THE TREE PROPERTY AND THE FAILURE OF SCH AT UNCOUNTABLE COFINALITY

DIMA SINAPOVA

Abstract. Given a regular cardinal λ and λ many supercompact cardinals, we describe a type of forcing such that in the generic extension there is a cardinal κ with cofinality λ, the Singular Cardinal Hypothesis at κ fails, and the tree property holds at κ^+.

1. Introduction

The relationship between the singular cardinal hypothesis (SCH), square principles, the tree property and large cardinals is important in singular cardinal arithmetic. The tree property at κ^+ states that there are no Aronszajn trees at κ^+ i.e. that every κ^+-tree has an unbounded branch. Recently an old question was answered by Neeman [5] in the negative: whether failure of SCH implies the existence of an Aronszajn tree. Previously the only known way to establish the tree property at a successor of a singular cardinal was due to Magidor-Shelah [4].

The result in Neeman [5] that the failure of SCH is consistent with the tree property was obtained at a cardinal of cofinality ω. Here we show that the failure of SCH is consistent with the tree property for cardinals of arbitrary cofinality.

Theorem 1. *Suppose that λ is a regular uncountable cardinal, $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ is a continuous sequence such that $\lambda < \kappa_0$, κ_0 and each $\kappa_{\alpha+1}$ are supercompact cardinals and let $\nu = \sup_\alpha \kappa_\alpha$. Then there is a generic extension in which:*

1. *$\kappa = \kappa_0$ is preserved and has cofinality λ,*
2. *the tree property holds at κ^+ and SCH fails at κ,*
3. *there is a very good scale and a bad scale at κ.*

The rest of the paper presents the proof of Theorem 1. In section 2 we define the forcing notion and give some basic properties about the forcing. The forcing that we will use combines ideas from Neeman [5] and Sinapova [6]. Also, we describe the very good scale and the bad scale in the generic extension. Scales are a central concept in PCF theory. The relationship between scales, the SCH, and square principles and singular arithmentic has been explored by Gitik, Cummings, Foreman, Magidor among others. In 2008 Gitik-Sharon [2] showed two important consistency results about scales: that failure of SCH does not imply weak square, and the existence of a very good scale does not imply weak square. The result was generalized by Sinapova [6] for singular cardinals of arbitrary cofinality. The Gitik-Sharon model provided much of the motivation behind the construction in Neeman [5]. Finally in section 3 we prove that the tree property holds. Both in the ω cofinality case and the uncountable cofinality case it remains open whether the result can be pushed down to small cardinals.

2. The construction

Let $\langle \kappa_\xi \mid \xi < \lambda \rangle$ be a continuous increasing sequence, such that each $\kappa_{\xi+1}$ is supercompact and $\lambda < \kappa_0$. Denote $\kappa = \kappa_0$. Let $\nu = \sup_{\xi<\lambda} \kappa_\xi$ and $\mu = \nu^+$. Using Laver's preparation, we may assume that the supercompactness of κ is indestructible under κ-directed closed forcing [3]. Let E be $Add(\kappa, \nu^{++})$ generic over V. Work in $V[E]$.

Proposition 2. *There is a sequence $\langle U_\xi \mid \xi < \lambda \rangle$, where each U_ξ is a normal measure on $\mathcal{P}_\kappa(\kappa_\xi)$, κ is κ_ξ-supercompact in $Ult(V[E], U_\xi)$, and the sequence is Mitchell order increasing, i.e. for $\xi < \eta < \lambda$, $U_\xi \in Ult(V[E], U_\eta)$, and there are functions $\langle f^\xi_\gamma \mid \gamma < \mu, \xi < \lambda \rangle$ from κ to κ, such that for all $\xi < \lambda, \gamma < \mu$, $j_{U_\xi}(f^\xi_\gamma)(\kappa) = \gamma$.*

Proof. We use the following claim:

Claim 3. *For all $\xi < \lambda$, for all $\mathcal{X} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa_\xi))$, there is a normal measure U_ξ on $\mathcal{P}_\kappa(\kappa_\xi)$, such that $\mathcal{X} \in Ult(V[E], U_\xi)$, and there are functions $\langle f_\gamma \mid \gamma < \mu \rangle$ from κ to κ, such that for all $\gamma < \mu$, $j_{U_\xi}(f_\gamma)(\kappa) = \gamma$.*

Proof. The proof of this claim adapts an argument due to Solovay, Reinhardt, and Kanamori [7]. For details, see Lemma 2 in [6]. □

Now, define the chain as follows. Suppose that we already have $\langle U_\eta \mid \eta < \xi \rangle$ and $\langle f^\eta_\gamma \mid \gamma < \mu, \eta < \xi \rangle$ as desired. Let $\bar{U}$ be a normal measure on $\mathcal{P}_\kappa(\kappa_\xi)$. We can code $\bar{U}$ and the U_η's by some $\mathcal{Y} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa_\xi))$. Apply the claim to find a normal measure U_ξ on $\mathcal{P}_\kappa(\kappa_\xi)$ with $\mathcal{Y} \in Ult(V[E], U_\xi)$ and functions $\langle f^\xi_\gamma \mid \gamma < \mu \rangle$ from κ to κ with $j_{U_\xi}(f^\xi_\gamma)(\kappa) = \gamma$ for each γ. Then $\bar{U} \in Ult(V[E], U_\xi)$, and so κ is κ_ξ-supercompact in $Ult(V[E], U_\xi)$.

□

Fix measures U_ξ, for $\xi < \lambda$ and functions $\langle f^\xi_\gamma \mid \gamma < \mu, \xi < \lambda \rangle$ as in the statement of the last proposition. For $\xi < \lambda$, let X_ξ be the set of $x \in \mathcal{P}_\kappa(\kappa_\xi)$ such that

(1) $x \cap \kappa =_{def} \kappa_x$ is an ordinal, $\lambda < \kappa_x$, and κ_x is $f^\xi_{\kappa_\xi}(\kappa_x)$ -supercompact,
(2) $(\forall \eta \leq \xi) o.t.(x \cap \kappa_\eta) = f^\xi_{\kappa_\eta}(\kappa_x)$,
(3) $(\forall \eta \leq \xi)(f^\xi_{\kappa_\eta}(\kappa_x)^{<\kappa_x} \leq f^\xi_{\kappa^+_\eta}(\kappa_x))$.

By standard reflection arguments $X_\xi \in U_\xi$. Here clause (3) is due to reflection of $\kappa_\eta^{<\kappa} = \kappa_\eta$ when η is a successor, and $\kappa_\eta^{<\kappa} = \kappa_\eta^+$ for η limit. Note that for $\eta < \eta' < \lambda$, if $x \in X_{\eta'}$, then $x \cap \kappa_\eta \in X_\eta$.

For $\xi < \eta < \lambda$, for $x \in X_\eta$, and $Y \subset \mathcal{P}_{\kappa_x}(\kappa_\xi \cap x)$, define $\overline{Y} \subset \mathcal{P}_{\kappa_x}(o.t.(x \cap \kappa_\xi))$, by $\overline{Y} = \{\{o.t.(\delta \cap x) \mid \delta \in y\} \mid y \in Y\}$. Since $U_\xi \in Ult(V[E], U_\eta)$, there is a function $x \mapsto \overline{U^\xi_{\eta,x}}$, such that $U_\xi = [x \mapsto \overline{U^\xi_{\eta,x}}]_{U_\eta}$ and each $\overline{U^\xi_{\eta,x}}$ is a normal measure on $\mathcal{P}_{\kappa_x}(f^\eta_{\kappa_\xi}(\kappa_x)) = \mathcal{P}_{\kappa_x}(o.t.(x \cap \kappa_\xi))$. We lift this measure to a normal measure $U^\xi_{\eta,x}$ on $\mathcal{P}_{\kappa_x}(\kappa_\xi \cap x)$. In particular, $\overline{U^\xi_{\eta,x}} = \{\overline{Y} \subset \mathcal{P}_{\kappa_x}(f^\eta_{\kappa_\xi}(\kappa_x)) \mid Y \in U^\xi_{\eta,x}\}$. Note that each $U^\xi_{\eta,x}$ is $\lambda-$ complete. For $\theta < \lambda$, let $B_\theta = \{z \in X_\theta \mid (\forall \xi, \eta)\xi < \eta < \theta \to \overline{U^\xi_{\theta,z}} = [x \mapsto \overline{U^\xi_{\eta,x}}]_{U^\eta_{\theta,z}}\}$. Adapting the arguments in [6] we have that each $B_\theta \in U_\theta$.

We are ready to define the main forcing $\mathbb{P}$. Conditions are of the form $p = \langle g, H\rangle$, where:

(1) $\mathrm{dom}(g)$ is a finite subset of λ and $\mathrm{dom}(H) = \lambda \setminus \mathrm{dom}(g)$,
(2) for each $\xi \in \mathrm{dom}(g)$, $g(\xi) \in B_\xi$,
(3) for $\xi < \eta$, in $\mathrm{dom}(g)$, we have $g(\xi) \prec g(\eta)$ i.e. $g(\xi) \subset g(\eta)$ and $o.t.(g(\xi)) < \kappa_{g(\eta)} = \kappa \cap g(\eta)$,
(4) for $\xi \notin \mathrm{dom}(g)$ and $\xi > \max(\mathrm{dom}(g))$, we have $H(\xi) \in U_\xi$, $H(\xi) \subset B_\xi$,
(5) for $\xi \notin \mathrm{dom}(g)$ and $\xi < \max(\mathrm{dom}(g))$, setting $\eta = \min(\mathrm{dom}(g) \setminus \xi)$, we have $H(\xi) \in U^\xi_{\eta,g(\eta)}$ (the normal measure on $\mathcal{P}_{\kappa_{g(\eta)}}(\kappa_\xi \cap g(\eta))$),
(6) for $\xi < \eta$, if $\xi \in \mathrm{dom}(g)$, $\eta \notin \mathrm{dom}(g)$, then for each $z \in H(\eta)$, $g(\xi) \prec z$.

We say that g is the *stem* of p.

$\langle g, H\rangle \leq \langle j, J\rangle$ iff

(1) $g \supset j$,
(2) for $\xi \in \mathrm{dom}(g) \setminus \mathrm{dom}(j)$, $g(\xi) \in J(\xi)$,
(3) for $\xi \notin \mathrm{dom}(g)$, we have $H(\xi) \subset J(\xi)$.

If $q \leq p$ and both conditions have the same stem, we say that q is a direct extension of p and write $q \leq^* p$.

$\mathbb{P}$ is a combination of the forcing notions from [5] and [6]. Adapting the arguments in [6] we get:

Proposition 4. *(Properties of the forcing notion)*

(1) *$\mathbb{P}$ has the ν^+ chain condition (since any two conditions with the same stem are compatible and the number of possible stems is ν).*
(2) *$\mathbb{P}$ satisfies the Prikry property, i.e. if ϕ is a formula and p is a condition, then there is $q \leq^* p$, such that q decides ϕ.*
(3) *A corollary to the above is that if ϕ is a formula, $p = \langle g, H\rangle$ is a condition and $\alpha \in \mathrm{dom}(g)$, then there is $q \leq^* p$, such that if $r \leq p$ decides ϕ, then $r \restriction \alpha^\frown q \restriction (\lambda \setminus \alpha)$ decides ϕ. Here we use that $\mathbb{P}^p_\alpha = \{r \restriction \alpha \mid r \leq p\}$ has size less than $\kappa_{g(\beta)}$, where $\beta = \min(\mathrm{dom}(g) \setminus \alpha + 1)$. All the measures used in p above α are $\kappa_{g(\beta)}$ complete, so we can apply the Prikry property to every element in $\mathbb{P}^p_\alpha$ and then intersect measure one sets.*

Let G be $\mathbb{P}$ generic and let $g^* = \bigcup_{\langle g,H\rangle \in G} g$. Then g^* is a function with domain λ and with $g^*(\alpha) \in \mathcal{P}_\kappa(\kappa_\alpha)$ for each $\alpha \in \lambda$. For each $\alpha < \lambda$, denote $x^*_\alpha = g^*(\alpha)$ and $\tau_\alpha = \kappa_{x^*_\alpha} = \kappa \cap x^*_\alpha$. Then $\nu = \bigcup_{\alpha<\lambda} x^*_\alpha$, and so the cofinality of κ and each $\kappa_{\alpha+1}$ is λ. In particular, in the generic extension $\kappa = \sup_{\alpha<\lambda} \tau_\alpha$. Below we summarize the preservation and collapsing of cardinals and cofinalities. The proof is an adaptation of the arguments in [6].

Proposition 5. *(Preservation of cardinals)*

1. *κ is preserved and has cofinality λ in the generic extension.*
2. *All cardinals and cofinalities below τ_0 are preserved.*
3. *For each $\alpha < \lambda$, τ_α is preserved.*
4. *Let τ be a cardinal in V with $\tau_\alpha < \tau \leq \sup_{\xi<\alpha} f^\alpha_{\kappa_\xi}(\tau_\alpha)$, for α limit. Then $\mathrm{card}^{V[E][G]}(\tau) = \tau_\alpha$, and if τ is regular in V, then in $V[E][G]$ the cofinality of τ is equal to $\mathrm{cf}(\alpha)$.*
5. *Let τ be a cardinal in V with $\kappa < \tau < \mu$. Then $\mathrm{card}^{V[E][G]}(\tau) = \kappa$, and if τ is regular in V, then in $V[E][G]$ the cofinality of τ is equal to λ.*
6. *Cardinals greater than or equal to μ are preserved. And so μ becomes the successor of κ.*

Remark 6. In particular, if τ is such that $\mathrm{cf}^V(\tau) \neq \mathrm{cf}^{V[E][G]}(\tau)$, then $\mathrm{cf}^{V[E][G]}(\tau) \leq \lambda$.

The reason behind item (4) is as follows: let $p = \langle g, H\rangle \in G$ with $\mathrm{dom}(g) = \{\alpha\}$ for a limit $\alpha < \lambda$. Below this condition we can factor the poset to $\mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ is defined from the normal measures $\overline{U^\xi_{\alpha,x^*_\alpha}}$ on $\mathcal{P}_{\tau_\alpha}(f^\alpha_{\kappa_\xi}(\tau_\alpha))$ for $\xi < \alpha$ and $\mathbb{P}_1$ is defined from the normal measures U_β, $\alpha < \beta < \lambda$. I.e. conditions in $\mathbb{P}_1$ are below $\langle 0, H \restriction (\lambda \setminus \alpha)\rangle$. Then $\mathbb{P}_0$ adds a generic sequence $\langle y_\xi \mid \xi < \alpha\rangle$, such that $\bigcup_{\xi<\alpha} y_\xi = \sup_{\xi<\alpha} f^\alpha_{\kappa_\xi}(\tau_\alpha)$. Thus $\sup_{\xi<\alpha} f^\alpha_{\kappa_\xi}(\tau_\alpha)$ is collapsed to τ_α. Moreover, $\mathbb{P}_0$ has the $\sup_{\xi<\alpha} f^\alpha_{\kappa_\xi}(\tau_\alpha)^+$ chain condition. For more details on this factoring see [6].

The factoring described above combined with the Prikry property gives that for limit $\alpha < \lambda$ and natural number k, cardinals τ with $\sup_{\xi<\alpha} f^\alpha_{\kappa_\xi}(\tau_\alpha)^+ \leq \tau < \tau_{\alpha+k}$ are preserved. This implies item (3) of the above proposition.

Corollary 7. *In the generic extension, $2^\kappa = \kappa^{++}$, and so the singular cardinal hypothesis fails at κ.*

Before we focus on the tree property, we turn our attention to scales in the generic extension. Scales are a central concept in PCF theory. The existence of a bad scale implies the failure of weak square. Actually it also implies that the approachability property fails. In both the models of Gitik-Sharon [2] and Neeman [5] there exists a very good scale and a bad scale. Gitik-Sharon [2] showed that starting from a supercompact there is a generic extension, in which the singular cardinal hypothesis fails at $\aleph_{\omega^2}$, there is a very good scale at $\aleph_{\omega^2}$ and the approachability property fails at $\aleph_{\omega^2}$. The proof of the existence of the bad scale in the Gitik-Sharon model is due to Cummings-Foreman [1]. The construction in [2] was generalized to a cardinal of arbitrary cofinality by Sinapova [6]. Adapting the arguments from the above papers, we get:

Proposition 8. *In $V[E][G]$, there is a bad scale and a very good scale at κ.*

As before, denote $\mu = (\nu^+)^V$. Using that κ is supercompact, in $V[E]$ we fix a bad scale $\langle G_\beta \mid \beta < \mu \rangle$ in $\prod_{\xi<\lambda} \kappa_\xi^+$. Moreover we can fix $\lambda < \tau < \kappa$, such that there is a stationary set of bad points of cofinality τ. We can define the forcing so that the generic sequence $\langle \tau_\xi \mid \xi < \lambda \rangle$ is above τ, and all of the measures are τ-complete. When defining the scales, we use the following key property (due to a density argument):

Proposition 9. *If $\langle A_\xi \mid \xi < \lambda \rangle \in V[E]$ is such that each $A_\xi \in U_\xi$, then $x_\xi^* \in A_\xi$ for all sufficiently large ξ.*

We also make use of a bounding lemma. For details of the proof, see [6].

Lemma 10. *(Bounding)*

(1) *Suppose that in $V[E][G]$, $f \in \prod_{\xi<\lambda} f^\xi_{\kappa_\xi^+}(\tau_\xi)$. Then there is a sequence of functions $\langle H_\eta \mid \eta < \lambda \rangle$ in $V[E]$, such that* $\mathrm{dom}(H_\eta) = X_\eta$ *and $f(\eta) < H_\eta(x_\eta^*)$ for all large $\eta < \lambda$.*
(2) *Suppose that in $V[E][G]$, $f \in \prod_{\xi<\lambda} f^\xi_\nu(\tau_\xi)^+$. Then there is a sequence of functions $\langle H_\eta \mid \eta < \lambda \rangle$ in $V[E]$, such that* $\mathrm{dom}(H_\eta) = X_\eta$ *and $f(\eta) < H_\eta(x_\eta^*)$ for all large $\eta < \lambda$.*

Proof of Proposition 8. The proof is an adaptation of the arguments in [1] and [6], so we only outline the main points. For more details, see sections 4 and 5 of [6]. Define in $V[E][G]$, $\langle g_\beta \mid \beta < \mu \rangle$ in $\prod_{\xi<\lambda} f^\xi_{\kappa_\xi^+}(\tau_\xi)$ by $g_\gamma(\xi) = f^\xi_{G_\gamma(\xi)}(\tau_\xi)$. By Proposition 9, it follows that the functions are increasing in the eventual domination order. Also, by the bounding lemma (1), we get that the sequence is cofinal. So, $\langle g_\beta \mid \beta < \mu \rangle$ is a scale.

Using the fact that all of the measures are τ-complete and Remark 6, we get that if a point of cofinality τ is bad in $V[E]$ for $\langle G_\beta \mid \beta < \mu \rangle$, then it is bad in $V[E][G]$ for $\langle g_\beta \mid \beta < \mu \rangle$ (and still has cofinality τ in $V[E][G]$). Since $\mathbb{P}$ has the μ-chain condition, and so preserves stationary sets, it follows that $\langle g_\beta \mid \beta < \mu \rangle$ is a bad scale.

Next we describe the very good scale. In $V[E][G]$, define $\langle h_\gamma \mid \gamma < \mu \rangle$ in $\prod_{\xi<\lambda} f^\xi_\nu(\tau_\xi)^+$, by $h_\gamma(\xi) = f^\xi_\gamma(\tau_\xi)$. By Proposition 9, the bounding lemma (2), and the completeness of the measures, we get that this is a very good scale.

□

3. The tree property

It remains to show that the tree property holds. Recall that we forced over V with $A = Add(\kappa, \nu^{++})$ to get $V[E]$. Let $\dot{T}$ in $V[E]$ be a $\mathbb{P}$ - name for a ν^+ tree with levels of size at most κ, such that this is forced by the empty condition. Furthermore we may assume that the empty condition forces that the elements of the α-th level of $\dot{T}$ are elements of $\{\alpha\} \times \kappa$ for $\alpha < \nu^+$. We will show that $T = \dot{T}_G$ has a cofinal branch in $V[E][G]$. The proof is motivated by Neeman [5].

Lemma 11. *There is $\vec{\eta} \in \lambda^{<\omega}$ and an unbounded $I \subset \nu^+$ (in $V[E]$), such that for all $\alpha < \beta$ in I, there are $\xi, \delta < \kappa$ and a condition $q = \langle g, H \rangle$ with* $\mathrm{dom}(g) = \vec{\eta}$*, such that $q \Vdash \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, \delta \rangle$.*

Proof. Let $j : V[E] \to M$ be ν^+ supercompact embedding with critical point κ. Let G^* be $j(\mathbb{P})$ - generic over M, such that the generic sequence determined by G^* is above ν^+. In particular, if $\langle x^*_\xi \mid \xi < \lambda \rangle$ is the generic sequence, where each $x^*_\xi \in \mathcal{P}_{j(\kappa)}(j(\kappa_\xi))$, and $\tau^*_\xi = j(\kappa) \cap x^*_\xi$ for each ξ, G^* is chosen to be such that $\nu^+ < \tau^*_0$. Set $T^* = j(\dot{T})_{G^*}$.

Let γ be such that $\sup j''\nu^+ < \gamma < j(\nu^+)$. Such a γ exists since M is closed under ν^+-sequences. Working in $M[G^*]$, fix a node $u \in T^*$ of level γ. Then for all $\alpha < \nu^+$ let $\xi_\alpha < j(\kappa)$ be such that $\langle j(\alpha), \xi_\alpha \rangle <_{T^*} u$, and let $p_\alpha \in G^*$ be such that $p_\alpha \Vdash \langle j(\alpha), \xi_\alpha \rangle <_{j(\dot{T})} \dot{u}$.

Since the generic sequence determined by G^* is above ν^+, we have that ν^+ is preserved in $M[G^*]$ and remains regular. So, there is an unbounded $I^* \subset \nu^+$ in $M[G^*]$ and a fixed $\vec{\eta} \in \lambda^{<\omega}$, such that for all $\alpha \in I^*$, $p_\alpha = \langle g_\alpha, H_\alpha \rangle$ where the domain of g_α is $\vec{\eta}$. Let b be a stem with this domain such that there is a condition in G^* with stem b.

Define $I = \{\alpha < \nu^+ \mid \exists p \in j(\mathbb{P}) \ \mathrm{stem}(p) = b \text{ and } \exists \xi < j(\kappa) p \Vdash \langle \alpha, \xi \rangle <_{j(\dot{T})} \dot{u}\}$. Then $I \in V[E]$ and $I^* \subset I$, so I is unbounded. Any two conditions with the same stem are compatible, so by elementarity of j and since $j(\dot{T})$ is forced to be a tree, we have that I is as desired. □

Lemma 12. *There is, in $V[E]$, an unbounded set $J \subset \nu^+$, a pair $\langle \bar{g}, \bar{H} \rangle$ and a sequence of nodes $\langle u_\alpha \mid \alpha \in J \rangle$, such that, setting* $\mathrm{dom}(\bar{g}) = \vec{\eta}$ *and* $\eta_0 = \max(\vec{\eta})$*, we have that $\bar{H}$ has domain $\eta_0 \setminus \vec{\eta}$ and for all $\alpha < \beta$ in J there is a condition p such that:*

- $\mathrm{stem}(p) = \bar{g}$, $p \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H} \rangle$,
- $p \Vdash u_\alpha <_{\dot{T}} u_\beta$.

Proof. Fix $\vec{\eta}$ and I as in the conclusion of the last lemma, and let $\eta_0 = \max(\vec{\eta})$. Let $\bar{j} : V \to N$ be a ν^+ - supercompact embedding with critical point κ_{η_0+1}. Using standard arguments, extend $\bar{j}$ to $j : V[E] \to N[E^*]$ where $j \in V[E][F]$, for F - $Add(\kappa_0, j(\nu^{++}))$ - generic over $V[E]$.

Let $\gamma \in j(I)$ be such that $\sup(j''\nu^+) < \gamma < j(\nu^+)$. By elementarity for all $\alpha \in I$ we can fix $\xi_\alpha, \delta_\alpha < \kappa_0$ and $p_\alpha = \langle g_\alpha, H_\alpha \rangle \in j(\mathbb{P})$ with domain of g_α equal to $\vec{\eta}$, such that $p_\alpha \Vdash \langle j(\alpha), \xi_\alpha \rangle <_{j(\dot{T})} \langle \gamma, \delta_\alpha \rangle$. I is cofinal in ν^+

and the number of possibilities for the part of the conditions below $\eta_0 + 1$ is less than κ_{η_0+1}. It follows that there is a cofinal $J \subset I$ in $V[E][F]$, fixed $\xi, \delta < \kappa_0$, and a fixed $\langle \bar{g}, \bar{H} \rangle$ such that for all $\alpha \in J$, $\delta_\alpha = \delta$, $\xi_\alpha = \xi$, and $p_\alpha \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H} \rangle$. Then for all $\alpha, \beta \in J$ with $\alpha < \beta$, there is a condition $p \in j(\mathbb{P})$ with stem $\bar{g}$ and $p \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H} \rangle$, such that $p \Vdash \langle j(\alpha), \xi \rangle <_{j(\dot{T})} \langle j(\beta), \xi \rangle$. Since $j(\langle \bar{g}, \bar{H} \rangle) = \langle \bar{g}, \bar{H} \rangle$, by elementarity, there is a condition $p \in \mathbb{P}$ such that $p \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H} \rangle$ and $p \Vdash \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, \xi \rangle$.

In $V[E]$ let $a = \{\langle g, H, \alpha, \beta, \xi \rangle \mid (\exists p \in \mathbb{P})(\mathrm{stem}(p) = g, p \restriction (\eta_0 + 1) = \langle g, H \rangle, p \Vdash \langle \alpha, \xi \rangle <_{\dot{T}} \langle \beta, \xi \rangle)\}$. Note that a can be coded by a subset of ν^+. So far we have shown that $V[E][F] \models (\exists g, H, J, \xi)\phi(a, g, H, J, \xi, \nu^+)$, where $\phi(a, g, H, J, \xi, \nu^+) =$ "J is an unbounded subset of ν^+ and for all $\alpha, \beta \in J$ with $\alpha < \beta$, $\langle g, H, \alpha, \beta, \xi \rangle \in a$".

Claim 13. $V[E] \models (\exists g, H, J, \xi)\phi(a, g, H, J, \xi, \nu^+)$

Proof. See Neeman [5]. The proof uses that a can be coded by a subset of ν^+ and the absoluteness of ϕ. □

□

Fix $\langle \bar{g}, \bar{H} \rangle$, J, and $u_\alpha = \langle \alpha, \xi \rangle$ for $\alpha \in J$ as in the conclusion of the above lemma. For a stem g and a formula ϕ, we say that $g \Vdash^* \phi$ iff there is a condition p with stem g such that $p \Vdash \phi$. By the Prikry property it follows that for all g and ϕ, either $g \Vdash^* \phi$ or $g \Vdash^* \neg\phi$.

The following proposition is due to Neeman [5].

Proposition 14. *Suppose that M is a model of ZFC, S is a tree of height θ in M, and $\mathbb{B} \in M$ is a poset such that $\mathbb{B} \times \mathbb{B}$ has the $\mathrm{cf}(\theta)$-chain condition and a power $\mathbb{B}^{|S|^+}$ does not collapse $|S|^+$. Then $\mathbb{B}$ does not add a branch through S.*

Lemma 15. *Suppose that g is a stem, $L \subset \nu^+$ is unbounded, and for all $\alpha < \beta$ with $\alpha, \beta \in L$, $g \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. Let $\eta > \max(\mathrm{dom}(g))$.*

Then there are $\rho < \nu^+$ and sets $\langle A_\alpha : \alpha \in L \setminus \rho \rangle$ such that:

1. *each $A_\alpha \in U_\eta$,*
2. *for all $\alpha < \beta$ in $L \setminus \rho$, for all $x \in A_\alpha \cap A_\beta$,*

$$g^\frown \langle \eta, x \rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

Proof. The proof follows closely the argument given in [5].

Let $\bar{j} : V \to N$ be a ν^+ - supercompact embedding with critical point $\kappa_{\eta+1}$. As in the previous lemma, extend $\bar{j}$ to $j : V[E] \to N[E^*]$ where $j \in V[E][F]$, for F - $Add(\kappa, j(\nu^{++}))$ - generic over $V[E]$. Let $\gamma \in j(L)$ be such that j"$\nu^+ < \gamma < j(\nu^+)$. We write u_γ for the γ^{th} member of the sequence $j(\langle u_\alpha : \alpha \in L \rangle)$. Note that $j(g) = g$. Then working in $V[E][F]$, by elementarity we can find conditions $\langle r_\alpha : \alpha \in L \rangle$, such that each $r_\alpha \in j(\mathbb{P})$, the stem of each r_α is g, and

$$r_\alpha \Vdash_{j(P)} j(u_\alpha) <_{j(\dot{T})} u_\gamma. \tag{1}$$

For each $\alpha \in L$, denote $r_\alpha = \langle g, H_\alpha \rangle$. Define $\alpha \mapsto A^*_\alpha$ (in $V[E][F]$) by $A^*_\alpha = H_\alpha(\eta) \in j(U_\eta)$. Note that although they have the same domain, $U_\eta \neq j(U_\eta)$. Actually, $\mathcal{P}(\mathcal{P}_\kappa(\kappa_\eta))^{V[E]} \neq \mathcal{P}(\mathcal{P}_\kappa(\kappa_\eta))^{V[E][F]}$, as the cardinality of the latter is $j(\nu^{++})$.

Equation 1 implies that for all $x \in A^*_\alpha$ we have that over $N[E^*]$.

$$g^\frown\langle \eta, x \rangle \Vdash^*_{j(\mathbb{P})} j(u_\alpha) <_{j(\dot{T})} u_\gamma \tag{2}$$

For $x \in \mathcal{P}_\kappa(\kappa_\eta)$, let $L_x = \{\alpha \in L \mid g^\frown\langle \eta, x \rangle \Vdash^*_{j(\mathbb{P})} j(u_\alpha) <_{j(\dot{T})} u_\gamma\}$. Note that for all $\alpha \in L$, if $x \in A^*_\alpha$, then $\alpha \in L_x$.

Claim 16. *If L_x is unbounded in ν^+, then $L_x \in V[E]$.*

Proof. Suppose that L_x is unbounded in ν^+.

Subclaim 17. *For all α, β in L with $\alpha < \beta$ and $\beta \in L_x$, we have that $\alpha \in L_x$ iff $g^\frown\langle \eta, x \rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.*

Proof. Let α, β be as above. So, $g^\frown\langle \eta, x \rangle \Vdash^*_{j(\mathbb{P})} j(u_\beta) <_{j(\dot{T})} u_\gamma$. Then $\alpha \in L_x$ iff $g^\frown\langle \eta, x \rangle \Vdash^*_{j(\mathbb{P})} j(u_\alpha) <_{j(\dot{T})} u_\gamma$ iff $g^\frown\langle \eta, x \rangle \Vdash^*_{j(\mathbb{P})} j(u_\alpha) <_{j(\dot{T})} j(u_\beta)$ iff $g^\frown\langle \eta, x \rangle \Vdash^*_{\mathbb{P}} u_\alpha <_{\dot{T}} u_\beta$.

□

Let S be the tree of attempts to construct L_x. I.e. S is the set of all bounded $v : \nu^+ \to L$ such that:

(1) v is increasing,
(2) for all α, β in L with $\alpha < \beta$ and $\beta \in \mathrm{ran}(v)$, we have that $\alpha \in \mathrm{ran}(b)$ iff $g^\frown\langle \eta, x \rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.

Then if we let $v^* : \nu^+ \to L$ enumerate L_x, we have an unbounded branch of S. Here we use that all initial segments of L_x are in $V[E]$. Applying Proposition 14 for $M = V[E]$, $\theta = \nu^+$, $\mathbb{B} = Add(\kappa, j(\nu^{++}))$, and S, we get that $L_x \in V[E]$.

□

For $x \in \mathcal{P}_\kappa(\kappa_\eta)$, let $K_x = \{C \in V[E] \mid C$ is cofinal in ν^+ and there is a $b \in Add(\kappa, j(\nu^{++}))$ with $b \Vdash \dot{L}_x = C\}$ if $g^\frown\langle \eta, x \rangle$ is a stem, and $K_x = \emptyset$ otherwise. Then each $K_x \in V[E]$ and since $Add(\kappa, j(\nu^{++}))$ has the κ^+ chain condition, we have that $\mathrm{card}\,(K_x) \leq \kappa$.

Claim 18. *For $C \in K_x$ and $\alpha < \beta < \nu^+$ such that both α and β are in L and $\beta \in C$, we have that $\alpha \in C$ iff $g^\frown\langle \eta, x \rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.*

Proof. similar as in the subclaim above. □

From the above claim it follows that any distinct C_1, C_2 in K_x are disjoint on a tail. For every x, and C_1, C_2 in K_x fix ρ_{x,C_1,C_2} to be such that above it, C_1 and C_2 are disjoint. Let $\rho = \sup\{\rho_{x,C_1,C_2} \mid x \in \mathcal{P}_\kappa(\kappa_\eta), C_1, C_2 \in K_x\}$. Then $\rho < \nu^+$, and for all x and $\alpha \in L \setminus \rho$, there is at most one $C \in K_x$ with $\alpha \in C$. Define $f(x, \alpha)$ to be this unique $C \in K_x$ if it exists and undefined otherwise. For $\alpha \in L \setminus \rho$ let $G_\alpha = \{x \in \mathcal{P}_\kappa(\kappa_\eta) \mid f(x, \alpha) \text{ is defined }\}$. Also for $\alpha, \alpha' \in L \setminus \rho$, define $G_{\alpha,\alpha'} = \{x \in G_\alpha \cap G_{\alpha'} \mid f(x, \alpha) = f(x, \alpha')\}$.

Claim 19. (1) *For each* α, $G_\alpha \in U_\eta$
(2) *For each* α, α', $G_{\alpha,\alpha'} \in U_\eta$

Proof. Otherwise, for some $\alpha \in L \setminus \rho$, $Y = \{x \in \mathcal{P}_\kappa(\kappa_\eta) \mid f(x, \alpha) \text{ is not defined }\} \in U_\eta$. Since $j(Y) = Y \in j(U_\eta)$, we have that for all $\beta \in L$, we can fix an element $x_\beta \in A^*_\alpha \cap A^*_\beta \cap Y$. L is unbounded in ν^+, so there is some unbounded $U \subset L$ and $x \in \mathcal{P}_\kappa(\kappa_\eta)$, such that for all $\beta \in U$, $x_\beta = x$.

Now, for all $\beta \in U$, since $x \in A^*_\beta$, we have that $g^\frown\langle \eta, x\rangle \Vdash^* j(u_\beta) <_{j(\dot{T})} u_\gamma$, so $\beta \in L_x$. I.e $U \subset L_x$, and so L_x is unbounded and thus it is in $V[E]$. Also, since $x \in A^*_\alpha$, it follows that $g^\frown\langle \eta, x\rangle \Vdash^* j(u_\alpha) <_{j(\dot{T})} u_\gamma$, and so $\alpha \in L_x$. In particular, $\alpha \in L_x \in K_x$, therefore $f(x, \alpha)$ is defined. But $x \in Y$, contradiction.

The proof of (2) is similar.

□

Let α_0 be the least element in $L \setminus \rho$. Define $A_\alpha = G_{\alpha_0,\alpha}$. The sets $\langle A_\alpha : \alpha \in L \setminus \rho\rangle$ are as desired. □

Lemma 20. *There are* $\rho < \nu^+$ *and conditions* $\langle p_\alpha : \alpha \in J \setminus \rho\rangle$ *such that:*

1. *each* p_α *has stem* $\bar{g}$ *and* $p_\alpha \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H}\rangle$,
2. *for all* $\alpha < \beta$ *in* $J \setminus \rho$, $p_\alpha \wedge p_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$.

Here $p \wedge q$ *denotes the weakest extension of* p *and* q.

Proof. Recall that $\langle \bar{g}, \bar{H}\rangle$, $\eta_0 = \max(\operatorname{dom}(g))$, J, and $u_\alpha = \langle \alpha, \xi\rangle$ are given by Lemma 12. I.e. $\bar{H}$ has domain $\eta_0 \setminus \vec{\eta}$ and for all $\alpha < \beta$ in J there is a condition p such that the stem of p is $\bar{g}$, $p \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H}\rangle$, and $p \Vdash u_\alpha <_{\dot{T}} u_\beta$.

First we make some remarks on taking diagonal intersections. Let $\eta < \lambda$, let S be a set of stems whose domain has maximum below η, and let $\langle A^g \mid g \in S\rangle$ be a sequence of U_η- measure one sets. For a stem g in S and $z \in \mathcal{P}_\kappa(\kappa_\eta)$, we write $g \prec z$ to denote that $g(\max\operatorname{dom}(g)) \prec z$, i.e. that $|g(\max\operatorname{dom}(g))| < \kappa_z$ and $g(\max\operatorname{dom}(g)) \subset z$. Note that $g \prec z$ iff $g^\frown\langle \eta, z\rangle$ is a stem. Then $A = \triangle_{g\in S} A^g = \{z \in \mathcal{P}_\kappa(\kappa_\eta) \mid z \in \bigcap_{g\prec z} A^g\}$ is the diagonal intersection of $\langle A^g \mid g \in S\rangle$ and $A \in U_\eta$.

Let $g \supset \bar{g}$ be a stem with $\max(\operatorname{dom}(g)) = \eta$. We say that g is *compatible* with $\bar{H}$ if for all $\xi \in \operatorname{dom}(g) \cap \eta_0$ with $\xi \notin \operatorname{dom}(\bar{g})$, $g(\xi) \in \bar{H}(\xi)$. Also for $\eta' \geq \eta$, we say that g is compatible with $\langle B_\xi \mid \eta_0 < \xi \leq \eta'\rangle$, where each $B_\xi \in U_\xi$, if for all $\eta_0 < \xi \in \operatorname{dom}(g)$, $g(\xi) \in B_\xi$. Note that if $\max(\operatorname{dom}(g)) = \eta_0$, then g is vacuously compatible with any sequence $\langle B_\xi \mid \eta_0 < \xi \leq \eta'\rangle$. We will define sequences $\langle \rho_\eta \mid \eta_0 < \eta < \lambda\rangle$, and $\langle A_\alpha(\eta) \mid \alpha \in J \setminus \rho_\eta, \eta_0 < \eta < \lambda\rangle$ by induction on η, such that each $A_\alpha(\eta) \in U_\eta$ and for all $\eta_0 \leq \eta < \lambda$ we have:

$(\dagger)_\eta$ For all stems $g \supset \bar{g}$ with $\max(\text{dom}(g)) = \eta$, and for all $\alpha < \beta$ in $J \setminus \rho_\eta$, if g is compatible with $\bar{H}$, $\langle A_\alpha(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, and $\langle A_\beta(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, then $g \Vdash^* u_\alpha <_{\dot{T}} u_\beta$.
Note that $(\dagger)_{\eta_0}$ holds by Lemma 12.

Suppose $\eta_0 < \eta < \lambda$ and suppose that we have defined ρ_ξ and $A_\alpha(\xi)$ for all $\xi < \eta$, $\alpha \in J \setminus \rho_\xi$ such that $(\dagger)_\xi$ holds for all $\eta_0 \leq \xi < \eta$.

For a stem $g \supset \bar{g}$ with $\max(\text{dom}(g)) < \eta$ set $J^g = \{\alpha \in J \setminus \sup_{\eta_0 < \xi < \eta} \rho_\xi \mid g$ is compatible with $\langle A_\alpha(\xi) \mid \eta_0 < \xi \leq \max(\text{dom}(g))\rangle\}$ if $\eta > \eta_0 + 1$, and if $\eta = \eta_0 + 1$, we set $J^g = J$. Define a function $g \mapsto \rho^g$ on stems extending $\bar{g}$ and compatible with $\bar{H}$ whose domain has maximum below η as follows:

- if J^g is bounded in ν^+, let $\rho^g < \nu^+$ be a bound,
- otherwise, let ρ^g and $\langle A^g_\alpha \mid \alpha \in J^g \setminus \rho^g \rangle$, be given by the previous lemma applied to g and J^g (here we use $(\dagger)_{\max(\text{dom}(g))}$ to get the assumptions of the lemma). Then, we have that:
  - each $A^g_\alpha \in U_\eta$,
  - for all $\alpha < \beta$ in $J \setminus \rho^g$, for all $y \in A^g_\alpha \cap A^g_\beta$,

$$g^\frown \langle \eta, y \rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta.$$

Let $\rho_\eta = \sup_g \rho^g$. Note that the number of all possible such stems is less that ν^+, so $\rho_\eta < \nu^+$. For each $\alpha \in J \setminus \rho_\eta$, define $H_\alpha = \{g \mid g$ is a stem, $g \supset \bar{g}, \max(\text{dom}(g)) < \eta, g$ is compatible with $\bar{H}$, $\langle A_\alpha(\xi) \mid \eta_0 < \xi \leq \max(\text{dom}(g))\rangle\}$. Then since $\alpha > \rho_\eta$, for each $g \in H_\alpha$, A^g_α is defined. For $\alpha \in J \setminus \rho_\eta$ define

$$A_\alpha(\eta) = \triangle_{g \in H_\alpha} A^g_\alpha.$$

Now we have to verify that $(\dagger)_\eta$ holds: suppose that $\alpha < \beta$ are in $J \setminus \rho_\eta$ and $g' \supset \bar{g}$ is compatible with $\langle A_\alpha(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, $\langle A_\beta(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, $\bar{H}$, and $\max(\text{dom}(g')) = \eta$. Then for some y, $g' = g \cup \langle \eta, y \rangle$. Since g' is compatible with $\langle A_\alpha(\xi) \mid \eta_0 < \xi \leq \eta \rangle$ and $\langle A_\beta(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, we have that $y \in A_\alpha(\eta) \cap A_\beta(\eta)$. Therefore since $g \prec y$, we have that $y \in A^g_\alpha \cap A^g_\beta$. Since $\alpha, \beta \in J^g$ and $\alpha, \beta \geq \rho_\eta \geq \rho^g$, it follows that ρ^g, A^g_α, and A^g_β were given by Lemma 15, so $g^\frown \langle \eta, y \rangle \Vdash^* u_\alpha <_{\dot{T}} u_\beta$ as desired.

Let $\rho = \sup\{\rho_\eta \mid \eta_0 < \eta < \lambda\}$. Define p_α for $\alpha < J \setminus \rho$ by:

- $p_\alpha \restriction (\eta_0 + 1) = \langle \bar{g}, \bar{H} \rangle$
- $p_\alpha(\eta) = A_\alpha(\eta)$, for $\eta_0 < \eta < \lambda$

Now suppose that $q \leq p_\alpha \wedge p_\beta$. Let g be the stem of q and let $\eta = \max(\text{dom}(g))$. Then g is compatible with $\bar{H}$, $\langle A_\alpha(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, and $\langle A_\beta(\xi) \mid \eta_0 < \xi \leq \eta \rangle$, so by $(\dagger)_\eta$ we have that $g \Vdash^* u_\alpha <_{\dot{T}} u_\beta$. Since any two conditions with the same stem are compatible, it follows that $q \not\Vdash u_\alpha \not<_{\dot{T}} u_\beta$. So, $p_\alpha \wedge p_\beta \Vdash u_\alpha <_{\dot{T}} u_\beta$ as desired.

□

Lastly, we show that $\{u_\alpha \mid p_\alpha \in G\}$ is an unbounded branch of T. It suffices to prove the following:

Proposition 21. *$B = \{\alpha < \nu^+ \mid p_\alpha \in G\}$ is unbounded.*

Proof. Otherwise, let $q \in G$ be such that $q \Vdash \dot{B}$ is bounded. Since both Lemma 11 and Lemma 12 can be done below any condition, we may assume that (by strengthening q if necessary) stem$(q) = \bar{g}$. $\mathbb{P}$ has the ν^+ chain condition, so for some $\alpha < \nu^+$, $q \Vdash \dot{B} \subset \alpha$. Let $\beta \in J \setminus \alpha$, and let r be a common extension of q and p_β. Then on one hand we have that $r \Vdash p_\beta \in \dot{G}$, but also $r \Vdash u_\beta \notin \dot{B}$. Contradiction.

□

Then $\{u_\alpha \mid \alpha \in B\}$ is an unbounded branch of T. This completes the proof of the tree property.

References

[1] James Cummings and Matthew Foreman, *Diagonal Prikry extensions*, ***J. of Symbolic Logic***, vol. 75(2010), no. 4, pp. 1383-1402.

[2] Moti Gitik and Assaf Sharon, *On SCH and the approachability property*, ***Proc. of the AMS***, vol. 136 (2008), pp. 311-320

[3] Richard Laver, *Making the supercompactness of κ indestructible under κ-directed closed forcing*, ***Israel J. of Math.***, vol. 29(1978), no. 4, pp. 385–388.

[4] Menachem Magidor and Saharon Shelah, *The tree property at successors of singular cardinals.*, ***Arch. Math. Logic*** , vol. 35 (1996), no. 5-6, pp. 385–404

[5] Itay Neeman, *Aronszajn trees and the failure of the singular cardinal hypothesis*, ***J. of Mathematical Logic*** vol. 9 (2009, published 2010) pp. 139-157.

[6] Dima Sinapova, *A model for a very good scale and a bad scale.*, J. Symbolic Logic, 73(4):1361–1372, 2008

[7] Robert M. Solovay, William N. Reinhardt and Akihiro Kanamori, *Strong axioms of infinity and elementary embeddings.*, ***Ann. Math. Logic***, vol. 13(1978), no. 1, pp. 73–116.

Department of Mathematics, University of California Irvine, Irvine, CA 92697-3875, U.S.A

E-mail address: dsinapov@math.uci.edu