THE TREE PROPERTY AND THE FAILURE OF THE SINGULAR CARDINAL HYPOTHESIS AT $\aleph_{\omega^2}$

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Abstract. We show that given ω many supercompact cardinals, there is a generic extension in which the tree property holds at $\aleph_{\omega^2+1}$ and the SCH fails at $\aleph_{\omega^2}$.

1. Introduction

The tree property at κ^+ states that every tree with height κ^+ and levels of size at most κ has an unbounded branch. Equivalently, there are no κ^+-Aronszajn trees. In the 1980's Woodin asked if the failure of the Singular Cardinal Hypothesis (SCH) at $\aleph_\omega$ implies the existence of an Aronszajn tree at $\aleph_{\omega+1}$. To motivate the question we note a few facts about Aronszajn trees. By Magidor and Shelah [9], the tree property holds at successors of singular limits of strongly compact cardinals. On the other hand, Solovay [12] showed that SCH holds above a strongly compact cardinal. Generalizing the original question, in 1989 Woodin and others asked if the failure of SCH at a singular cardinal κ of cofinality ω implies the existence of an Aronszajn tree at κ^+.

The first progress on Woodin's question was made by Gitik-Sharon [6], who showed the consistency of the failure of SCH at a singular cardinal κ together with the non-existence of *special* κ^+-Aronszajn trees. They also pushed down their result to $\kappa = \aleph_{\omega^2}$. Then in 2009, Neeman [10] obtained the failure of the singular cardinal hypothesis at some large singular cardinal κ, together with the full tree property at κ^+. It remained open whether this construction can be pushed down to smaller cardinals. In this paper we show that his result can indeed be obtained at smaller cardinals. In particular, we prove that it is consistent relative to large cardinals that $\aleph_{\omega^2}$ is a strong limit cardinal, $2^{\aleph_{\omega^2}} > \aleph_{\omega^2+1}$, and the tree property holds at $\aleph_{\omega^2+1}$. It still remains open whether an analogous result holds for $\aleph_\omega$.

Theorem 1. *Suppose that in V, $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals and GCH holds. Then there is a generic extension in which:*

The author thanks Itay Neeman for his invaluable comments and corrections.

(1) $\kappa_0 = \aleph_{\omega^2}$,
(2) *the tree property holds at* $\aleph_{\omega^2+1}$,
(3) *SCH fails at* κ.

The rest of the paper presents the proof of Theorem 1. In section 2 we define the forcing notion and give some basic properties about the forcing. Section 3 deals with a preservation lemma, which will be used to show the tree property. Finally in section 4 we prove that the tree property holds at $\aleph_{\omega^2+1}$.

2. The forcing

Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals. We start by using Laver's forcing to make κ_0 indestructably supercompact while maintaining GCH above κ_0. Let V be the resulting model. Denote $\kappa = \kappa_0$, $\nu = \sup_n \kappa_n$ and $\mu = \nu^+$. First we force with $\mathbb{C}$, where $\mathbb{C}$ is the full support iterated collapse to make each κ_n be the n-th successor of κ. Let H be $\mathbb{C}$-generic over V. Then we force with $Add(\kappa, \nu^{++})$. Let E be $Add(\kappa, \nu^{++})$ generic over $V[H]$. Work in $V[H][E]$.

Proposition 2. *There is a normal measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$, U, such that for each $n < \omega$ if we let U_n be the projection of U to $\mathcal{P}_\kappa(\kappa^{+n})$ and $j_n = j_{U_n}$, then there exists a $(Col(\kappa^{+\omega+2}, < j_n(\kappa)))^{N_n}$ - generic filter, K_n, over $N_n = Ult(V[H][E], U_n)$.*

Proof. See Gitik-Sharon [6]. $\square$

Let U be a normal measure on $\mathcal{P}_\kappa(\kappa^{+\omega+1})$ given by the above proposition. For each n, let U_n be the projection of U to $\mathcal{P}_\kappa(\kappa^{+n})$ and $j_n = j_{U_n}$. Using standard reflection arguments, we choose sets $X_n \in U_n$ for $n < \omega$, such that for all $x \in X_n$:

- $\kappa \cap x = \kappa_x$ is κ_x^{+n} -supercompact.
- $(\forall k \leq n) o.t.(x \cap \kappa^{+k}) = \kappa_x^{+k}$. In particular, $o.t.(x) = \kappa_x^{+n}$.

Fix generic filters $\langle K_n \mid n < \omega \rangle$ as in the above proposition. We are ready to define the main forcing. Basically, we take the forcing in Gitik-Sharon [6] with collapses using the filters $\langle K_n \mid n < \omega \rangle$.

We use the notation $x \prec y$ to denote that $x \subset y$ and $|x| < \kappa_y$.

Definition 3. *Conditions in $\mathbb{P}$ are of the form $p = \langle d, \langle p_n \mid n < \omega \rangle \rangle$, where for some integer $l = lh(p)$ (the length of p), we have:*

(1) *For $n < l$, $p_n = \langle x_n, c_n \rangle$ such that:*
   - *$x_n \in X_n$ and for $i < n$, $x_i \prec x_n$,*
   - *if $n < l - 1$, then $c_n \in Col(\kappa_{x_n}^{+\omega+2}, < \kappa_{x_{n+1}})$, and $c_{l-1} \in Col(\kappa_{x_{l-1}}^{+\omega+2}, < \kappa)$.*

(2) *For* $n \geq l$, $p_n = \langle A_n, C_n \rangle$ *such that:*
- $A_n \in U_n$, $A_n \subset X_n$, *and* $x_{l-1} \prec y$ *for all* $y \in A_n$.
- C_n *is a function with domain* A_n, *for* $y \in A_n$, $C_n(y) \in Col(\kappa_y^{+\omega+2}, <\kappa)$,
- $[C_n]_{U_n} \in K_n$.

(3) *if* $l > 0$, *then* $d \in Col(\omega, \kappa_{x_0}^{+\omega})$, *otherwise* $d \in Col(\omega, \kappa)$.

For a condition p, *we will use the notation* $p = \langle d^p, \langle p_n \mid n < \omega \rangle \rangle$, $p_n = \langle x_n^p, c_n^p \rangle$ *for* $n < lh(p)$, *and* $p_n = \langle A_n^p, C_n^p \rangle$ *for* $n \geq lh(p)$. *The stem of* p *is* $h = \langle d^p, \langle p_n \mid n < lh(p) \rangle \rangle$. *Sometimes we will also denote the stem of* p *by* $\langle d^p, \langle \vec{x}, \vec{c} \rangle \rangle$, *where* $\vec{x}$ *and* $\vec{c}$ *are with length* $lh(p)$, *and for* $i < lh(p)$, $p_i = \langle x_i, c_i \rangle$.

$q = \langle d^q, \langle q_n \mid n < \omega \rangle \rangle \leq p = \langle d^p, \langle p_n \mid n < \omega \rangle \rangle$ *if* $lh(q) \geq lh(p)$ *and:*

- $d^q \supset d^p$,
- *for all* $n < lh(p)$, $x_n^p = x_n^q, c_n^q \supset c_n^p$,
- *for* $lh(p) \leq n < lh(q)$, $x_n^q \in A_n^p$ *and* $c_n^q \supset C_n^p(x_n^q)$,
- *for* $n \geq lh(q)$, $A_n^q \subset A_n^p$ *and for all* $y \in A_n^q$, $C_n^q(y) \supset C_n^p(y)$

We say that q *is a direct extension of* p, *denoted by* $q \leq^* p$, *if* $q \leq p$ *and* $lh(q) = lh(p)$. *For two stems* h_1 *and* h_2, *we say that* h_1 *is stronger or an extension of* h_2 *if there are conditions* $p_1 \leq p_2$ *with stems* h_1 *and* h_2 *respectively.*

Note that any two conditions with the same stem are compatible. That is since the collapsing part is taken to be in the filters K_n. For two conditions p, q with the same stem we define $p \wedge q$ to be the weakest common extension.

Let G be $\mathbb{P}$ generic over $V[H][E]$, and let $\langle x_n \mid n < \omega \rangle$, where each $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$, be the components derived from the generic set G. Set $\lambda_n = x_n \cap \kappa$. By Gitik-Sharon [6], we get:

Proposition 4.

(1) *If* $\langle A_n \mid n < \omega \rangle \in V[H][E]$ *is a sequence of sets such that every* $A_n \in U_n$, *then for all large* n, $x_n \in A_n$.

(2) $\bigcup_n x_n = \nu = (\kappa^{+\omega})^{V[H][E]}$.

(3) *For each* $n \geq 0$, *the cofinality of* $\kappa_n = (\kappa^{+n})^{V[H][E]}$ *in* $V[H][E][G]$ *is* ω.

(4) *Since any two conditions with the same stem are compatible,* $\mathbb{P}$ *has the* $\mu = (\kappa^{+\omega+1})^{V[H][E]}$ *chain condition. So, cardinals greater than or equal to* μ *are preserved.*

(5) *In* $V[H][E][G]$, $2^\kappa = \mu^+$ *and GCH holds from* μ *upward.*

(6) $\mathbb{P}$ *has the Prikry property. I.e. if* p *is a condition with length at least* 1 *and* ϕ *is a formula, then there is a direct extension* $p' \leq^* p$ *which decides* ϕ.

Remark 5. The main point in the proof of the Prikry property is the diagonal lemma, which states that for $p \in \mathbb{P}$ with length at least 1 and $lh(p) < n < \omega$ if H is a set of stems of length $lh(p)+n$ and $\langle q^h \mid h \in H \rangle$ are conditions stronger than p such that each q^h has a stem h, then there is $q \leq^* p$ such that if $r \leq q$ is a condition of length at least $lh(p) + n$, then $r \leq q^h$ for some $h \in H$.

Remark 6. Using the closure of the collapsing posets, we get the following corollary to the Prikry property: If p is a condition with length $n + 1$ and ϕ is a formula, then there is a direct extension $q \leq^* p$, such that $q \restriction n = p$ and if $r \leq p$ decides ϕ, then $r \restriction n^\frown q \restriction (\omega \setminus n)$ decides ϕ.

The last property implies that all cardinals χ, such that $\lambda_n \leq \chi \leq \lambda_n^{+\omega+2}$ for some $n > 0$ are preserved. In particular, in $V[H][E][G]$ every λ_n for $n > 0$ is a cardinal. And so, κ remains a cardinal, as well. Also, $\mathbb{P}$ preserves $(\lambda_0^{+\omega+1})^{V[H][E]}$, $(\lambda_0^{+\omega+2})^{V[H][E]}$, and collapses $(\lambda_0^{+\omega})^{V[H][E]}$ to ω. So in $V[H][E][G]$, $\kappa = \aleph_{\omega^2}$, $\mu = \kappa^+$, $(\lambda_0^{+\omega+1})^{V[H][E]} = \aleph_1$, and $2^{\aleph_{\omega^2}} = \aleph_{\omega^2+2}$. So, SCH fails at $\aleph_{\omega^2}$.

3. The preservation lemma

In this section we prove a preservation lemma, which will be used to show the tree property. The proof of this lemma is motivated by the Preservation Theorem in Magidor-Shelah [9]. The main difference is that instead of trees, here we are working with narrow systems, which are defined below. Throughout this section V will denote some arbitrary ground model. We say that a poset is χ-closed if it is closed under sequences of length χ. We start with defining the notion of a narrow system. This is the same as the definition given in Magidor-Shelah [9].

Definition 7. $S = \langle I \times \kappa, \mathcal{R} \rangle$ *is a narrow system of height* ν^+ *and levels of size* $\kappa < \nu$ *if:*

- I *is an unbounded subset of* ν^+, *and for each* $\alpha \in I$, $S_\alpha = \{\alpha\} \times \kappa$ *is the* α*-level of* S,
- $\mathcal{R}$ *is a set of transitive binary relations on* $I \times \kappa$, *such that for all* $R \in \mathcal{R}$, *if* $\langle \alpha, \delta \rangle R \langle \beta, \xi \rangle$, *then* $\alpha < \beta$.
- $|\mathcal{R}| < \nu$.
- *For every* $\alpha < \beta$ *in* I, *there are* $u \in S_\alpha$, $v \in S_\beta$, *and* $R \in \mathcal{R}$ *such that* $\langle u, v \rangle \in R$.

- *For all $R \in \mathcal{R}$, if u_1, u_2 are distinct elements of $I \times \kappa$ such that $\langle u_1, v\rangle \in R$ and $\langle u_2, v\rangle \in R$, then $\langle u_1, u_2\rangle \in R$ or $\langle u_2, u_1\rangle \in R$.*

For $a_1, a_2 \in S$ and $R \in \mathcal{R}$ we write $a_1 \perp_R a_2$ if $\langle a_1, a_2\rangle \notin R$ and $\langle a_2, a_1\rangle \notin R$, and in that case say that a_1, a_2 are R-incomparable.

A **branch** *of S is a set $b \subset \bigcup_{\alpha\in I} S_\alpha$ such that for some $R \in \mathcal{R}$, we have that for all $u, v \in b$, $\langle u, v\rangle \in R$ or $\langle v, u\rangle \in R$. In this case we say that b is a branch through R (or with respect to R). We say that $b \subset I \times \kappa$ is unbounded if for unboundedly many $\alpha \in I$, $b \cap S_\alpha \neq \emptyset$.*

Theorem 8. *Suppose that ν is a singular cardinal of cofinality ω, $\kappa, \tau < \nu$ are regular cardinals, and in V, $\mathbb{Q}$ is κ^+ c.c notion of forcing and $\mathbb{R}$ is a $\max(\kappa, \tau)^+$ closed notion of forcing. Let E be $\mathbb{Q}$-generic over V and let F be $\mathbb{R}$-generic over $V[E]$. Suppose that $S = \langle I, \mathcal{R}\rangle$ is a narrow system in $V[E]$ of height ν^+, levels of size κ, and with $\mathcal{R} = \langle R_\sigma \mid \sigma < \tau\rangle$. Suppose that in $V[E][F]$ there are (not necessarily all unbounded) branches $\langle b_{\sigma,\delta} \mid \sigma \in L,\ \delta < \kappa\rangle$, such that:*

1. *every $b_{\sigma,\delta}$ is a branch through R_σ, and for some $\langle \sigma, \delta\rangle \in L \times \kappa$, $b_{\sigma,\delta}$ is unbounded.*
2. *for all $\alpha \in I$, there is $\langle \sigma, \delta\rangle \in L \times \kappa$, such that $S_\alpha \cap b_{\sigma,\delta} \neq \emptyset$.*

Then S has an unbounded branch in $V[E]$.

Proof. Let $\sigma < \tau, \delta < \kappa$. For $\alpha < \nu^+$, $p \in \mathbb{Q}$, and $r_1, r_2 \in \mathbb{R}$, we say that $p \Vdash_{\mathbb{Q}}$ "r_1, r_2 force contradictory values about $\dot b_{\sigma,\delta} \cap S_{<\alpha}$" if there are nodes $u_1 = \langle \alpha_1, \xi_1\rangle, u_2 = \langle \alpha_2, \xi_2\rangle$ with $\alpha_1 < \alpha_2 < \alpha$, such that:

- $p \Vdash_{\mathbb{Q}}$ "$\alpha_1, \alpha_2 \in \dot I$ and $\langle u_1, u_2\rangle \notin \dot R_\sigma$"
- $p \Vdash_{\mathbb{Q}}$ "$r_1 \Vdash_{\mathbb{R}} u_1 \in \dot b_{\sigma,\delta}$ and $r_2 \Vdash_{\mathbb{R}} u_2 \in \dot b_{\sigma,\delta}$".

The following lemma uses an argument from a branch lemma by Spencer Unger [13]. Versions of this lemma were also proved by Baumgartner, Foreman and Spinas while they were working on [1].

Lemma 9. *(Splitting Lemma) Let $\sigma < \tau, \delta < \kappa$. Suppose that for some $\beta < \nu^+$, $p \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot b_{\sigma,\delta}$ is an unbounded branch or $r \Vdash_{\mathbb{R}} \dot b_{\sigma,\delta} \subset S_{<\beta}$", and p forces that $\dot S$ has no unbounded branches through $\dot R_\sigma$ in $V^{\mathbb{Q}}$. Let $r_1, r_2 \in \mathbb{R}$ be stronger than r. Then there are $r_1^* \le r_1$, $r_2^* \le r_2$, a maximal antichain $A \subset \mathbb{Q}$ of conditions below p and $\alpha < \nu^+$ with $\alpha \ge \beta$, such that for all $p' \in A$, $p' \Vdash_{\mathbb{Q}}$ "r_1^* and r_2^* force contradictory values about $S_{<\alpha} \cap \dot b_{\sigma,\delta}$", or $p' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot b_{\sigma,\delta} \subset S_{<\beta}$".*

Proof. First we show the following claim.

Claim 10. *For any $r_1' \le r_1$ and $r_2' \le r_2$ the set $D^{r_1', r_2'} = \{p' \le p \mid p'' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot b_{\sigma,\delta} \subset S_{<\beta}$" or $(\exists r_1'' \le r_1')(\exists r_2'' \le r_2')(\exists \alpha < \nu^+)(p' \Vdash_{\mathbb{Q}}$ "r_1'', r_2'' force contradictory values for $\dot b_{\sigma,\delta} \cap S_{<\alpha}$")$\}$ is dense below p.*

Proof. Let $r_1' \leq r_1$, $r_2' \leq r_2$, $p' \leq p$. Let $q \leq p'$ be such that $q \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta}$ is unbounded", or $q \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta}$". If the latter holds, than $q \in D^{r_1',r_2'}$, so we are done.

Now suppose that $q \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta}$ is unbounded". Let E' be $\mathbb{Q}$-generic over V with $q \in E'$. In $V[E']$ define $E_1 = \{a \in I \times \kappa \mid (\exists r'' \leq r_1')(r'' \Vdash a \in \dot{b}_{\sigma,\delta})\}$ and $E_2 = \{a \in I \times \kappa \mid (\exists r'' \leq r_2')(r'' \Vdash a \in \dot{b}_{\sigma,\delta})\}$. Then since in $V[E']$, r forces that $\dot{b}_{\sigma,\delta}$ is unbounded, we have that both $\{\alpha \in I \mid E_1 \cap S_\alpha \neq \emptyset\}$ and $\{\alpha \in I \mid E_2 \cap S_\alpha \neq \emptyset\}$ are unbounded in I. Also since S has no unbounded branches through R_σ in $V[E']$, we have that for any $a \in E_1$, there are R_σ incomparable nodes c, d in E_1 with $\langle a, c\rangle \in R_\sigma$ and $\langle a, d\rangle \in R_\sigma$. The same holds for E_2. So, we can find R_σ incomparable nodes $c_1 \in E_1$ and $c_2 \in E_2$ as follows. Let a_1, a_2 be R_σ-incomparable nodes in E_1 and let $c_2 \in E_2$ be of level above the levels of a_1, a_2. Then c_2 must be R_σ-incomparable with at least one of these nodes. Let c_1 be a_1 if $a_1 \perp_{R_\sigma} c_2$, otherwise let $c_1 = a_2$. Then there are $p'' \leq q$, $r_1'' \leq r_1'$, $r_2'' \leq r_2'$, and $\alpha < \nu^+$, such that $p'' \Vdash_{\mathbb{Q}}$ "r_1'', r_2'' force contradictory values for $\dot{b}_{\sigma,\delta} \cap S_{<\alpha}$", as witnessed by c_1 and c_2. $\square$

The rest of the proof of the lemma uses Spencer Unger's diagonal construction argument to get the antichain and r_1^*, r_2^* as desired. The only difference is that here all antichains are below p, and $r_1^0 \leq r_1$, $r_2^0 \leq r_2$. We include it for completeness.

Inductively construct antichains $\langle A_\xi \mid \xi < \kappa^+\rangle$ in $\mathbb{Q}$, conditions $\langle r_1^\xi, r_2^\xi \mid \xi < \kappa^+\rangle$ in $\mathbb{R}$ and $\langle \alpha_\xi \mid \xi < \kappa^+\rangle$ as follows. Let $A_0 = \emptyset$, $\alpha_0 = \beta$, $r_1^0 = r_1$, and $r_2^0 = r_2$. For δ limit, set $A_\delta = \bigcup_{\xi<\delta} A_\xi$ and r_1^δ, r_2^δ to be lower bounds of $\langle r_1^\xi \mid \xi < \delta\rangle$ and $\langle r_2^\xi \mid \xi < \delta\rangle$, respectively. Also set $\alpha_\delta = \sup_{\xi<\delta} \alpha_\xi$. At successor stages, suppose we have defined A_ξ. If A_ξ is a maximal antichain below p, then set $A_{\xi+1} = A_\xi$, $\alpha_{\xi+1} = \alpha_\xi$, $r_1^{\xi+1} = r_1^\xi$ and $r_2^{\xi+1} = r_2^\xi$. Otherwise, let $q \leq p$ be such that q is incompatible with every condition in A_ξ. Applying the claim, let $q' \leq q$ be such that $q' \in D^{r_1^\xi, r_2^\xi}$. If $q' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta}$", set $\alpha_{\xi+1} = \alpha_\xi$, $r_1^{\xi+1} = r_1^\xi$, and $r_2^{\xi+1} = r_2^\xi$. Otherwise let $\alpha_{\xi+1} > \alpha_\xi$, $r_1^{\xi+1} \leq r_1^\xi$, and $r_2^{\xi+1} \leq r_2^\xi$ be such that $q' \Vdash_{\mathbb{Q}}$ "$r_1^{\xi+1}, r_2^{\xi+1}$ force contradictory values for $\dot{b}_{\sigma,\delta} \cap S_{<\alpha_{\xi+1}}$". Set $A_{\xi+1} = A_\xi \cup \{q'\}$.

Since $\mathbb{Q}$ has the κ^+ chain condition, for some $\delta < \kappa^+$, we have that A_δ is a maximal antichain below p. Set $\alpha = \alpha_\delta$, $r_1^* = r_1^\delta$ and $r_2^* = r_2^\delta$. Then A_δ, α, r_1^*, and r_2^* are as desired.

$\square$

Let $\lambda = \max(\kappa, \tau)^+$. The Splitting lemma yields the following corollary.

Corollary 11. *Let $\sigma < \tau, \delta < \kappa$. Suppose that for some $\beta < \nu^+$, $p \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta}$ is an unbounded branch or $r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta}$", and p forces that $\dot{S}$ has no unbounded branches through $\dot{R}_\sigma$ in $V^{\mathbb{Q}}$. Let $\langle r_i \mid i < \lambda\rangle$ be conditions in $\mathbb{R}$ that are stronger than r. Then there is a sequence $\langle r'_i \mid i < \lambda\rangle$ and $\alpha < \nu^+$, such that each $r'_i \leq r_i$ and for all $i < j < \lambda$, there is a maximal antichain $A_{ij} \subset \mathbb{Q}$ of conditions below p, such that for all $p' \in A_{ij}$, $p' \Vdash_{\mathbb{Q}}$ "r'_i and r'_j force contradictory values about $S_{<\alpha} \cap \dot{b}_{\sigma,\delta}$", or $p' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta}$".*

Proof. By induction on $\eta < \lambda$, we build sequences $\langle\langle r^\eta_i \mid i < \eta\rangle \mid \eta < \lambda\rangle$ of conditions in $\mathbb{R}$, such that:

(1) for all $i < \lambda$, $\langle r^\eta_i \mid \eta > i\rangle$ is decreasing,
(2) for all $i < j < \lambda$, there is a maximal antichain A_{ij} in $\mathbb{Q}$ of conditions below p and $\alpha_{ij} < \nu^+$, such that for all $p' \in A_{ij}$, $p' \Vdash_{\mathbb{Q}}$ "r^{j+1}_i, r^{j+1}_j force contradictory values about $\dot{b}_{\sigma,\delta} \cap S_{<\alpha_{ij}}$" or $p' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta}$".

Set $r^1_0 = r_0$. At limit stages, suppose $\rho < \lambda$ is limit and we have defined $\langle\langle r^\eta_i \mid i < \eta\rangle \mid \eta < \rho\rangle$. For $i < \rho$, set r^ρ_i to be stronger than each r^η_i for $i < \eta < \rho$.

For successor stages, suppose we have defined $\langle r^\rho_i \mid i < \rho\rangle$. Define $\langle r^{\rho+1}_i \mid i \leq \rho\rangle$ as follows. Inductively apply the splitting lemma for all $i < \rho$ to get a decreasing sequences of conditions $\langle q_i \mid i < \rho\rangle$ in $\mathbb{R}$ below r_ρ and $r^{\rho+1}_i \leq r^\rho_i$, such that for every $i < \rho$, there is $\alpha_{i\rho} < \nu^+$ and an antichain $A_{i\rho}$ of conditions below p, such that for all $p' \in A_{i\rho}$, $p' \Vdash_{\mathbb{Q}}$ "$r^{\rho+1}_i, q_i$ force contradictory values about $\dot{b}_{\sigma,\delta} \cap S_{<\alpha_{i\rho}}$" or $p' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta}$". Then set $r^{\rho+1}_\rho$ to be stronger than each q_i. This completes the construction.

Finally let r'_i be stronger than each r^η_i for $i < \eta < \lambda$, and $\alpha = \sup_{i<j<\lambda} \alpha_{ij}$.

□

We return to the proof of the theorem. Suppose for contradiction that S has no unbounded branch in $V[E]$. Since $\mathbb{R}$ is $\max(\kappa, \tau)^+$ distributive in $V[E]$, we can find $r \in F$ and $\beta_0 < \nu^+$, such that for all $\langle\sigma, \delta\rangle \in \tau \times \kappa$, either $r \Vdash$ "$\dot{b}_{\sigma,\delta}$ is unbounded", or $r \Vdash$ "$\dot{b}_{\sigma,\delta} \subset S_{<\beta_0}$". Also by (2) from the assumptions of the theorem, by further strengthening r, we can assume: (†) $r \Vdash$ "for all $\alpha \in I$, there is $\langle\sigma, \delta\rangle$ such that

$S_\alpha \cap \dot{b}_{\sigma,\delta} \neq \emptyset$".

Now let $p \in E$ be such that p forces that for all $\langle\sigma, \delta\rangle \in \tau \times \kappa$, either $r \Vdash$ "$\dot{b}_{\sigma,\delta}$ is unbounded", or $r \Vdash$ "$\dot{b}_{\sigma,\delta} \subset S_{\beta_0}$". Suppose also that p forces that $\dot{S}$ has no unbounded branches in $V^{\mathbb{Q}}$. Working in V, we will define a sequence $\langle r_i \mid i < \lambda\rangle$ of conditions stronger than r as follows. For every $\langle \sigma, \delta\rangle \in \tau \times \kappa$ using the last corollary we build conditions $\langle r_i^{\sigma,\delta} \mid i < \lambda\rangle$ stronger than r and $\beta^{\sigma,\delta} < \nu^+$, such that:

(1) for every i, $\langle r_i^{\sigma,\delta} \mid \langle \sigma, \delta\rangle \in \tau \times \kappa\rangle$ is decreasing according to some enumeration of $\tau \times \kappa$,
(2) for every $\langle \sigma, \delta\rangle \in \tau \times \kappa$ and every $i < j < \lambda$, there is a maximal antichain $A_{i,j}^{\sigma,\delta}$ in $\mathbb{Q}$ of conditions below p, such that, for every $p' \in A_{ij}^{\sigma,\delta}$, $p' \Vdash_{\mathbb{Q}}$ "$r_i^{\sigma,\delta}$ and $r_j^{\sigma,\delta}$ force contradictory values about $\dot{b}_{\sigma,\delta} \cap S_{<\beta^{\sigma,\delta}}$" or $p' \Vdash_{\mathbb{Q}}$ "$r \Vdash_{\mathbb{R}} \dot{b}_{\sigma,\delta} \subset S_{<\beta_0}$".

Then for every $i < \lambda$, set r_i to be stronger than all of $\langle r_i^{\sigma,\delta} \mid \langle \sigma, \delta\rangle \in \tau \times \kappa\rangle$.

Work in $V[E]$. Let $\beta \in I$ be such that $\beta > \sup_{\sigma,\delta} \beta^{\sigma,\delta}$. For all $i < \lambda$, let $r_i' \leq r_i$ be such that for some $\xi_i, \sigma_i, \delta_i$,

$$r_i' \Vdash \langle \beta, \xi_i\rangle \in \dot{b}_{\sigma_i,\delta_i}.$$

We can find such r_i' by (†).

Since $\lambda > \max(\tau, \kappa)$, for some $\xi < \kappa$ and $\langle \sigma, \delta\rangle \in \tau \times \kappa$, there are distinct $i < j < \lambda$, such that $\xi_i = \xi_j = \xi$, $\sigma_i = \sigma_j = \sigma$ and $\delta_i = \delta_j = \delta$. Let $p' \in E \cap A_{ij}^{\sigma,\delta}$. Then there is $p'' \leq p'$, such that:

(1) $p'' \Vdash_{\mathbb{Q}}$ "r_i, r_j force contradictory information about $\dot{b}_{\sigma,\delta} \cap S_{<\beta}$" or $p'' \Vdash_{\mathbb{Q}}$ "r forces that $\dot{b}_{\sigma,\delta} \subset S_{<\beta_0}$",
(2) $p'' \Vdash_{\mathbb{Q}} \beta \in \dot{I}, \dot{r}_i' \leq r_i, \dot{r}_j' \leq r_j$",
(3) $p'' \Vdash_{\mathbb{Q}}$ "$\dot{r}_i'$ forces that $\langle \beta, \xi\rangle \in \dot{b}_{\sigma,\delta}$",
(4) $p'' \Vdash_{\mathbb{Q}}$ "$\dot{r}_j'$ forces that $\langle \beta, \xi\rangle \in \dot{b}_{\sigma,\delta}$"

If $p'' \Vdash_{\mathbb{Q}}$ "r forces that $\dot{b}_{\sigma,\delta} \subset S_{<\beta_0}$", then by (2), we get that $p'' \Vdash_{\mathbb{Q}}$ "$\dot{r}_i'$ forces that $\dot{b}_{\sigma,\delta} \subset S_{<\beta_0}$", but this is a contradiction with (3).

On the other hand if $p'' \Vdash_{\mathbb{Q}}$ "r_i, r_j force contradictory information about $\dot{b}_{\sigma,\delta} \cap S_{<\beta}$", then by (2), it follows that $p'' \Vdash_{\mathbb{Q}}$ "$\dot{r}_i', \dot{r}_j'$ force contradictory information about $\dot{b}_{\sigma,\delta} \cap S_{<\beta}$". But since $\dot{b}_{\sigma,\delta}$ is forced to be a branch, then by (3) and (4) we have that p'' must force that $\dot{r}_i'$ and $\dot{r}_j'$ force the same values for $\dot{b}_{\sigma,\delta}$ at levels below β. Contradiction.

□

Remark 12. We can actually get something stronger. Starting from the assumptions of the Preservation Theorem, we can show that for some $\sigma < \tau$ and $\delta < \kappa$, $b_{\sigma,\delta}$ is unbounded, and there is an unbounded branch b in $V[E]$ through R_σ with $b_{\sigma,\delta} \subset b$. To do this, we just modify the assumptions of the Splitting Lemma to state that p forces that either

$r \Vdash \dot{b}_{\sigma,\delta} \subset S_{<\beta}$ or $r \Vdash$ "$\dot{b}_{\sigma,\delta}$ is unbounded and $\dot{S}$ has no unbounded branch b in $V^{\mathbb{Q}}$ through $\dot{R}_\sigma$, such that $\dot{b}_{\sigma,\delta} \subset b$". Then carry over the same assumption to the corollary to the splitting lemma. The rest of the argument is exactly the same. We will use this stronger version in Lemma 16 of next section.

4. The tree property

In this section we will show that in $V[H][E][G]$ the tree property holds at κ^+. Let $\dot{T}$ in $V[H][E]$ be a $\mathbb{P}$ - name for a ν^+ tree with levels of size at most κ, such that this is forced by the empty condition. Denote the α-th level of T by T_α. We may assume that $T_\alpha = \{\alpha\} \times \kappa$ for $\alpha < \nu^+$. We will show that T has a cofinal branch in $V[H][E][G]$. The outline of our proof is motivated by Neeman [10]. The main difference is that we have to deal with the poset $\mathbb{C}$ and rely on the Preservation Lemma from the last section.

Lemma 13. *There is $n < \omega$ and an unbounded $I \subset \nu^+$ in $V[H][E]$, such that for all $\alpha < \beta$ in I, there are $\xi, \delta < \kappa$ and a condition q with length n, such that $q \Vdash \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, \delta \rangle$.*

Proof. Recall that U was the normal measure on $\mathcal{P}_\kappa(\nu^+)$ fixed in advance and each U_n is the projection of U to $\mathcal{P}_\kappa(\kappa_n)$. Let $j = j_U : V[H][E] \to M$. Let G^* be $j(\mathbb{P})$ - generic over M and $T^* = j(\dot{T})_{G^*}$ be such that the first element of the generic sequence added by G^* is κ. We can arrange that since $\kappa \in j(X_0)$. Then $(\nu^+)^V$ remains a regular cardinal in $M[G^*]$.

Fix a node $u \in T^*$ of level γ, where $\sup(j''\nu^+) < \gamma < j(\nu^+)$. Then for all $\alpha < \nu^+$ let $\xi_\alpha < j(\kappa)$ be such that $\langle j(\alpha), \xi_\alpha \rangle <_{T^*} u$ and $p_\alpha \in G^*$ be such that $p_\alpha \Vdash \langle j(\alpha), \xi_\alpha \rangle <_{j(\dot{T})} u$. Then in $M[G^*]$ there is an unbounded $I^* \subset \nu^+$ and a fixed n, such that for all $\alpha \in I^*$, p_α has length n.

Denoting $p_\alpha = \langle d_\alpha, \langle p_{\alpha i} \mid i < \omega \rangle \rangle$, by further shrinking I^* we can assume that for some $d \in Col(\omega, \kappa^{+\omega})$, for each $\alpha \in I^*$, $d_\alpha = d$. Also, for each $\alpha \in I^*$ and $i < n$, denote $p_{\alpha i} = \langle y_i, c_i^\alpha \rangle$. Note that by choice of G^*, we have that $y_0 = \kappa$. Let $b = \langle d, \langle \vec{y}, \vec{c} \rangle \rangle$ be a stem in $j(\mathbb{P})$ with length n such that $\vec{y} = \langle y_i \mid i < n \rangle$ and $\vec{c} = \langle c_i \mid i < n \rangle$ where each $c_i = \bigcup_\alpha c_i^\alpha$. We can take this union since for $0 < i < n$, c_i belongs to a poset which is $< (j(\kappa) \cap y_i)^{+\omega+2}$ closed, and $c_0 \in Col(\kappa^{+\omega+2}, < j(\kappa)_{y_1})$. In particular, the closure is larger than $\nu^+ = \kappa^{+\omega+1}$.

Define $I = \{\alpha < \nu^+ \mid \exists p \in j(\mathbb{P}) \text{ stem}(p) = b, \text{ and } \exists \xi < j(\kappa) p \Vdash \langle j(\alpha), \xi \rangle <_{j(\dot{T})} u\}$. Then $I \in V[H][E]$ and $I^* \subset I$, so I is unbounded. So, I is as desired. $\square$

Remark 14. Note that for any condition p, we can strengthen Lemma 13 to get an unbounded set I and $n < \omega$, such that for all $\alpha < \beta$ in I, there are $\xi, \delta < \kappa$ and a condition $q \leq p$ with length n, such that $q \Vdash \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, \delta \rangle$. To do this, if $k = lh(p)$, we choose G^*, so that the k^{th} element of the generic sequence added by G^* below $j(\kappa)$ is κ and below that we take the stem of p. More precisely, if $\langle x_i^* \mid i < \omega \rangle$ is the G^*-generic sequence, we arrange so that $j(\kappa) \cap x_k^* = \kappa$, and for $i < k$, $x_i^* = j(x_i^p) = j\text{"}x_i^p$.

Let $\bar{n}$ and I be as in the conclusion of the above lemma. We will say that a stem $h \Vdash^* \phi$ if there is a condition p, such that the stem of p is h and $p \Vdash \phi$. Since any two conditions with the same stem are compatible, we have that if $h \Vdash^* \phi$, then $h \nVdash^* \neg\phi$

Lemma 15. *There is, in $V[H][E]$, an unbounded set $J \subset \nu^+$, a stem h of length $\bar{n}$, and a sequence of nodes $\langle u_\alpha \mid \alpha \in J \rangle$ with every u_α of level α, such that for all $\alpha < \beta$ in J there is a condition p with stem h, such that $p \Vdash u_\alpha <_{\dot{T}} u_\beta$.*

Proof. Let $j : V \to N$ be a ν^+ - supercompact embedding with critical point $\kappa_{\bar{n}+2}$. Using standard arguments, extend j to $j' : V[H] \to N[H^*]$, where H^* is $j(\mathbb{C})$ generic over V. We can arrange so that $H^* = H * E * H'$, where H' is $\mathbb{C}'$- generic over $V[H][E]$ (see [8]). We have that $\mathbb{C}'$ is $< \kappa_{\bar{n}+1}$ distributive in $V[H][E]$, and it is $< \kappa_{\bar{n}+1}$ closed in $V[H]$. Let F be $Add(\kappa, j(\nu^{++}))$ generic over $V[H][E]$. Then F is also generic over $V[H][E][H']$, since $Add(\kappa, j(\nu^{++}))$ has the κ^+ chain condition. Define $E^* = \{f : j(\nu^{++}) \times \kappa \to \kappa \mid |f| < \kappa, f \restriction (j\text{"}\nu^{++}) \in j'\text{"}E, f \restriction (j(\nu^{++}) \setminus j\text{"}\nu^{++}) \in F\}$. Then E^* is $Add(\kappa, j(\nu^{++}))$ generic over $V[H^*] = V[H][E][H']$, such that $j'\text{"}E \subset E^*$. It follows that we can extend j to $j^* : V[H][E] \to N[H^*][E^*]$ where $j^* \in V[H][E][F][H'] = V[H][E][H'][F]$.

Let $\gamma \in j^*(I)$ be such that $\sup(j''\nu^+) < \gamma < j(\nu^+)$. By elementarity for all $\alpha \in I$ we can fix $\xi_\alpha, \delta_\alpha < \kappa$ and $p_\alpha \in j^*(\mathbb{P})$ with length $\bar{n}$ such that $p_\alpha \Vdash_{j^*(\mathbb{P})} \langle j^*(\alpha), \xi_\alpha \rangle <_{j^*(\dot{T})} \langle \gamma, \delta_\alpha \rangle$. Let h_α be the stem of p_α. Note that the function $\alpha \mapsto \langle \xi_\alpha, \delta_\alpha, h_\alpha \rangle$ is in $V[H][E][F][H']$.

First we will use the Preservation Lemma to show that J as in the statement of the lemma exists in $V[H][E][F]$. Then we will use the arguments in Neeman [10] to get $J \in V[H][E]$.

The number of possible stems in $j^*(\mathbb{P})$ of length $\bar{n}$ is less than $\kappa_{\bar{n}}$. Then since ν^+ is regular in $V[H][E][F]$ and $\mathbb{C}'$ does not add sequences of length less than $\kappa_{\bar{n}}$, we have that in $V[H][E][F][H']$ there is a cofinal $J \subset I$, $\delta < \kappa$, and a stem h such that for all $\alpha \in J$, $\delta_\alpha = \delta$ and $h_\alpha = h$.

We consider the narrow system $S = \langle I, \mathcal{R}\rangle$ of height ν^+ and levels of size κ, in $V[H][E]$ (and so in $V[H][E][F]$), where:

- $\mathcal{R} = \langle R_h \mid h \text{ is a stem of length } \bar{n}\rangle$; $|\mathcal{R}| < \kappa_{\bar{n}}$.
- For nodes a, b, we say that $\langle a, b\rangle \in R_h$ iff $h \Vdash^* a <_{\dot{T}} b$

Note that $\{h \mid h \text{ is a stem of length } \bar{n}\} \in V[H]$. Also each R_h is transitive since conditions with the same stem are compatible and $\dot{T}$ is forced to be a tree by the empty condition. Apply the preservation lemma to S for $V_1 = V[H]$, $\mathbb{Q} = Add(\kappa, j(\nu^{++}))$, and $\mathbb{R} = \mathbb{C}'$, and the branches:

$$b_{h,\delta} =_{def} \{\langle \alpha, \xi\rangle \mid h \Vdash^*_{j^*(\mathbb{P})} \langle j(\alpha), \xi\rangle <_{j^*(\dot{T})} \langle \gamma, \delta\rangle\}.$$

Note that the preservation lemma works for any $\mathcal{R} = \langle R_\sigma \mid \sigma \in L\rangle$ with the index set L in the ground model for the lemma, which in this case is V_1.

We get that S has an unbounded branch in $V[H][E][F]$. I.e. in $V[H][E][F]$, there are an unbounded $J \subset I$, $\alpha \mapsto \xi_\alpha$ and a stem h such that for all $\alpha, \beta \in J$ with $\alpha < \beta$, we have that $h \Vdash^* \langle \alpha, \xi_\alpha\rangle <_{\dot{T}} \langle \beta, \xi_\beta\rangle$.

Then by the argument in Lemma 3.2 from [10], we can get such $J, h, \alpha \mapsto \xi_\alpha$ in $V[H][E]$. Setting $u_\alpha = \langle \alpha, \xi_\alpha\rangle$ for $\alpha \in J$, we get that for all $\alpha < \beta$ in J there is a condition p with stem h which forces that $u_\alpha <_{\dot{T}} u_\beta$. $\square$

Fix $\bar{n}$, $\bar{h}$, J, and $\alpha \mapsto u_\alpha$ as in the conclusion of the above lemma. By shrinking J we may assume that for some $\xi < \kappa$, each $u_\alpha = \langle \alpha, \xi\rangle$.

Lemma 16. *Suppose that h is a stem of length k, $L \subset \nu^+$ is unbounded, and for all $\alpha < \beta$ with $\alpha, \beta \in L$, $h \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. Then there are $\rho < \nu^+$ and sets $\langle A_\alpha, C_\alpha : \alpha \in L \setminus \rho\rangle$ in $V[H][E]$ such that:*

1. *Each $A_\alpha \in U_k$, $\mathrm{dom}(C_\alpha) = A_\alpha$, $C_\alpha(x) \in Col(\kappa_x^{+\omega+2}, < \kappa)$ for $x \in A_\alpha$, and $[C_\alpha]_{U_k} \in K_k$.*
2. *For all $\alpha < \beta$ in $L \setminus \rho$, for all $x \in A_\alpha \cap A_\beta$ such that $C_\alpha(x)$ and $C_\beta(x)$ are compatible,*

$$h^\frown\langle x, C_\alpha(x) \cup C_\beta(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

Proof. Let $j : V \to N$ be a ν^+ - supercompact embedding with critical point κ_{k+3}. As in the previous lemma, extend j to $j' : V[H] \to N[H^*]$, where $H^* = H * E * H'$ is $j(\mathbb{C})$ generic over V, and H' is $\mathbb{C}'$- generic over $V[H][E]$. We have that $\mathbb{C}'$ is $< \kappa_{k+2}$ distributive in $V[H][E]$ and $< \kappa_{k+2}$ closed in $V[H]$. Let F be $Add(\kappa, j(\nu^{++}))$ generic over $V[H][E]$. Then F is also generic over $V[H][E][H']$, and so $E^* =_{def} \{f : j(\nu^{++}) \times \kappa \to \kappa \mid |f| < \kappa, f \restriction (j"\nu^{++}) \in j'" E, f \restriction (j(\nu^{++}) \setminus j"\nu^{++}) \in F\}$ is $Add(\kappa, j(\nu^{++}))$ generic over $V[H^*] = V[H][E][H']$, such that $j'" E \subset$

E^*. It follows that we can extend j to $j^* : V[H][E] \to N[H^*][E^*]$ where $j^* \in V[H][E][F][H'] = V[H][E][H'][F]$.

First we will adapt the arguments in Neeman [10] to include the collapses and use them to show that there are $\rho < \nu^+$, $\alpha \mapsto A^*_\alpha$ and $\alpha \mapsto C^*_\alpha$ as in the statement of the lemma in $V[H][E][H']$. Then we will use the Preservation Lemma to show that we can find these objects in $V[H][E]$.

For all $\alpha < \beta$ in L, by the assumptions of the lemma, we can fix (in $V[H][E]$) $[C_{\alpha,\beta}]_{U_k} \in K_k$, such that for almost all $x \in \mathcal{P}_\kappa(\kappa_k)$, $h^\frown\langle x, C_{\alpha\beta}(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. Since K_k is closed under sequences of length $\kappa^{+\omega+1}$, let $[C^*]_{U_k} \in K_k$ be stronger than $[C_{\alpha,\beta}]_{U_k}$ for all $\alpha < \beta$ in L. Then for all $\alpha < \beta$ in L, there is $r \in \mathbb{P}$ with stem h, such that $r \Vdash u_\alpha <_{\dot{T}} u_\beta$, and denoting $r_k = \langle A^r_k, C^r_k\rangle$, we have that $[C^r_k]_{U_k} = [C^*]_{U_k}$. Without loss of generality we can assume that $\mathrm{dom}(C^*) = \mathcal{P}_\kappa(\kappa_k)$.

Claim 17. *In $V[H][E][H']$, there are $\rho < \nu^+$, $\alpha \mapsto A^*_\alpha$ and $\alpha \mapsto C^*_\alpha$ that satisfy the conclusion of the lemma.*

Proof. The argument follows closely Lemma 3.5 in [10], so we only outline the main points. The difference here is that we are also dealing with collapses. Let $\gamma \in j^*(L)$ be such that $j``\nu^+ < \gamma < j(\nu^+)$. We write u_γ for the γ^{th} member of the sequence $j^*(\langle u_\alpha : \alpha \in L\rangle)$. By elementarity of j^*, we can find conditions $\langle r_\alpha : \alpha \in L\rangle$ in $N[H^*][E^*]$, such that each $r_\alpha \in j^*(\mathbb{P})$, the stem of r_α is h and

$$r_\alpha \Vdash_{j^*(\mathbb{P})} j(u_\alpha) <_{j^*(\dot{T})} u_\gamma.$$

We may assume that for each α, $[C^{r_\alpha}_k]_{j^*(U_k)} = j^*([C^*]_{U_k}) = [C^*]_{j^*(U_k)}$. For $x \in \mathcal{P}_\kappa(\kappa_k)$, set $J_x = \{\alpha \in L \mid h^\frown\langle x, C^*(x)\rangle \Vdash^* j(u_\alpha) <_{j^*(\dot{T})} u_\gamma\}$. Also, let $K_x = \{C \subset L \mid C \text{ is unbounded and } (\exists b \in Add(\kappa, j(\nu^{++})))(b \Vdash \dot{J}_x = C)\}$. Then each K_x and $x \mapsto K_x$ is in $V[H][E][H']$. We have that (see [10]):

- If J_x is unbounded in ν^+, then $J_x \in V[H][E][H']$.
- If $C_1 \neq C_2$ are both in K_x, then they are disjoint on a tail end.
- For all $C \in K_x$, if $\alpha < \beta$ in C, then $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.

Then let $\rho < \nu^+$ be such that for all x and $C_1, C_2 \in K_x$, C_1 and C_2 are disjoint above ρ. For $\alpha \in L \setminus \rho$ and $x \in \mathcal{P}_\kappa(\kappa_k)$ define $f(x, \alpha)$ to be the unique $C \in K_x$ such that $\alpha \in C$ if such a C exists and undefined otherwise. Note that if J_x is unbounded in ν^+ and $\alpha \in J_x$, then $f(x, \alpha) = J_x$. Let $\alpha_0 = \min(L \setminus \rho)$ and for $\alpha \in L \setminus \rho$, set:

$$A^*_\alpha = \{x \in \mathcal{P}_\kappa(\kappa_k) \mid f(x, \alpha_0) = f(x, \alpha)\}.$$

By the arguments in [10] and since for each α, $[C_k^{r_\alpha}] = [C^*]$, each $A^*_\alpha \in U_k$. We remark that here when adapting the argument from [10], we use that H' adds no sequences of length κ_k and that ν^+ is regular in $V[H][E]$. Set $C^*_\alpha = C^* \restriction A^*_\alpha$. Then ρ, $\alpha \mapsto A^*_\alpha$ and $\alpha \mapsto C^*_\alpha$ are as desired.

□

Fix $\rho < \nu^+$, $\alpha \mapsto A^*_\alpha$ and $\alpha \mapsto C^*_\alpha$ as in the above claim. Note that for each α, $C^*_\alpha = C^* \restriction A^*_\alpha$. Let $L' = L \setminus \rho$. For every $x \in \mathcal{P}_\kappa(\kappa_k)$, let $b_x = \{\alpha \in L \setminus \rho \mid x \in A^*_\alpha\}$. Note that each $b_x \in V[H][E][H']$ and for all $\alpha < \beta$ in b_x, $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.

Claim 18. *For every $A \in U_k$, there is $x \in A$, such that b_x is unbounded and there is an unbounded set $b \subset \nu^+$ in $V[H][E]$, such that:*

- *$b_x \subset b$,*
- *for all $\alpha < \beta$ in b, $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.*

Proof. Let $\mathcal{R} = \{R_x \mid x \in \mathcal{P}_\kappa(\kappa_k)\}$; $|\mathcal{R}| = \kappa_k$. We consider the narrow system $S = \langle L', \mathcal{R}\rangle$, where for $\alpha \in L'$, $S_\alpha = \{\alpha\} \times \kappa$ and for nodes a, b, we say that $\langle a, b\rangle \in R_x$ iff:

$$h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$$

Each R_x is a transitive relation since $\dot{T}$ is forced to be a tree by the empty condition and any two conditions with the same stem are compatible. For $x \in \mathcal{P}_\kappa(\kappa_k)$, set $b^*_x = \{u_\alpha \mid \alpha \in b_x\}$ if $x \in A$, and $b^*_x = \emptyset$ otherwise. We have that $\{b^*_x \mid x \in \mathcal{P}_\kappa(\kappa_k)\}$ are branches in $V[H][E][H']$, such that:

1. each b^*_x is a branch through R_x.
2. for all $\alpha \in L'$, there is x, such that $b^*_x \cap S_\alpha \neq \emptyset$ (just take any $x \in A^*_\alpha \cap A$),
3. for some x, b^*_x is unbounded.

(3) holds because otherwise, for all x, let α_x be a bound for b_x (in $V[H][E][H']$) and let $\alpha = \sup_{x \in \mathcal{P}_\kappa(\kappa_k)} \alpha_x$. Since H' does not add sequences of length κ_{k+1} and ν^+ is regular in $V[H][E]$, we have that $\alpha < \nu^+$; contradiction.

We apply the preservation lemma to S for $V_1 = V[H]$, $\mathbb{Q} = Add(\kappa, \nu^{++})$, $\mathbb{R} = \mathbb{C}'$, and these branches to get an unbounded branch through S in $V[H][E]$. So, in $V[H][E]$ we have an unbounded $b \subset L$ and $x \in \mathcal{P}_\kappa(\kappa_k)$, such that $b_x \subset b$ and b_x is unbounded. Here we use the stronger version of the preservation lemma, see Remark 12. Since b_x is unbounded, we know that $x \in A$. So, since the preservation lemma produces a branch through R_x, we have that for all $\alpha < \beta$ in b, $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. □

For every $x \in \mathcal{P}_\kappa(\kappa_k)$, let $\dagger_x$ be the statement: b_x is unbounded, and there is an unbounded set $b \subset \nu^+$ in $V[H][E]$ with $b_x \subset b$, such that for all $\alpha < \beta$ in b, $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.

Corollary 19. *$A =_{def} \{x \in \mathcal{P}_\kappa(\kappa_k) \mid \dagger_x$ holds $\} \in V[H][E]$ and $A \in U_k$.*

Proof. A is in $V[H][E]$ since $\mathbb{C}'$ is distributive enough. Now suppose for contradiction that A is not measure one. Then $Y = \{x \mid \dagger_x$ does not hold $\} \in U_k$. Apply the above claim to Y to get a contradiction. □

For every $x \in A$, let $L_x \subset L$ witness $\dagger_x$. I.e. $L_x \in V[H][E]$, it is unbounded in ν^+, $b_x \subset L_x$, b_x is unbounded, and for all $\alpha < \beta$ in L_x, we have that $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. Note that by distributivity of $\mathbb{C}'$, we have that $x \mapsto L_x$ is in $V[H][E]$. Finally, for $\alpha \in L'$, define:

$$A_\alpha = \{x \in A \mid \alpha \in L_x\}.$$

Claim 20. *For all $\alpha \in L'$, $A_\alpha \in U_k$*

Proof. Suppose otherwise. Then $Y = \mathcal{P}_\kappa(\kappa_k) \setminus A_\alpha \in U_k$. Let $x \in Y \cap A^*_\alpha \cap A$. Then $\dagger_x$ holds, and $\alpha \in b_x \subset L_x$. But we assumed that $x \notin A_\alpha$, contradiction □

For $\alpha \in L'$, define $C_\alpha = C^* \restriction A_\alpha$. Now suppose $\alpha < \beta$ in L', $x \in A_\alpha \cap A_\beta$ and $c \leq C_\alpha(x) \cup C_\beta(x) = C^*(x)$. Then $\alpha, \beta \in L_x$, and so $h^\frown\langle x, C^*(x)\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. So, $h^\frown\langle x, c\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.

□

Lemma 21. *There is some $\rho < \nu^+$ and a sequence of conditions $\langle p_\alpha \mid \alpha \in J \setminus \rho\rangle$ with stem $\bar{h}$ such that for $\alpha < \beta$ in $J \setminus \rho$, $p_\alpha \wedge p_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$. (Recall that $p_\alpha \wedge p_\beta$ denotes the weakest common extension of p_α and p_β.)*

Proof. The proof is an adaptation of the argument given in [10].

First we make some remarks on taking diagonal intersections. Let H be a set of stems of length n, and let $\langle A^h \mid h \in H\rangle$ be a sequence of U_n- measure one sets. For a stem $h = \langle d, \langle \vec{y}, \vec{c}\rangle\rangle$ in H and $z \in \mathcal{P}_\kappa(\kappa_n)$, we write $h \prec z$ to denote that $y_{n-1} \prec z$, i.e. that $|y_{n-1}| < \kappa_z$ and $y_{n-1} \subset z$. Note that $h \prec z$ iff for some c, $h^\frown\langle z, c\rangle$ is a stem. Then $A = \triangle_{h\in H} A^h = \{z \in \mathcal{P}_\kappa(\kappa_n) \mid z \in \bigcap_{h\prec z} A^h\}$ is the diagonal intersection of $\langle A^h \mid h \in H\rangle$ and $A \in U_n$.

We will define sequences $\langle \rho_n \mid \bar{n} \leq n < \omega\rangle$, $\langle A^n_\alpha \mid \alpha \in J \setminus \rho_n, \bar{n} \leq n < \omega\rangle$, and $\langle C^n_\alpha \mid \alpha \in J \setminus \rho_n, \bar{n} \leq n < \omega\rangle$ by induction on n, such that for all n:

(1) For all $\alpha \in J \setminus \rho_n$, we have that $A^n_\alpha \in U_n$, $[C^n_\alpha]_{U_n} \in K_n$, $\mathrm{dom}(C^n_\alpha) = A_\alpha$, and $C^n_\alpha(x) \in Col(\kappa_x^{+\omega+2}, < \kappa)$ for $x \in A^n_\alpha$.

(2) For all $\alpha < \beta$ in $J \setminus \rho_n$, for all stems $h = \langle d, \langle \vec{x}, \vec{c} \rangle \rangle$ of length $n+1$ extending $\bar{h}$, such that for $\bar{n} \leq i \leq n$, $x_i \in A^i_\alpha \cap A^i_\beta$ and $c_i \leq C^i_\alpha(x_i) \cup C^i_\beta(x_i)$,
$$h \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

Let $\rho_{\bar{n}}$ and $\langle A^{\bar{n}}_\alpha, C^{\bar{n}}_\alpha \mid \alpha \in J \setminus \rho_{\bar{n}} \rangle$ be given by the above lemma applied to $\bar{h}$. Then by the conclusion of the lemma, both of the conditions above for $\bar{n}$ are satisfied. Now suppose we have defined ρ_n and $\langle A^n_\alpha, C^n_\alpha \mid \alpha \in J \setminus \rho_n \rangle$ such that (1) and (2) above hold for n. We have to define ρ_{n+1} and A^{n+1}_α, C^{n+1}_α for $\alpha \in J \setminus \rho_{n+1}$.

For a stem $h = \langle d^h, \langle \vec{x}, \vec{c} \rangle \rangle$ of length $n+1$ extending $\bar{h}$, we say that h fits α iff each $x_i \in A^i_\alpha$ and $c_i \leq C^i_\alpha(x_i)$ for $i \leq n$. Set
$$J^h = \{\alpha \in J \setminus \rho_n \mid h \, fits \, \alpha\}.$$

Define a function $h \mapsto \rho^h$ on stems of length $n+1$ extending $\bar{h}$ as follows:

- if J^h is bounded in ν^+, let $\rho^h < \nu^+$ be a bound,
- otherwise, let ρ^h and $\langle A^h_\alpha, C^h_\alpha \mid \alpha \in J^h \setminus \rho^h \rangle$ be given by the previous lemma applied to h and J^h (here we use the inductive assumption for n).

Set $\rho_{n+1} = \sup\{\rho^h \mid h \text{ is a stem of length } n+1 \text{ extending } \bar{h}\}$. For $\alpha \in J \setminus \rho_{n+1}$, set $H_\alpha(n+1) = \{h \mid h \text{ has length } n+1, \text{ extends } \bar{h}, \text{ and fits } \alpha\}$. For each $\alpha \in J \setminus \rho_{n+1}$, let
$$A^{n+1}_\alpha = \triangle_{h \in H_\alpha(n+1)} A^h_\alpha.$$

Also set $[C_\alpha]_{U_{n+1}} = \bigcup\{[C^h_\alpha]_{U_{n+1}} \mid h \in H_\alpha(n+1)\} \in K_{n+1}$. By shrinking A^{n+1}_α (via diagonal intersections), we can arrange that for all $x \in A^{n+1}_\alpha$, $C_\alpha(x) = \bigcup\{C^h_\alpha(x) \mid h \prec x\}$.

It remains to check that (1) and (2) hold for $n+1$. The first condition holds by construction. For the second condition, we have to show that for all $\alpha < \beta$ in $J \setminus \rho_{n+1}$, for all stems t of length $n+2$, which extend $\bar{h}$ and fit both α and β, we have that
$$t \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

So, suppose that $\alpha < \beta$ are in $J \setminus \rho_{n+1}$ and $h^\frown \langle x, c \rangle$ is a stem as above. Since h fits α and β and $\rho^h < \alpha, \beta$, we have that J^h is unbounded and ρ^h was given by applying the previous lemma. Then $h \in H_\alpha(n+1) \cap H_\beta(n+1)$ and since we took diagonal intersections, it follows that $x \in A^h_\alpha \cap A^h_\beta$ and $c \leq C^h_\alpha(x) \cup C^h_\beta(x)$. So, by construction of

$A^h_\alpha, A^h_\beta, C^h_\alpha, C^h_\beta$, we have that $h^\frown\langle x, c\rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. This completes the construction.

Now let $\rho = \sup_n \rho_n$ and set p_α for $\alpha \in J \setminus \rho$ to be such that:

- the stem of p_α is $\bar{h}$,
- for all $n \geq \bar{n}$, let $p_{\alpha n} = \langle A^n_\alpha, C^n_\alpha\rangle$.

Then $\langle p_\alpha \mid \alpha \in J \setminus \rho\rangle$ is as desired. For if $\alpha < \beta$ are in $J \setminus \rho$, and $q \leq p_\alpha \wedge p_\beta$, then by the construction, we have that $\mathrm{stem}(q) \Vdash^* u_\alpha <_{\dot{T}} u_\beta$, and so $q \nVdash u_\alpha \not<_{\dot{T}} u_\beta$. It follows that $p_\alpha \wedge p_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$. $\square$

Lastly, we show that $\{u_\alpha \mid \alpha < \nu^+, p_\alpha \in G\}$ is an unbounded branch of T. It suffices to prove the following:

Proposition 22. *$B = \{\alpha < \nu^+ \mid p_\alpha \in G\}$ is unbounded.*

Proof. Otherwise, let $q \in G$ be such that $q \Vdash \dot{B}$ is bounded. Since both Lemma 13 and Lemma 15 can be done below any condition, we may assume that (by strengthening q if necessary) $\mathrm{stem}(q) = \bar{h}$. $\mathbb{P}$ has the ν^+ chain condition, so for some $\alpha < \nu^+$, $q \Vdash \dot{B} \subset \alpha$. Let $\beta \in J \setminus \alpha$, and let r be a common extension of q and p_β. Then on one hand we have that $r \Vdash p_\beta \in \dot{G}$, but also $r \Vdash u_\beta \notin \dot{B}$. Contradiction.

$\square$

Then $\{u_\alpha \mid \alpha \in B\}$ is an unbounded branch of T. This completes the proof of the tree property. To summarize, starting from ω many supercompact cardinals, we have constructed a generic extension in which there are no Aronszajn trees at $\aleph_{\omega^2+1}$ and the SCH fails at $\aleph_{\omega^2}$.

We conclude this paper with an open problem. It is still not known whether it is consistent to have the tree property at $\aleph_{\omega+1}$ and not SCH at $\aleph_\omega$.

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