (C_\alpha) \leq \kappa \).
Claim 2.8. In $V$ there is a sequence $\langle D^*_\alpha \mid \alpha < \mu \rangle$ of club subsets of $\mathcal{P}_\kappa(\kappa^+)$, such that

1. For each $\alpha < \mu$, $D^*_\alpha \subset D_\alpha$, 
2. For all $\alpha < \beta < \mu$, there is $\gamma < \kappa^+$, such that $D^*_\beta \setminus D^*_\alpha \subset \{ x \mid \gamma \notin x \}$
3. For all $\alpha < \beta < \mu$, with $\beta \in S$, if $\alpha \in C_\beta$, $\text{o.t.}(C_\beta) < \kappa$, then $D^*_\beta \subset D^*_\alpha$.

Proof. Work in $V$. For every $\beta < \mu$ fix a one-to-one function $f_\beta : \beta \to \kappa^+$. Let $D^*_0 = D_0$. Suppose we have defined $\langle D^*_\alpha \mid \alpha < \beta \rangle$. If $\beta \in S$ and $\text{o.t.}(C_\beta) < \kappa$, set

$$D^*_\beta = D_\beta \cap \{ x \mid x \in \bigcap_{\gamma \in x} D^*_{f_\beta^{-1}(\gamma)} \} \cap \bigcap_{\alpha \in C_\beta} D^*_\alpha.$$ 

Otherwise, let $D^*_\beta = D_\beta \cap \{ x \mid x \in \bigcap_{\gamma \in x} D^*_{f_\beta^{-1}(\gamma)} \}.$

$\langle D^*_\alpha \mid \alpha < \mu \rangle$ is as desired. The $\gamma$ in item (2) is $\gamma = f_\beta(\alpha)$. 

Let $\langle D^*_\alpha \mid \alpha < \mu \rangle$ be as in the claim, but with the additional requirement that for $x \in D^*_\alpha$, $\text{sup}(x)$ is a closure point of the algebra defining $D_\alpha$. For each $\alpha$, let $D'_\alpha := \{ \text{sup}(x) \mid x \in D^*_\alpha \}$. Then $D'_\alpha$ is a club subset of $\kappa^+$. Let $\langle \lambda_n \mid n < \omega \rangle$ be a cofinal sequence in $\kappa^+$ given by the last theorem applied to the closure of $D'_\alpha$. Actually if we arrange that $\text{cf}^W(\lambda_n) = \omega_1$, then for all $\alpha$, for all large $n$, $\lambda_n \in D'_\alpha$.

Suppose that $\langle \kappa_n \mid n < \omega \rangle \in W$ is a cofinal sequence in $\kappa$. Define $\gamma^*_n := \text{sup}\{ \eta \in (\kappa_n, \kappa_{n+1}) \mid (\exists x \in D^*_\alpha)(x \cap \kappa = \eta, \text{sup}(x) = \lambda_n) \}$. Let $d_{\alpha} = \{ n \mid \gamma^*_n > 0 \}$. By choosing the $\kappa_n$’s fast enough, we make sure that $d_{\alpha}$ is nonempty. We have that:

1. If $\beta \in S$, $\alpha \in \text{lim}(C_\beta)$, $\text{o.t.}(C_\beta) < \kappa$, then $d_\beta \subset d_\alpha$.
2. For every $\eta \in (\kappa_n, \kappa_{n+1})$, such that $(\exists x \in D^*_\alpha)(x \cap \kappa = \eta, \text{sup}(x) = \lambda_n)$, there is a unique witness $x_\eta$ such that $\eta_1 < \eta_2 \rightarrow x_{\eta_1} \subset x_{\eta_2}$.

Then taking $x$ to be their union, we have that $x \in D^*_\alpha$, and $x \cap \kappa = \gamma^*_n, \text{sup}(x) = \lambda_n$.

Denote $E_\alpha := \langle \gamma^*_n \mid n \in d_{\alpha} \rangle$. For $\alpha < \beta$, we say that there is a major change between $\alpha$ and $\beta$ if $E_\alpha \Delta E_\beta$ is infinite. Note that for $\alpha < \beta < \beta'$, if there is a major change between $\alpha$ and $\beta$, then there is a major change between $\alpha$ and $\beta'$.

Claim 2.9. There is some $\alpha$, such that for every $\alpha < \beta < \mu$, there is no major change between $\alpha$ and $\beta$.

Proof. Otherwise for every $\alpha$, let $f(\alpha) > \alpha$ be such that there is a major change between $\alpha$ and $f(\alpha)$. The set $C := \{ \alpha < \mu \mid (\forall \beta < \alpha)(f(\beta) < \alpha) \}$ is a club, so pick $\alpha \in \text{lim}(C) \cap S \cap \text{cof}^W(\omega_1)$. Let $\langle \alpha_i \mid i < \omega_1 \rangle$ be a cofinal sequence through $\text{lim}(C_\alpha)$. Then if $i < j < \omega_1$, $D^*_{\alpha_j} \subset D^*_{\alpha_i}$, and so $d_{\alpha_j} \subset d_{\alpha_i}$. Then $\langle d_{\alpha_i} \mid i < \omega_1 \rangle$ must be eventually constant. So, let $d$ be such that for all large $i$, $d = d_{\alpha_i}$. 


Now for every $n \in d$, the sequence $\langle \gamma_n^\alpha \mid i < \omega_1 \rangle$ is weakly decreasing, so for some $i_n < \omega_1$ and $\bar{\gamma}_n$, we have that for all $i \geq i_n$, $\gamma_n^\alpha = \bar{\gamma}_n$. Set $i = \sup_{n \in d} i_n < \omega_1$. Let $E := \{\bar{\gamma}_n \mid n \in d\}$. Then for all $\xi \geq i$, $E_{\alpha_\xi} = E$.

But if $i < \xi < \xi'$, since $\alpha_{\xi'} \in C$, we have $f(\alpha_\xi) < \alpha_{\xi'}$, so there is a major change between $\alpha_\xi$ and $\alpha_{\xi'}$. But $E_{\alpha_\xi} = E = E_{\alpha_{\xi'}}$. Contradiction

Now let $\alpha$ be as in the claim. Consider $E_\alpha = \langle \gamma_n^\alpha \mid n \in d_\alpha \rangle$ and $\langle \lambda_n \mid n < \omega \rangle$. By applying the same argument as in the end of the last theorem, we can arrange that $\text{cf}(\gamma_n^\alpha) \neq \omega$. Set $x_n \in D$ to be (the unique) such that $x_n \cap \kappa = \gamma_n^\alpha$ and $\text{sup}(x_n) = \lambda_n$. Then $\langle x_n \mid n < \omega \rangle$ is as desired.

3. Consequences on square properties

**Theorem 3.1.** Suppose $V \subset W$ are such that $\kappa$ is an inaccessible cardinal in $V$, a singular cardinal in $W$, such that for $k \leq n$, $W \models \text{cf}((\kappa^+)^k) = \omega$, and $(\kappa^{+n+1})^V = (\kappa^+)^W$. Then in $W$, $\Box_{\kappa, \omega}$ holds.

**Proof.** Work in $V$. Denote $\mu := \kappa^{+n+1}$. Let $\chi$ be some big enough cardinal, and $<_\chi$ be a well-ordering of $H_\chi$. For $\alpha < \mu$ with $\text{cf}(\alpha) < \kappa$, let $\langle M_\alpha^\delta \mid \delta < \kappa^{+n} \rangle$ be a continuous $\subset$-increasing sequence of elementary submodels of $\langle H_\chi, \in, <_\chi \ldots \rangle$, such that:

1. $\alpha, \kappa \in M_\alpha^\delta$,
2. for each $\delta < \kappa^{+n}$, $|M_\alpha^\delta| < \kappa^{+n}$,
3. for each $\delta < \kappa^{+n}$, $M_\alpha^\delta \cap \kappa^{+n} \in \kappa^{+n}$.

**Claim 3.2.** For all $\alpha < \mu$, $\delta < \kappa^{+n}$ with $\text{cf}(\delta) > \omega$, if $M_\alpha^\delta \cap \alpha$ is cofinal in $\alpha$, then $M_\alpha^\delta \cap \alpha$ is $\omega$-closed.

**Proof.** Suppose for contradiction that $\langle \beta_i \mid i < \omega \rangle$ is an increasing sequence of points in $M_\alpha^\delta \cap \alpha$, such that $\beta := \sup_i \beta_i \notin M_\alpha^\delta \cap \alpha$. Let $\beta^*$ be the least ordinal in $M_\alpha^\delta \cap \alpha$ above $\beta$. If $\text{cf}(\beta^*) = \lambda < \kappa^{+n}$, then $\lambda \in M_\alpha^\delta \cap \kappa^{+n} \in \kappa^{+n}$. Then $\lambda + 1 \subset M_\alpha^\delta$, and so $M_\alpha^\delta$ is cofinal in $\beta^*$, which is a contradiction.

It follows that the cofinality of $\beta^*$ must be $\kappa^{+n}$. Say $M_\alpha^\delta \models h : \kappa^{+n} \rightarrow \beta^*$ is cofinal. For every $i$, let $\beta_i^* \in \text{ran}(h) \setminus \beta_i$. The order type of $\text{ran}(h)$ is $\kappa^{+n} \cap M_\alpha^\delta$. And $\text{cf}(\kappa^{+n} \cap M_\alpha^\delta) = \text{cf}(\delta) > \omega$. So there is something in the range of $h$ above $\beta$. Contradiction with the choice of $\beta^*$.

**Claim 3.3.** Let $\alpha < \mu$. $D_\alpha := \{\text{sup}(M_\delta^\alpha \cap \kappa^{+n}) \mid \delta < \kappa^{+n}\}$ is club in $\kappa^{+n}$.

**Proof.** $D_\alpha$ is closed by construction since the sequence is continuous. To see that it is unbounded, note that $\kappa^{+n} \subset \bigcup_\delta M_\delta^\alpha$. Note that this also implies that $\alpha \subset \bigcup_\delta M_\delta^\alpha$, since there is a function in $H_\chi$ from $\kappa^{+n}$ onto $\alpha$.

By Theorem 2.2 there is a sequence $\langle \lambda_i \mid i < \omega \rangle \in W$, such that for all large $i$, $\lambda_i \in D_\alpha$ and for all $i$, $\text{cf}(\lambda_i) > \omega$. For $\alpha < \mu$ with $\text{cf}(\alpha) < \kappa$, define
\( C_\alpha := \{ M^\beta_\delta \cap \alpha \mid \alpha \leq \beta < \mu, \delta < \kappa^+ n, M^\beta_\delta \) is cofinal in \( \beta \) and \( \alpha, (\exists i) M^\beta_\delta \cap (\kappa^+ n)^V = \lambda_i \}. \) Here \( M^\beta_\delta \cap \alpha \) denotes the closure of \( M^\beta_\delta \cap \alpha \). Otherwise, \( \text{cf}^W (\alpha) = \omega \) and we let \( C_\alpha \) be a singleton of some \( \omega \)-sequence cofinal in \( \alpha \).

Claim 3.4. \( 1 \leq |C_\alpha| \leq \omega \).

Proof. Each \( C_\alpha \) is nonempty because for all large \( i, \lambda_i \in D_\alpha \). Now suppose \( \beta, \beta', \delta, \delta' \) are such that:

- \( M^\beta_\delta \cap (\kappa^+ n)^V = M^\beta_{\delta'} \cap (\kappa^+ n)^V = \lambda_i \),
- \( M^\beta_\delta, M^\beta_{\delta'} \) are both cofinal in \( \alpha \).

We want to show that \( M^\beta_\delta \cap \alpha = M^\beta_{\delta'} \cap \alpha \). If \( \text{cf}(\alpha) = \omega \), then \( \alpha \in M^\beta_\delta, M^\beta_{\delta'} \).

By elementarity, since \( M^\beta_\delta \cap (\kappa^+ n)^V = M^\beta_{\delta'} \cap (\kappa^+ n)^V \), they both have the same functions from \( \kappa^+ n \) to \( \alpha \), and the result follows. Now suppose that \( \text{cf}(\alpha) > \omega \). We have \( \text{cf}(\delta) = \text{cf}(\delta') = \text{cf}(\lambda_i) > \omega \). Then \( M^\beta_\delta \cap \alpha, M^\beta_{\delta'} \cap \alpha \) are both \( \omega \)-clubs in \( \alpha \), and so \( M^\beta_\delta \cap \alpha, M^\beta_{\delta'} \cap \alpha \) is an \( \omega \)-club in \( \alpha \). By the above for any \( \eta \in M^\beta_\delta \cap M^\beta_{\delta'} \cap \alpha, M^\beta_\delta \cap \eta = M^\beta_{\delta'} \cap \eta \), and so \( M^\beta_\delta \cap \alpha = M^\beta_{\delta'} \cap \alpha \).

□ □

We conclude with some questions.

**Question 1.** Can we replace \( \kappa^+ m \) in Theorem 3.1 with any regular \( V \)-cardinal \( \tau \)?

We have to require that \( \tau \) is regular, because in the case of singular \( \tau \), there is a counterexample in Gitik-Sharon[4]. Another question is whether we can obtain the existence of pseudo Prikry sequence for more general domains:

**Question 2.** Can we replace \( \kappa^+ m \) in Theorem 2.6 with any (possibly singular) \( V \)-cardinal \( \tau \)?

**References**