A MODEL FOR A VERY GOOD SCALE AND A BAD SCALE

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Abstract. Given a supercompact cardinal κ and a regular cardinal $\lambda < \kappa$, we describe a type of forcing such that in the generic extension the cofinality of κ is λ, there is a very good scale at κ, a bad scale at κ, and SCH at κ fails. When creating our model we have great freedom in assigning the value of 2^κ, and so we can make SCH hold or fail arbitrarily badly.

1. Introduction

The relationship between Jensen's square principle, the Singular Cardinal Hypothesis (SCH), very good scales and large cardinals is important in singular cardinal arithmetic. Recently two old questions were answered by a paper of Gitik and Sharon [3]: that failure of SCH does not imply weak square, and the existence of a very good scale does not imply weak square. Their result that the failure of weak square is consistent with failure of SCH and existence of a very good scale was obtained at a cardinal of cofinality ω.

This paper was inspired by work of Cummings and Foreman, who, analyzing the Gitik–Sharon model, showed that the failure of weak square in this model is in fact a consequence of the existence of a bad scale at κ [1]. Precisely, in the Gitik–Sharon model obtained after forcing at κ, the following holds in the forcing extension:

(1) κ has cofinality ω,
(2) SCH fails at κ,
(3) there is a very good scale at κ, and
(4) there is a bad scale at κ.

The existence of the bad scale implies failure of weak square.

The natural question is whether we can get the same result for larger cofinalities; it turns out that we can. The main result we will present is that if in the ground model GCH holds, κ is a supercompact cardinal and λ is a regular cardinal less than κ, then there is a generic extension in which the cofinality of κ is λ, SCH fails at κ, there is a very good scale at κ, and there is a bad scale at κ (and so weak square at κ fails). The forcing we use combines ideas from Magidor's forcing for changing cofinalities of cardinals [6] and the forcing described in Gitik–Sharon paper. We note that both in the Gitik–Sharon model and here we have great freedom in assigning the value of 2^κ. Thus SCH can hold or fail arbitrarily badly.

The notion of a scale is a central concept in PCF theory. Let κ be a singular cardinal and consider an increasing sequence $\langle \kappa_\eta \mid \eta < \mathrm{cf}(\kappa)\rangle$ that is unbounded in κ. For functions f and g in $\prod_{\eta<\mathrm{cf}(\kappa)} \kappa_\eta$, we say that $f <^* g$ if there exists $\delta < \mathrm{cf}(\kappa)$ such that for every $\eta > \delta$, $f(\eta) < g(\eta)$.

A *scale of length* κ^+ is a sequence of functions $\langle f_\alpha \mid \alpha < \kappa^+\rangle$ from $\prod_{\eta<\mathrm{cf}(\kappa)} \kappa_\eta$ which is increasing and cofinal with respect to $<^*$. We say that $\gamma < \kappa^+$ of cofinality between $\mathrm{cf}(\kappa)$ and κ is a *good point* iff there exists an $A \subseteq \gamma$ that is unbounded in γ and $\zeta < \mathrm{cf}(\kappa)$ such that for all $\alpha, \beta \in A$ and $\eta < \mathrm{cf}(\kappa)$, if $\alpha < \beta$ and $\zeta < \eta$, then $f_\alpha(\eta) < f_\beta(\eta)$. If "unbounded in γ" is replaced by "club in γ", then γ is a *very good point.* The given scale is (*very*) *good* iff modulo the club filter on κ^+, almost every point of cofinality greater than $\mathrm{cf}(\kappa)$ and less than κ is (very) good. *Bad scales* that are those that are not good.

Theorem 1. *Suppose κ is supercompact, λ is a regular cardinal less than κ, and GCH holds. Then there is a generic extension, in which κ has cofinality λ, there is a very good scale at κ, there is a bad scale at κ, and SCH fails at κ.*

The rest of the paper presents the proof of Theorem 1. We shall work with a slightly different assumption on κ then the one in the theorem. Precisely, we work throughout the paper under the assumption that, in the ground model, κ is supercompact, $2^\kappa = \mu^+$ where $\mu = \kappa^{+\lambda+1}$, and $(\kappa^{+\alpha})^{<\kappa} \leq \kappa^{+\alpha+1}$, for each limit $\alpha < \lambda$. Moreover there are functions $f_\gamma : \kappa \to \kappa$ for $\gamma < \mu$ and a supercompactness measure U on $\mathcal{P}_\kappa(\tau)$, where $\tau = 2^{(\kappa^{+\lambda})}$, such that $j_U(f_\gamma)(\kappa) = \gamma$ for each γ. This situation can be arranged, starting from the assumption in the theorem, as follows:

First use Laver's forcing to make κ indestructably supercompact [5]. Let $j : V \to M$ be a τ-supercompactness embedding with critical point κ. Let $\mathbb{P}$ be the poset consisting of partial functions from $\kappa^{+\lambda+2} \times \kappa$ to κ ordered by extension and let H be $\mathbb{P}$ generic. For $\gamma < \kappa^{+\lambda+2}$ define $f_\gamma : \kappa \to \kappa$ to be the γ-th generic function. Using the Laver indestructibility we can extend j to $j^* : V[G] \to M[G^*]$. We do this as in the proof of Laver indestructibility except that when choosing the master condition we choose it to force that $j(f_\gamma)(\kappa) = \gamma$ for each $\gamma < \kappa^{+\lambda+2}$. Now define $U = \{X \in \mathcal{P}_\kappa(\tau) \mid j^{*}{}''\tau \in j^*(X)\}$. Then for $\gamma < \kappa^{+\lambda+2}$, $j_U(f_\gamma)(\kappa) = \gamma$.

Also, for limit $\alpha < \lambda$, $(\kappa^{+\alpha})^{<\kappa} = \kappa^{+\alpha+1}$, since the forcing to make $2^\kappa = \mu^+$ is $< \kappa$ closed. When α is a successor we get that $(\kappa^{+\alpha})^{<\kappa} = \kappa^{+\alpha}$.

2. Measures

In this section we introduce the normal measures which will be used in defining the main forcing. We are interested in normal measures U_α on

$\mathcal{P}_\kappa(\kappa^{+\alpha})$, $\alpha < \lambda$, such that for $\alpha < \beta < \lambda$, $U_\alpha \in Ult_{U_\beta}$. Here we show that such a chain exists, and we prove some propositions about the measures.

The following lemma follows an argument due to Solovay, Reinhardt, and Kanamori [8].

Lemma 2. *For all $\xi < \lambda$, for all $\mathcal{X} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\xi}))$, there is a normal measure U_ξ on $\mathcal{P}_\kappa(\kappa^{+\xi})$, such that $\mathcal{X} \in Ult_{U_\xi}$ and there are functions $\langle F_\gamma \mid \gamma < \mu\rangle$ from κ to κ, such that for all $\gamma < \mu$, $j_{U_\xi}(F_\gamma)(\kappa) = \gamma$.*

Proof. Suppose not. Fix $\xi < \lambda$, such that the statement $\phi(\mathcal{X}, \kappa, \xi, \lambda)$ holds for some $\mathcal{X}$, where $\phi(\mathcal{X}, \kappa, \xi, \lambda) \equiv$ "$\mathcal{X} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\xi}))$ and for all normal measures U_ξ on $\mathcal{P}_\kappa(\kappa^{+\xi}), \mathcal{X} \notin Ult_{U_\xi}$ or $\nexists\langle F_\gamma \mid \gamma < \kappa^{+\lambda+1}\rangle$ from κ to κ, such that for all $\gamma < \kappa^{+\lambda+1}$, $j_{U_\xi}(F_\gamma)(\kappa) = \gamma$." Fix such $\mathcal{X}$.

Let $j : V \to M$ be a τ-supercompactness embedding with critical point κ as in the preparation of the ground model. Namely, for each $\gamma < \kappa^{+\lambda+2}$, $j(f_\gamma)(\kappa) = \gamma$. Since $M^\tau \subset M$, $\mathcal{X} \in M$. Also, if $M \models U_\xi$ is a normal measure on $\mathcal{P}_\kappa(\kappa^{+\xi})$, then in V, U_ξ is a normal measure on $\mathcal{P}_\kappa(\kappa^{+\xi})$. If $\mathcal{X} \notin Ult_{U_\xi}$, then $\mathcal{X} \notin Ult^M_{U_\xi}$. It follows that $M \models (\exists\mathcal{X})\phi(\mathcal{X}, \kappa, \xi, \lambda)$.

Let $U_\xi = \{X \subset \mathcal{P}_\kappa(\kappa^{+\xi}) \mid j"\kappa^{+\xi} \in j(X)\}$, U_ξ is a normal measure on $\mathcal{P}_\kappa(\kappa^{+\xi})$. Define $k : Ult_{U_\xi} \to M$ by $k([f]) = j(f)(j"\kappa^{+\xi})$. Then k is elementary, $j = k \circ j_{U_\xi}$, and $k(\eta) = \eta$, for all $\eta \le \kappa^{+\xi}$. Also since every element of $\mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\xi}))$ belongs to Ult_{U_ξ}, we have that $k(\mathcal{X}) = \mathcal{X}$ for all $\mathcal{X} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\xi}))$ in Ult_{U_ξ}.

By elementarity of k and since $k(\kappa) = \kappa$, $Ult_{U_\xi} \models (\exists\mathcal{X})\phi(\mathcal{X}, \kappa, \xi, \lambda)$. Fix a witness $\mathcal{X}' \in Ult_{U_\xi}$.

Again, by elementarity of k, since $k(\mathcal{X}') = \mathcal{X}'$ and $Ult_{U_\xi} \models \phi(\mathcal{X}', \kappa, \xi, \lambda)$, it follows that $M \models \phi(\mathcal{X}', \kappa, \xi, \lambda)$.

$\mathcal{X}' \in Ult_{U_\xi}$, and $U_\xi \in M$. So, it follows that $M \models$ " $\nexists\langle F_\gamma \mid \gamma < \mu\rangle$ from κ to κ, such that for all $\gamma < \mu$, $j_{U_\xi}(F_\gamma)(\kappa) = \gamma$."

For $\gamma < \mu$ define F_γ as follows. Let g be such that $[g]_{U_\xi} = \gamma$ and set $\delta = j(g)(j"\kappa^{+\xi})$. Let $X = \{x \in \mathcal{P}_\kappa(\kappa^{+\xi}) \mid g(x) = f_\delta(\kappa \cap x)\}$. By definition of U_ξ and since $j(g)(j"\kappa^{+\xi}) = \delta = j(f_\delta)(\kappa)$, it follows that $X \in U_\xi$. So, $j_{U_\xi}(f_\delta)(\kappa) = \gamma$. Set $F_\gamma = f_\delta$. Since $M^\tau \subset M$, $\langle F_\gamma \mid \gamma < \mu\rangle \in M$, contradiction. □

Proposition 3. *There is a chain U_α, $\alpha < \lambda$, such that each U_α is a normal measure on $\mathcal{P}_\kappa(\kappa^{+\alpha})$, for $\alpha < \beta < \lambda$, $U_\alpha < U_\beta$ (i.e. $U_\alpha \in Ult_{U_\beta}$) and functions $\langle F^\xi_\gamma \mid \gamma < \mu, \xi < \lambda\rangle$ from κ to κ, such that for all $\xi < \lambda, \gamma < \mu$, $j_{U_\xi}(F^\xi_\gamma)(\kappa) = \gamma$.*

Proof. Define the chain as follows. Suppose that we already have U_η, $\eta < \xi$ and $\langle F^\eta_\gamma \mid \gamma < \mu, \eta < \xi\rangle$. We can code this sequence by some $\mathcal{Y} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\xi}))$. Apply the argument of the previous lemma to find a normal measure U_ξ on $\mathcal{P}_\kappa(\kappa^{+\xi})$ with $\mathcal{Y} \in Ult_{U_\xi}$ and functions $\langle F^\xi_\gamma \mid \gamma < \mu\rangle$ from κ to κ with $j_{U_\xi}(F^\xi_\gamma)(\kappa) = \gamma$ for each γ. □

Fix measures U_α, for $\alpha < \lambda$ and functions $\langle F^\xi_\gamma \mid \gamma < \mu, \xi < \lambda \rangle$ as in the statements of the last proposition. For each $\alpha < \lambda$ let X_α be the set of $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$ such that

(1) $x \cap \kappa = \kappa_x$ is $\kappa_x^{+\alpha}$ -supercompact
(2) $ot(x) = \kappa_x^{+\alpha}$
(3) $(\forall \gamma < \alpha) ot(x \cap \kappa^{+\gamma}) = \kappa_x^{+\gamma}$
(4) $(\forall \gamma \leq \alpha)((\kappa_x^{+\gamma})^{<\kappa_x} \leq \kappa_x^{+\gamma+1})$

By standard reflection arguments $X_\alpha \in U_\alpha$.

For $\alpha < \beta$, for $x \in X_\beta$, and $Y \subset \mathcal{P}_{\kappa_x}(\kappa^{+\alpha} \cap x)$, define $\overline{Y} \subset \mathcal{P}_{\kappa_x}(\kappa_x^{+\alpha})$, by $\overline{Y} = \{\{o.t.(\xi \cap x) \mid \xi \in y\} \mid y \in Y\}$. $U_\alpha \in Ult_{U_\beta}$ and hence there is a function $x \mapsto \overline{U^\alpha_{\beta,x}}$ such that $U_\alpha = [x \mapsto \overline{U^\alpha_{\beta,x}}]_{U_\beta}$. We may assume that each $\overline{U^\alpha_{\beta,x}}$ is a measure on $\mathcal{P}_{\kappa_x}(\kappa_x^{+\alpha})$, and so there is a normal measure $U^\alpha_{\beta,x}$ on $\mathcal{P}_{\kappa_x}(\kappa^{+\alpha} \cap x)$ such that $\overline{U^\alpha_{\beta,x}} = \{\overline{Y} \subset \mathcal{P}_{\kappa_x}(\kappa_x^{+\alpha}) \mid Y \in U^\alpha_{\beta,x}\}$. Note that each $U^\alpha_{\beta,x}$ is $\lambda-$ complete.

Claim 4. *Let $\alpha < \beta < \gamma$, $z \in X_\gamma$, then,*

(1) *if $A \subset \mathcal{P}_\kappa(\kappa^{+\alpha})$, then $A = [x \mapsto \overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})}]_{U_\beta}$.*
(2) *if $A \subset \mathcal{P}_{\kappa_z}(z \cap \kappa^{+\alpha})$, then $\overline{A} = [x \mapsto \overline{A \cap P_{\kappa_x}(x \cap \kappa^{+\alpha})}]_{U^\beta_{\gamma,z}}$.*

Proof. (1) For $y \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $y = [x \mapsto \{o.t.(\xi \cap x) \mid \xi \in y\}]_{U_\beta}$. If $y \in A$,

$\{x \in \mathcal{P}_\kappa(\kappa^{+\beta}) \mid \{o.t.(\xi \cap x) \mid \xi \in y\} \in \overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})}\} =$
$\{x \in \mathcal{P}_\kappa(\kappa^{+\beta}) \mid y \in A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})\} =$
$\{x \in \mathcal{P}_\kappa(\kappa^{+\beta}) \mid y \in \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})\} \in U_\beta$

Also, if $y \in [x \mapsto \overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})}]_{U_\beta}$, then $\{x \mid y \in A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})\} = \{x \mid \{o.t.(\xi \cap x) \mid \xi \in y\} \in \overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})}\} \in U_\beta$, so $y \in A$.

(2) Similar as above. □

Corollary 5. *If $\alpha < \beta < \gamma$, $z \in X_\gamma$, then*

(1) *if $\mathcal{Y} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\alpha}))$, then $\mathcal{Y} = [x \mapsto \{\overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})} \mid A \in \mathcal{Y}\}]_{U_\beta}$.*
(2) *if $\mathcal{Y} \subset \mathcal{P}(\mathcal{P}_{\kappa_z}(z \cap \kappa^{+\alpha}))$, then $\overline{\mathcal{Y}} = [x \mapsto \{\overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})} \mid A \in \mathcal{Y}\}]_{U^\beta_{\gamma,z}}$.*

Lemma 6. *For $\alpha < \beta < \gamma$ and for U_γ-almost all $z \in X_\gamma$, $U^\alpha_{\gamma,z} < U^\beta_{\gamma,z}$, and $\overline{U^\alpha_{\gamma,z}} = [x \mapsto \overline{U^\alpha_{\beta,x}}]_{U^\beta_{\gamma,z}}$*

Proof. By absoluteness, $Ult_{U_\gamma} \models U_\alpha < U_\beta$ and $U_\alpha = [x \mapsto \overline{U^\alpha_{\beta,x}}]_{U_\beta}$.

So, for U_γ-almost all $z \in X_\gamma$, $U^\alpha_{\gamma,z} < U^\beta_{\gamma,z}$. By the above corollary, each $\overline{U^\alpha_{\gamma,z}} = [x \mapsto \{\overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})} \mid A \in U^\alpha_{\gamma,z}\}]_{U^\beta_{\gamma,z}}$.

Also, since $U_\alpha = [z \mapsto \overline{U^\alpha_{\gamma,z}}]_{U_\gamma}$ again by the last corollary for almost all $z \in X_\gamma$, $U^\alpha_{\gamma,z} = \{A \cap \mathcal{P}_{\kappa_z}(z \cap \kappa^{+\alpha}) \mid A \in U_\alpha\}$. Then for almost all $z \in X_\gamma$,

$\overline{U^\alpha_{\gamma,z}} = [x \mapsto \{\overline{A \cap \mathcal{P}_{\kappa_z}(z \cap \kappa^{+\alpha}) \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})} \mid A \in U_\alpha\}]_{U^\beta_{\gamma,z}} =$
$= [x \mapsto \{\overline{A \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})} \mid A \in U_\alpha\}]_{U^\beta_{\gamma,z}} =$

$= [x \mapsto \overline{U^\alpha_{\beta,x}}]_{U^\beta_{\gamma,z}}$ □

For $\gamma < \lambda$, let $B_\gamma = \{z \in X_\gamma \mid (\forall \alpha, \beta)\alpha < \beta < \gamma \to \overline{U^\alpha_{\gamma,z}} = [x \mapsto \overline{U^\alpha_{\beta,x}}]_{U^\beta_{\gamma,z}}\}$. By taking intersections of measure one sets, we have that $B_\gamma \in U_\gamma$.

Remark 7. It follows that if $\alpha < \beta$ and $A \in U_\alpha$, then

$$(\forall_{U_\beta} x) A \cap P_{\kappa_x}(x \cap \kappa^{+\alpha}) \in U^\alpha_{\beta,x}$$

Similarly, if $\alpha < \beta < \gamma$, $z \in B_\gamma$, and $A \in U^\alpha_{\gamma,z}$, then

$$(\forall_{U^\beta_{\gamma,z}} x) A \cap P_{\kappa_x}(x \cap \kappa^{+\alpha}) \in U^\alpha_{\beta,x}$$

3. The main construction

Before we define the forcing conditions, we briefly discuss the relation between scales and large cardinals. Shelah showed that for κ supercompact, if $\nu > \kappa$ is such that $\mathrm{cf}(\nu) < \kappa$, then there are no good scales at ν [7].

Lemma 8. *Suppose $\langle G_\beta \mid \beta < \mu\rangle$ is a scale in $\prod_{\alpha<\lambda} \kappa^{+\alpha+1}$. There exists an inaccessible $\delta < \kappa$, such that there are stationary many bad points $\beta < \mu$ with $\mathrm{cf}(\beta) = \delta^{+\lambda+1}$.*

Proof. Suppose otherwise. For each inaccessible $\delta < \kappa$ fix a club C_δ in μ, such that all points in C_δ with cofinality $\delta^{+\lambda+1}$ are good for the scale. Let $C = \bigcap_{\delta<\kappa} C_\delta$, which is also club since $\mathrm{cf}(\mu) = \mu > \kappa$.

Let $j : V \longrightarrow M$ be μ-supercompact measure on κ, and let $\rho = sup(j\text{"}\mu)$ Then we have,

$M \models \rho \in j(C)$, $\mathrm{cf}(\rho) = \mathrm{cf}(\mu) = \kappa^{+\lambda+1}$,

and since $\kappa < j(\kappa)$ we have by elementarity that $M \models \rho$ is good.

But if we define g to be the function $\alpha \mapsto sup(j\text{"}\kappa^{+\alpha+1})$, then g is an exact upper bound for $\langle j(G)_\eta \mid \eta < \rho\rangle$ with non-uniform cofinality, so ρ cannot be good. Contradiction. □

For the rest of the proof fix $\langle G_\beta \mid \beta < \mu\rangle$ as above and δ as in the conclusion of the lemma.

Before we give the definition of the main forcing let us recall some relevant types of forcings:

(1) Magidor forcing adds a club set of order type λ in κ, starting with a Mitchell order increasing sequence $\langle U_\alpha \mid \alpha < \lambda\rangle$ of normal measures on κ.
(2) Supercompact Prikry forcing adds an increasing ω-sequence of sets $x_n \in (\mathcal{P}_\kappa(\eta))^V$ with $\eta = \bigcup_n x_n$, starting from a supercompactness measure U on $\mathcal{P}_\kappa(\eta)$.
(3) Gitik-Sharon forcing adds an increasing ω-sequence of sets $x_n \in (\mathcal{P}_\kappa(\kappa^{+n}))^V$ with $\kappa^{+\omega} = \bigcup_n x_n$, starting from a sequence $\langle U_n \mid n < \omega\rangle$ where each U_n is a supercompactness measure on $\mathcal{P}_\kappa(\kappa^{+n})$.

We can also define a "supercompact Magidor forcing", starting from an increasing sequence $\langle U_\alpha \mid \alpha < \lambda\rangle$ of supercompactness measures on $\mathcal{P}_\kappa(\eta)$ which adds an increasing and continuous λ-sequence of sets $x_\alpha \in \mathcal{P}_\kappa(\eta)$ with $\eta = \bigcup_{\alpha<\lambda} x_\alpha$. The main forcing described below starts from an increasing sequence $\langle U_\alpha \mid \alpha < \lambda\rangle$ where each U_α is a supercompactness measures on $\mathcal{P}_\kappa(\kappa^{+\alpha})$ and adds an increasing and continuous λ-sequence of sets $x_\alpha \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, for $\alpha < \lambda$ such that $\kappa^{+\lambda} = \bigcup_{\alpha<\lambda} x_\alpha$.

The main forcing:

Conditions are of the form $p = \langle g, H\rangle$, where:

(1) $\mathrm{dom}(g)$ is a finite subset of λ, and $\mathrm{dom}(H) = \lambda \setminus \mathrm{dom}(g)$.
(2) for each $\alpha \in \mathrm{dom}(g)$, $g(\alpha) \in B_\alpha$, and $\kappa \cap g(\alpha) = \kappa_{g(\alpha)} > \delta^{+\lambda+1}$.
(3) for $\alpha < \beta$, in $\mathrm{dom}(g)$, we have $g(\alpha) \subset g(\beta)$, $ot(g(\alpha)) < \kappa_{g(\beta)}$.
(4) for $\alpha \notin \mathrm{dom}(g)$ and $\alpha > \max(\mathrm{dom}(g))$, we have $H(\alpha) \in U_\alpha$, $H(\alpha) \subset B_\alpha$.
(5) for $\alpha \notin \mathrm{dom}(g)$ and $\alpha < \max(\mathrm{dom}(g))$, setting $\beta = \min(\mathrm{dom}(g) \setminus \alpha)$, we have $H(\alpha) \in U^\alpha_{\beta,g(\beta)}$ (the normal measure on $\mathcal{P}_{\kappa_{g(\beta)}}(\kappa^{+\alpha} \cap g(\beta))$)
(6) for $\alpha < \beta$, if $\alpha \in \mathrm{dom}(g)$, $\beta \notin \mathrm{dom}(g)$, then for each $z \in H(\beta)$, $g(\alpha) \subset z$ and $o.t.(g(\alpha)) < \kappa_z$ (this requirement is needed mainly for technical reasons).

$\langle g, H\rangle \leq \langle j, J\rangle$ iff $g \supset j$, for $\alpha \in \mathrm{dom}(g) \setminus \mathrm{dom}(j)$, $g(\alpha) \in J(\alpha)$, and for $\alpha \notin \mathrm{dom}(g)$, we have $H(\alpha) \subset J(\alpha)$

Proposition 9. *$\mathbb{P}$ has the $\mu = \kappa^{+\lambda+1}$ chain condition.*

Proof. For $\alpha < \lambda$, the number of possibilities for $g(\alpha)$ is at most $\mathrm{card}(\mathcal{P}_\kappa(\kappa^{+\alpha})) \leq \kappa^{+\alpha+1}$, and so $\mathrm{card}(\{g \mid \exists H \langle g, H\rangle \in \mathbb{P}\}) = \kappa^{+\lambda}$. Any two conditions $\langle g, H\rangle$, $\langle g, J\rangle$ are compatible. $\square$

Let G be $\mathbb{P}$ generic. Let $g^* = \bigcup_{\langle g,H\rangle \in G} g$. Then g^* is an increasing function with domain contained in λ and with $g^*(\alpha) \in \mathcal{P}_\kappa(\kappa^{+\alpha})$ for each $\alpha \in \mathrm{dom}(g^*)$.

Proposition 10. $\mathrm{dom}(g^*) = \lambda$.

Proof. Let $\alpha < \lambda$, we claim that $D_\alpha = \{\langle g, H\rangle \mid \alpha \in \mathrm{dom}(g)\}$ is dense.

Let $\langle g, H\rangle \in \mathbb{P} \setminus D_\alpha$. Let $\eta = \max(\mathrm{dom}(g) \cap \alpha)$. We will choose $x \in H(\alpha)$ as follows:

Case 1. $\mathrm{dom}(g) \setminus \alpha = \emptyset$. Then $H(\alpha) \in U_\alpha$. By Remark 7, for each ρ such that $\eta < \rho < \alpha$, we have $(\forall_{U_\alpha} x) H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x) \in U^\rho_{\alpha,x}$. By intersecting measure one sets, choose $x \in H(\alpha)$, such that for all ρ with $\eta < \rho < \alpha$, $H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x) \in U^\rho_{\alpha,x}$.

Case 2. $\mathrm{dom}(g) \setminus \alpha \neq \emptyset$, so let $\beta = \min(\mathrm{dom}(g) \setminus \alpha)$ and $y = g(\beta)$. Then $H(\alpha) \in U^\alpha_{\beta,y}$, and for ρ with $\eta < \rho < \alpha$, $H(\rho) \in U^\rho_{\beta,y}$. So, for each such ρ, by Remark 7 we have $(\forall_{U^\alpha_{\beta,y}} x) H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x) \in U^\rho_{\alpha,x}$. Again by intersecting measure one sets, choose $x \in H(\alpha)$, such that for all ρ with $\eta < \rho < \alpha$, $H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x) \in U^{\rho}_{\alpha,x}$.

Now set $g' = g \cup \{\langle \alpha, x\rangle\}$, and for $\rho \notin \mathrm{dom}(g) \cup \{\alpha\}$, $H'(\rho) = H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x) \in U^{\rho}_{\alpha,x}$ if $\eta < \rho < \alpha$, and $H'(\rho) = H(\rho)$ otherwise. Then $\langle g', H'\rangle \leq \langle g, H\rangle$ and $\langle g', H'\rangle \in D_\alpha$. $\square$

Set $x_\alpha = g^*(\alpha)$, and $\kappa_\alpha = \kappa \cap x_\alpha$.

Proposition 11. *$V[G] \models \mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ and $\mathrm{cf}((\kappa^{+\alpha+1})^V) = \mathrm{cf}(\lambda)$ for each $\alpha < \lambda$.*

Proof. It is enough to show that $\kappa^{+\lambda} = \bigcup_{\alpha<\lambda} x_\alpha$. Let $\eta < \kappa^{+\lambda}$, we claim that $D_\eta = \{\langle g, H\rangle \mid \eta \in \bigcup_{\alpha\in\mathrm{dom}(g)} g(\alpha)\}$ is dense.

Let $\langle g, H\rangle \in \mathbb{P} \setminus D_\eta$, $\beta = \max(\mathrm{dom}(g))$, and γ be such that $\beta < \gamma < \lambda$, and $\eta < \kappa^{+\gamma}$.

For ρ, such that $\beta < \rho < \gamma$, $H(\rho) \in U_\rho = [x \mapsto \overline{U^{\rho}_{\gamma,x}}]_{U_\gamma}$, and $H(\rho) = [x \mapsto \overline{H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x)}]_{U_\gamma}$. So, $Z_\rho = \{x \mid H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x) \in U^{\rho}_{\gamma,x}\} \in U_\gamma$.

Let $Z = \bigcap_{\beta<\rho<\gamma} Z_\rho$. $Z \in U_\gamma$, so we can choose $x \in Z$, such that $\eta \in x$. Define $\langle g', H'\rangle$, by $g' = g \cup \{\langle \gamma, x\rangle\}$, and for $\rho \notin \mathrm{dom}(g)$, let $H'(\rho)$ be $H(\rho) \cap \mathcal{P}_{\kappa_x}(\kappa^{+\rho} \cap x)$, if $\beta < \rho < \gamma$, and $H(\rho)$ otherwise. Then $\langle g', H'\rangle \in D_\eta$, and $\langle g', H'\rangle \leq \langle g, H\rangle$. $\square$

By a similar argument if $\alpha < \lambda$ is limit, then $x_\alpha = \bigcup_{\xi<\alpha} x_\xi$. (If $\eta < x_\alpha$, $\langle g, H\rangle \in G$ with $\alpha \in \mathrm{dom}(g)$, then the set $D_\eta = \{\langle g', H'\rangle \mid \eta \in \bigcup_{\xi\in\mathrm{dom}(g),\xi<\alpha} g(\xi)\}$ is dense below $\langle g, H\rangle$.) It follows that $sup_{\xi<\alpha}\kappa_\xi = \kappa_\alpha$.

We have to show that our forcing preserves κ. To prove this we will show that $\mathbb{P}$ satisfies the Prikry property. First we will show that below any condition we can factor the poset into $\mathbb{P}_0 \times ...\mathbb{P}_n$ where each $\mathbb{P}_i$ is below a condition of the form $\langle 0, H\rangle$.

Recall that the main forcing was defined starting from a supercompact κ and limit λ, normal measures U_α on $\mathcal{P}_\kappa(\kappa^{+\alpha})$, sets $B_\alpha \in U_\alpha$ for $\alpha < \lambda$ and functions $x \mapsto \overline{U^{\alpha}_{\beta,x}}$ such that

(1) for $\alpha < \beta$, $U_\alpha = [x \mapsto \overline{U^{\alpha}_{\beta,x}}]_{U_\beta}$
(2) for $\alpha < \beta < \gamma$ and $z \in B_\gamma$, $\overline{U^{\alpha}_{\gamma,z}} = [x \mapsto \overline{U^{\alpha}_{\beta,x}}]_{U^{\beta}_{\gamma,z}}$.

More generally we can define our forcing for any supercompact ν and limit α with the appropriate measures and functions satisfying (1) and (2).

Let $\langle g, H\rangle \in \mathbb{P}$ and suppose $\mathrm{dom}(g) = \{\alpha\}$ for some limit $\alpha < \lambda$ and $g(\alpha) = x$. Below this condition we can factor the poset to $\mathbb{P}_0 \times \mathbb{P}_1$ as follows:

For $\xi < \alpha$, let $v_\xi = \overline{U^{\xi}_{\alpha,x}}$. Then each v_ξ is a normal measure on $\mathcal{P}_{\kappa_x}(\kappa_x^{+\xi})$. Also for $\xi < \eta < \alpha$, let $y \mapsto \overline{v^{\xi}_{\eta,y}}$ be the function such that $v_\xi = [y \mapsto \overline{v^{\xi}_{\eta,y}}]_{v_\eta}$, where each $\overline{v^{\xi}_{\eta,y}}$ is a normal measure on $\mathcal{P}_{\kappa_x\cap y}((\kappa_x \cap y)^{+\xi})$. Applying the previous claims we can find sets $b_\rho \subset \overline{H(\rho)}$, $b_\rho \in v_\rho$, such that for $\xi < \eta < \rho < \alpha$ and $y \in b_\rho$, $\overline{v^{\xi}_{\rho,y}} = [z \mapsto \overline{v^{\xi}_{\eta,z}}]_{v^{\eta}_{\rho,y}}$. Here we use the notation defined in section 2, namely $v^{\xi}_{\rho,y}$ is obtained by lifting $\overline{v^{\xi}_{\rho,y}}$ to a normal measure on $\mathcal{P}_{\kappa_x \cap y}(\kappa_x^{+\xi} \cap y)$ by using the order isomorphism between $\kappa_x^{+\xi} \cap y$ and $(\kappa_x \cap y)^{+\xi}$.

Let $\mathbb{P}_0$ be defined from:

- the normal measures $v_\xi = \overline{U^{\xi}_{\alpha,x}}$ on $\mathcal{P}_{\kappa_x}(\kappa_x^{+\xi})$ for $\xi < \alpha$
- the sets $b_\xi \subset \overline{H(\xi)}$, $b_\xi \in v_\xi$.
- functions $y \mapsto \overline{v^{\xi}_{\eta,y}}$ for $\xi < \eta < \alpha$, where $v_\xi = [y \mapsto \overline{v^{\xi}_{\eta,y}}]_{v_\eta}$.

$\mathbb{P}_0$ adds a generic sequence $\langle y_\xi \mid \xi < \alpha \rangle$ such that $\bigcup_{\xi<\alpha} y_\xi = \kappa_x^{+\alpha}$. Using the order isomorphism between $\kappa_x^{+\alpha}$ and x, we can lift this chain to a chain $\langle y^*_\xi \mid \xi < \alpha \rangle$ whose union is x.

Let $\mathbb{P}_1$ be defined from the measures U_β, $\alpha < \beta < \lambda$ and the functions $x \mapsto \overline{U^{\beta}_{\gamma,x}}$ for $\alpha < \beta < \gamma < \lambda$.

By lifting the sets b_ξ, $\xi < \alpha$, we can find a condition $\langle g, H' \rangle \leq \langle g, H \rangle$ such that the forcing below $\langle g, H' \rangle$ is isomorphic to $\mathbb{P}_0 \times \mathbb{P}_1$. Here H' is such that for $\xi < \alpha$, $\overline{H'(\xi)} = b_\xi$. Conditions in $\mathbb{P}_0$ are below $\langle 0, \overline{H' \restriction \alpha} \rangle$ and conditions in $\mathbb{P}_1$ are below $\langle 0, H' \restriction (\lambda \setminus \alpha) \rangle$.

Similarly, if $\langle g, H \rangle \in \mathbb{P}$ is such that $\mathrm{dom}(g)$ has size n, there is a stronger condition $\langle g, H' \rangle$ such that we can factor the forcing below this condition as the product of $n+1$ forcings as above.

Proposition 12. *(Diagonalization Lemma) Let $\langle 0, H \rangle \in \mathbb{P}$, $\alpha < \lambda$, and $A \in U_\alpha$. Suppose $\langle g_x, H_x \rangle$ for $x \in A$ are conditions with $g_x = \{\langle \alpha, x \rangle\}$ and $\langle g_x, H_x \rangle \leq \langle 0, H \rangle$. Then there is a condition $\langle 0, H' \rangle \leq \langle 0, H \rangle$, such that if $\langle j, J \rangle \leq \langle 0, H' \rangle$ with $\alpha \in \mathrm{dom}(j)$, then there is an $x \in A$ for which $\langle j, J \rangle \leq \langle g_x, H_x \rangle$.*

Proof. For $\xi < \lambda$ define $H'(\xi)$ as follows:

(1) If $\xi < \alpha$, let $B_\xi = [x \mapsto \overline{H_x(\xi)}]_{U_\alpha}$. By previous lemmas, we have that $B_\xi \in U_\xi$. Also, for almost all $x \in A$, $B_\xi \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\xi}) = H_x(\xi)$. Let, $A_\xi = \{x \in A \mid B_\xi \cap \mathcal{P}_{\kappa_x}(x \cap \kappa^{+\xi}) = H_x(\xi)\}$. Now, set $H'(\xi) = B_\xi$, and $H'(\alpha) = H(\alpha) \cap \bigcap_{\xi<\alpha} A_\xi$.

(2) If $\xi > \alpha$, set $H'(\xi) = \triangle_{x \in A} H_x(\xi) = \{z \mid z \in \bigcap_{x \in A, x \prec z} H_x(\xi)\}$. Here "$x \prec z$" means $x \subset z$ and $o.t.(x) < \kappa_z$.

Then $\langle g, H' \rangle$ is stronger than $\langle g, H \rangle$ (since each $\langle g_x, H_x \rangle$ was). Also, suppose that $\langle j, J \rangle \leq \langle 0, H' \rangle$ with $\alpha \in \mathrm{dom}(g)$. Let $x = j(\alpha)$. Then $\langle j, J \rangle \leq \langle g_x, H_x \rangle$. □

Proposition 13. *(The Prikry property) Let $p = \langle g, H \rangle \in \mathbb{P}$, $\alpha < \lambda$ and let Φ be a statement in the forcing language. Then there is a condition $\langle g, H' \rangle \leq \langle g, H \rangle$, such that $\langle g, H' \rangle \parallel \Phi$.*

Proof. Using product factoring it is enough to show this for $p = \langle 0, H \rangle$.

Suppose that there is no direct extension of $\langle 0, H \rangle$ which forces the negation of Φ. We claim that then there is a finite sequence $\langle \alpha_1, ..., \alpha_k \rangle$ of points in λ, such that for any $\langle 0, H'\rangle \leq \langle 0, H\rangle$, there is a condition $\langle i, I\rangle \leq \langle 0, H'\rangle$ with $\mathrm{dom}(i) = \{\alpha_1, ..., \alpha_k\}$ which forces Φ. Otherwise, for each finite sequence $\overrightarrow{\alpha}$ of points in λ, we can fix a direct extension $q_{\overrightarrow{\alpha}}$ of $\langle 0, H\rangle$ as a witness. By taking intersection of measure one sets let q be stronger than each $q_{\overrightarrow{\alpha}}$. Then there is no condition stronger than q which forces Φ, contradiction.

Fix such a sequence $\overrightarrow{\alpha}$ and let k be the size of $\overrightarrow{\alpha}$. We will show the proof for $k = 1$, the general case is similar using induction and product factoring. Say $\overrightarrow{\alpha} = \{\alpha\}$.

By previous lemmas $A = \{x \mid (\exists \langle j, J\rangle \leq \langle 0, H\rangle) j = \{\langle \alpha, x\rangle\}\} \in U_\alpha$. For each $x \in A$, fix a witness $\langle g_x, H_x\rangle$. By shrinking H_x if necessary, we can assume that if there is a condition $\langle g_x, J\rangle \leq \langle g_x, H_x\rangle$ with $\langle g_x, J\rangle \Vdash \Phi$, then $\langle g_x, H_x\rangle \Vdash \Phi$.

Set $A^+ = \{x \in A \mid \langle g_x, H_x\rangle \Vdash \Phi\}$, and $A^- = \{x \in A \mid \langle g_x, H_x\rangle \nVdash \Phi\}$. Since U_α is an ultrafilter, one of these is in U_α, Let $A' = A^+$ if $A^+ \in U_\alpha$, and $A' = A^-$ if $A^- \in U_\alpha$.

Let $\langle 0, H'\rangle$ be as in the conclusion of the diagonalization lemma applied to $\langle 0, H\rangle$ and $\langle g_x, H_x\rangle$, $x \in A'$.

Then $\langle 0, H'\rangle \leq \langle 0, H\rangle$ and is such that if $\langle j, J\rangle \leq \langle g, H'\rangle$ and $\alpha \in \mathrm{dom}(j)$, then there is an $x \in A'$ such that $\langle j, J\rangle \leq \langle g_x, H_x\rangle$. Also note that $H'(\alpha) \subset A'$.

By our choice of α, there is a condition $\langle i, I\rangle \leq \langle 0, H'\rangle$ with $\mathrm{dom}(i) = \{\alpha\}$ and $\langle i, I\rangle \Vdash \Phi$. Then we have $x = i(\alpha) \in H'(\alpha) \subset A'$, and and by definition of $\langle 0, H'\rangle$, $\langle i, I\rangle \leq \langle g_x, H_x\rangle$. Since $i = g_x$ and $\langle i, I\rangle \Vdash \Phi$ by the way we chose each H_x, we have that $\langle g_x, H_x\rangle$ forces Φ. So, $A' = A^+$.

We have to show that $\langle 0, H'\rangle$ forces Φ. Otherwise there is a condition $\langle j, J\rangle \leq \langle 0, H'\rangle$ which forces the negation of Φ. We may assume that $\alpha \in \mathrm{dom}(j)$. Setting $y = j(\alpha)$, we get that $y \in A'$ (since $y = j(\alpha) \in H'(\alpha) \subset A'$), and by definition of $\langle 0, H'\rangle$, we have $\langle j, J\rangle \leq \langle g_y, H_y\rangle$. But $y \in A' = A^+$, contradiction.

□

Corollary 14. *Let $\langle g, H\rangle \in \mathbb{P}$, $\alpha \in \mathrm{dom}(g)$, α limit, and let Φ be a statement in the forcing language. Then there is a condition $\langle g, H'\rangle \leq \langle g, H\rangle$, such that if $\langle j, J\rangle \leq \langle g, H\rangle$ decides Φ, then $\langle j \restriction \alpha, J \restriction \alpha\rangle^\frown\langle g \restriction (\lambda \setminus \alpha), H' \restriction (\lambda \setminus \alpha)\rangle$ decides Φ.*

Proof. By shrinking H if necessary, we can factor the poset below $\langle g, H\rangle$ as $\mathbb{P}_0 \times \mathbb{P}_1$. Conditions in $\mathbb{P}_0$ are of the form $\langle j, J\rangle$ where $\mathrm{dom}(j)$ is a finite subset of α and $\mathrm{dom}(g) \restriction \alpha \subset \mathrm{dom}(j)$. More precisely, using the notation above, the conditions in $\mathbb{P}_0$ are below $\langle g \restriction \alpha, \overline{H \restriction \alpha}\rangle$. Conditions in $\mathbb{P}_1$ are below $\langle g \restriction (\lambda \setminus \alpha), H \restriction (\lambda \setminus \alpha)\rangle$.

Applying the Prikry property, for each $q \in \mathbb{P}_0$ we can get a condition $p_q \in \mathbb{P}_1$ such that $\mathrm{dom}(p_q) = \mathrm{dom}(g) \restriction (\lambda \setminus \alpha)$ and $\langle q, p_q\rangle$ decides Φ. The size of $\mathbb{P}_0$ is at most $2^{\kappa_{g(\alpha)}^{+\alpha}}$, which is less than $\kappa_{g(\beta)}$, where $\beta = \min(\mathrm{dom}(g) \setminus \alpha{+}1)$. So, by intersecting measure one sets we can find a condition p' such that p'

is a direct extension of $\langle g, H\rangle$ and for each $q \in \mathbb{P}_0$, $p' \restriction (\lambda \setminus \alpha) \leq p_q$. Then p' is the desired condition. $\square$

For $\alpha < \lambda$, let $\mathbb{P}_{\alpha,x} = \{\langle g \restriction \alpha+1, H \restriction \alpha+1\rangle \mid \langle g, H\rangle \in \mathbb{P}, g(\alpha) = x\}$. Also set $G_\alpha = \{\langle g \restriction \alpha+1, H \restriction \alpha+1\rangle \mid \langle g, H\rangle \in G\}$. Then G_α is generic in $\mathbb{P}_{\alpha,x_\alpha}$. Also, $\mathbb{P}_{\alpha,x_\alpha}$ has the $\kappa_\alpha^{+\alpha+1}$ chain condition.

Proposition 15. *Let $\tau < \kappa$ be a cardinal in V, such that for some limit $\alpha < \lambda$ and natural number k, $\kappa_\alpha^{+\alpha+1} \leq \tau < \kappa_{\alpha+k}$. Then $\mathbb{P}$ preserves τ. Moreover, $\mathrm{cf}^V(\tau) = \mathrm{cf}^{V[G]}(\tau)$.*

Proof. We will show that if $a \subset \tau$ and $a \in V[G]$, then $a \in V[G_\alpha]$.

Fix $\langle g, H\rangle \in G$ with $\{\alpha, \alpha+1, ..., \alpha+k\} \subset \mathrm{dom}(g)$. For each $\rho < \tau$, let $\langle g, H_\rho\rangle \leq \langle g, H\rangle$ be such that if $\langle j, J\rangle \leq \langle g, H\rangle$ decides "$\rho \in \dot{a}$", then $\langle j \restriction \alpha, J \restriction \alpha\rangle^\frown\langle g \restriction (\lambda \setminus (\alpha+1)), H_\rho \restriction (\lambda \setminus (\alpha+1))\rangle$ decides "$\rho \in \dot{a}$".

Set $H'(\xi) = \bigcap_{\rho<\tau} H_\rho(\xi)$, if $\xi > \alpha, \xi \notin \mathrm{dom}(g)$, and $H'(\xi) = H(\xi)$, if $\xi \in \alpha \setminus \mathrm{dom}(g)$. Then,

$a = \{\rho < \tau \mid (\exists q \in G_\alpha) q^\frown\langle g \restriction (\lambda \setminus (\alpha+1)), H' \restriction (\lambda \setminus (\alpha+1))\rangle \Vdash \rho \in \dot{a}\} \in V[G_\alpha]$.

$\mathbb{P}_{\alpha,x_\alpha}$ has the $\kappa_\alpha^{+\alpha+1}$ chain condition, so τ is a cardinal in $V[G_\alpha]$. By the above τ is still a cardinal in $V[G]$. $\square$

So, since a limit of cardinals is a cardinal, in $V[G]$ each $\kappa_{\alpha+1}$ is a cardinal, and since for limit α, $sup_{\xi<\alpha}\kappa_\xi = \kappa_\alpha$, and $sup_{\xi<\lambda}\kappa_\xi = \kappa$, we have that in $V[G]$ each κ_α is a cardinal and κ is a cardinal. By Proposition 11, we get $V[G] \models \kappa^+ = \mu = (\kappa^{+\lambda+1})^V$. Below we summarize facts about collapsing of cardinals and change of cofinalities.

- Let τ be a cardinal in V such that $\kappa_\alpha < \tau < \kappa_\alpha^{+\alpha+1}$, for α limit. Then $\mathrm{card}^{V[G]}(\tau) = \kappa_\alpha$, and if τ is regular in V, then in $V[G]$ the cofinality of τ is equal to $\mathrm{cf}(\alpha)$.
- Let τ be a cardinal in V such that $\kappa < \tau < \kappa^{+\lambda+1}$. Then $\mathrm{card}^{V[G]}(\tau) = \kappa$, and if τ is regular in V, then in $V[G]$ the cofinality of τ is equal to λ.

In particular, if τ is such that $\mathrm{cf}^V(\tau) \neq \mathrm{cf}^{V[G]}(\tau)$, then $\mathrm{cf}^{V[G]}(\tau) \leq \lambda$.

Proposition 16. *If $\langle A_\alpha \mid \alpha < \lambda\rangle \in V$ is such that each $A_\alpha \in U_\alpha$, then $x_\alpha \in A_\alpha$ for all sufficiently large α.*

Proof. $D = \{\langle g, H\rangle \mid (\exists\beta < \lambda) \max(\mathrm{dom}(g)) < \beta, (\forall\alpha > \beta) H(\alpha) \subset A_\alpha\}$ is dense $\square$

Proposition 17. $V[G] \models A \subset ON, o.t.(A) = \tau, \lambda < \mathrm{cf}^V(\tau) = \tau \leq \delta^{+\lambda+1}$, *then there is a $B \in V$ such that $B \subset A$, and B is unbounded in A.*

Proof. Recall that δ was fixed to be such that the scale $\langle G_\beta \mid \beta < \mu\rangle$ in $\prod_{\alpha<\lambda} \kappa^{+\alpha+1}$ has stationary many bad points of cofinality $\delta^{+\lambda+1}$ and for $\langle g, H\rangle \in \mathbb{P}$, $\alpha \in \mathrm{dom}(g)$, we have $\kappa_{g(\alpha)} > \delta^{+\lambda+1}$. Also note that in $V[G]$ $\mathrm{cf}(\tau) = \tau > \lambda$.

For $p \in \mathbb{P}$ let $A_p = \{\alpha \mid p \Vdash \alpha \in \dot{A}\}$; $A = \bigcup_{p \in G} A_p$. Fix $\alpha < \lambda$ such that $\bigcup_{\langle g,H\rangle \in G, \alpha = \max(\mathrm{dom}(g))} A_p$ unbounded in A.

The number of possibilities for $\mathrm{dom}(g)$ with maximum α is less than λ, so we can fix a set $D = \{\alpha_0, ..., \alpha_n\}$ where $\alpha_n = \alpha$, so that $A' = \bigcup_{\langle g,H\rangle \in G, \mathrm{dom}(g) = D} A_p$ is unbounded in A. I.e. we can fix g (by taking $g(\alpha_i) = x_{\alpha_i}$).

In $V[G]$ let $f : \tau \to A'$ enumerate A'. Then by definition of A', for each $\gamma < \tau$, fix $\langle g, H_\gamma\rangle$ deciding $f(\gamma)$.

For $\alpha \in \lambda \setminus \mathrm{dom}(g)$, set $H(\alpha) = \bigcap_{\gamma < \tau} H_\gamma(\alpha)$. If $\alpha < \max(\mathrm{dom}(g))$ and $\beta = \min(\mathrm{dom}(g) \setminus \alpha)$, then $\tau \leq \delta^{+\lambda+1} < \kappa_{g(\beta)}$, and so $H(\alpha) \in U^{\alpha}_{\beta, g(\beta)}$. Also if $\alpha > \max(\mathrm{dom}(g))$, $H(\alpha) \in U_\alpha$. So, $\langle g, H\rangle$ is a condition and it decides f. □

Lemma 18. *Let $\eta : \lambda \to \lambda + 2$, with $\alpha < \eta(\alpha)$, and $\eta(\alpha)$ a successor ordinal. Let in $V[G]$, $h \in \prod_{\alpha<\lambda} \kappa_\alpha^{+\eta(\alpha)}$, then there is a sequence of functions $\langle H_\alpha \mid \alpha < \lambda\rangle$ in V, such that* $\mathrm{dom}(H_\alpha) = B_\alpha$*, $H_\alpha(x) < \kappa_x^{+\eta(\alpha)}$, and $h(\alpha) < H_\alpha(x_\alpha)$ for all large α.*

Proof. For simplicity suppose $\langle 0, H\rangle \in G$ forces that $\dot{h}$ is as in the statement of the lemma. For $\alpha < \lambda$, and $x \in B_\alpha$, define:

$H_\alpha(x) = \sup\{\gamma \mid \gamma < \kappa_x^{+\eta(\alpha)}, (\exists \langle g, H\rangle \in \mathbb{P})\alpha = \max(\mathrm{dom}(g)), g(\alpha) = x$ and $\langle g, H\rangle \Vdash \dot{h}(\alpha) = \gamma\} + 1$

Now, for $\beta < \alpha$ in the domain of g, the possible values of $g(\beta)$ are at most $\mathrm{card}(\mathcal{P}_{\kappa_x}(\kappa^{+\beta} \cap x)) = (\kappa_x^{+\beta})^{<\kappa_x} \leq \kappa_x^{+\beta+1} \leq \kappa_x^{+\alpha} < \kappa_x^{+\eta(\alpha)}$, where $x = g(\alpha)$. Since $\eta(\alpha)$ is a successor, it follows that $H_\alpha(x) < \kappa_x^{+\eta(\alpha)}$.

For each $\alpha < \lambda$ and $x \in H(\alpha)$, let $g_{x,\alpha} = \{\langle \alpha, x\rangle\}$. By the Prikry property and the definition of $H_\alpha(x)$, we can find $\langle g_{x,\alpha}, H_{x,\alpha}\rangle$ such that $\langle g_{x,\alpha}, H_{x,\alpha}\rangle$ forces that $\dot{h}(\alpha) < H_\alpha(x)$. To do this, for each $\gamma < \kappa_x^{+\eta(\alpha)}$, let $\langle g_{x,\alpha}, H^\gamma_{x,\alpha}\rangle$ be given by Corollary 14 applied to "$\dot{h}(\alpha) = \gamma$". Then let $H_{x,\alpha}(\xi) = \bigcap_{\gamma < \kappa_x^{+\eta(\alpha)}} H^\gamma_{x,\alpha}(\xi)$ for $\xi \neq \alpha$.

Apply the diagonalization lemma to $\langle g_{x,\alpha}, H_{x,\alpha}\rangle$, $x \in H(\alpha)$ and get $p_\alpha = \langle 0, H_\alpha\rangle \in G$ to be such that if $\langle j, J\rangle \leq p_\alpha$ with $\alpha \in \mathrm{dom}(j)$, then $\langle j, J\rangle \leq \langle g_{x,\alpha}, H_{x,\alpha}\rangle$, where $x = j(\alpha)$. Then p_α forces that $\dot{h}(\alpha) < H_\alpha(\dot{x}_\alpha)$. By intersecting measure one sets, choose p to be stronger than each p_α. Then p forces that $\dot{h}(\alpha) < H_\alpha(\dot{x}_\alpha)$ for all $\alpha < \lambda$. □

4. The Bad Scale

Let in V, $\langle G_\beta \mid \beta < \mu\rangle$ be the bad scale in $\prod_{\alpha<\lambda} \kappa^{+\alpha+1}$ which we fixed in the beginning of the last section. Define in $V[G]$, $\langle g_\beta \mid \beta < \mu\rangle$ in $\prod_{\alpha<\lambda} \kappa_\alpha^{+\alpha+1}$ as follows:

$\forall \alpha < \lambda$, $\forall \eta < \kappa^{+\alpha+1}$, fix $F^\eta_\alpha : B_\alpha \longrightarrow V$, such that $\forall x F^\eta_\alpha(x) < \kappa_x^{+\alpha+1}$, and $[F^\eta_\alpha]_{U_\alpha} = \eta$. For $\beta < \mu$, set $g_\beta(\alpha) = F_\alpha^{G_\beta(\alpha)}(x_\alpha)$.

If $\beta < \gamma < \lambda$, then for all large α, $G_\beta(\alpha) < G_\gamma(\alpha)$, so for all large α, $[F_\alpha^{G_\beta(\alpha)}]_{U_\alpha} < [F_\alpha^{G_\gamma(\alpha)}]_{U_\alpha}$, so by Proposition 16, for all large α, $F_\alpha^{G_\beta(\alpha)}(x_\alpha) < F_\alpha^{G_\gamma(\alpha)}(x_\alpha)$, so $\langle g_\beta \mid \beta < \mu\rangle$ is increasing.

Also, if in $V[G]$, $h \in \prod_{\alpha<\lambda} \kappa_\alpha^{+\alpha+1}$, fix $\langle H_\alpha \mid \alpha < \lambda\rangle \in V$ as in the conclusion of Lemma 18. In V, define $h^* \in \prod_{\alpha<\lambda} \kappa^{+\alpha+1}$ by $h^*(\alpha) = [H_\alpha]_{U_\alpha}$. Then $h^* \in \prod_{\alpha<\lambda} \kappa^{+\alpha+1}$, and so we can fix $\beta < \mu$, such that $h^* < G_\beta$. Then for all large α, $[H_\alpha]_{U_\alpha} = h^*(\alpha) < G_\beta(\alpha) = [F_\alpha^{G_\beta(\alpha)}]_{U_\alpha}$, and so for all large α, $H_\alpha(x_\alpha) < F_\alpha^{G_\beta(\alpha)}(x_\alpha)$, so $h <^* g_\beta$.

Thus, $\langle g_\beta \mid \beta < \mu\rangle$ is a scale. It remains to show that it is not good.

Lemma 19. *Suppose $\beta < \mu$ with $\mathrm{cf}(\beta) = \delta^{+\lambda+1}$ is a good point for $\langle g_\gamma \mid \gamma < \mu\rangle$ in $V[G]$. Then β is a good point in V for $\langle G_\gamma \mid \gamma < \mu\rangle$.*

Proof. First note that $\mathrm{cf}(\beta) = \delta^{+\lambda+1}$ in V as well. Fix unbounded $A^* \subset \beta$, and $\nu < \lambda$ witnessing goodness of β in $V[G]$. Then by Proposition 17, there is an unbounded set $A \subset A^*$ in V. Then A and ν witness goodness, so let $p = \langle h, H\rangle \in G$ be such that $p \Vdash (\forall \alpha > \nu)\langle g_\gamma(\alpha) \mid \gamma \in A\rangle$ is increasing.

Claim 20. *$\forall \alpha > \max(\nu, \max(\mathrm{dom}(h)))$, for U_α-almost all $y \in H(\alpha)$, we have that $\langle F_\alpha^{G_\gamma(\alpha)}(y) \mid \gamma \in A\rangle$ is increasing.*

Proof. Otherwise, for some $\alpha > \max(\nu, \max(\mathrm{dom}(h)))$, we can find $y_\alpha \in H(\alpha)$, such that $\langle F_\alpha^{G_\gamma(\alpha)}(y_\alpha) \mid \gamma \in A\rangle$ is not increasing and $(\exists q \le p) q = \langle h', H'\rangle, \alpha \in \mathrm{dom}(h'), h'(\alpha) = y_\alpha$. Fix such a condition q. Then

$(\forall \gamma < \mu) q \Vdash \dot{g}_\gamma(\alpha) = F_\alpha^{G_\gamma(\alpha)}(y_\alpha)$,

$q \le p \Rightarrow q \Vdash \langle \dot{g}_\gamma(\alpha) \mid \gamma \in A\rangle$ is increasing.

But, $\langle F_\alpha^{G_\gamma(\alpha)}(y_\alpha) \mid \gamma \in A\rangle$ is not increasing. Contradiction. □

So, we have that for all large α, $\langle [F_\alpha^{G_\gamma(\alpha)}]_{U_\alpha} \mid \gamma \in A\rangle$ is increasing, and so for all large α, $\langle G_\gamma(\alpha) \mid \gamma \in A\rangle$ is increasing. Thus, β is good for $\langle G_\gamma \mid \gamma < \mu\rangle$ in V. □

Since $\mathbb{P}$ has the μ chain condition and there are stationary many bad points with cofinality $\delta^{+\lambda+1}$ in V for $\langle G_\gamma \mid \gamma < \mu\rangle$, we get that $\langle g_\gamma \mid \gamma < \mu\rangle$ is not good.

5. Defining the very good scale

Recall that in V we have $\langle F_\gamma^\xi \mid \gamma < \mu, \xi < \lambda\rangle$, each $F_\gamma^\xi : \kappa \to \kappa$, such that for all $\xi < \lambda, \gamma < \mu$, $j_{U_\xi}(F_\gamma^\xi)(\kappa) = \gamma$. Since $j_{U_\xi}(F_\gamma^\xi)(\kappa) < \kappa^{+\lambda+1}$, without loss of generality we may assume that for all $\eta < \kappa$, $F_\gamma^\xi(\eta) < \eta^{+\lambda+1}$.

In $V[G]$, define $\langle f_\gamma \mid \gamma < \mu\rangle$ in $\prod_{\xi<\lambda} \kappa_\xi^{+\lambda+1}$, by $f_\gamma(\xi) = F_\gamma^\xi(\kappa_\xi)$.

Proposition 21. *$\langle f_\gamma \mid \gamma < \mu\rangle$ is a scale*

Proof. If $\gamma < \delta < \mu$,

then, $\forall\xi$, $j_{U_\xi}(F^\xi_\gamma)(\kappa) = \gamma < \delta = j_{U_\xi}(F^\xi_\delta)(\kappa)$

$\Rightarrow \forall\xi\ \forall_{U_\xi} x$, $F^\xi_\gamma(\kappa_x) < F^\xi_\delta(\kappa_x)$,

so by Proposition 16, for all large $\xi < \lambda$, $F^\xi_\gamma(\kappa_\xi) < F^\xi_\delta(\kappa_\xi)$, i.e. $f_\gamma <^* f_\delta$. So the f_γ's are increasing.

Suppose $h \in \prod_{\xi<\lambda} \kappa_\xi^{+\lambda+1}$, fix $\langle H_\xi \mid \xi < \lambda\rangle$ in V as in the conclusion of Lemma 18. For $\xi < \lambda$, let $\gamma_\xi = [y \mapsto H_\xi(y)]_{U_\xi} < \kappa^{+\lambda+1} = \mu$. Fix $\gamma < \mu$, such that for all $\xi < \lambda$, $\gamma_\xi < \gamma$, and so $\gamma_\xi < j_{U_\xi}(F^\xi_\gamma)(\kappa) = \gamma$,

$\Rightarrow (\forall\xi < \lambda)(\forall_{U_\xi} x) H_\xi(x) < F^\xi_\gamma(\kappa_x)$

$\Rightarrow$ for all large $\xi < \lambda$, $H_\xi(x_\xi) < F^\xi_\gamma(\kappa_\xi)$, i.e. for all large ξ, $h(\xi) < f_\gamma(\xi)$, and so the f_γ's are cofinal. $\square$

Proposition 22. *$\langle f_\gamma \mid \gamma < \mu\rangle$ is very good.*

Proof. Let $\gamma < \mu$ with $\lambda < \mathrm{cf}(\gamma) < \kappa$ (in $V[G]$, and so in V, since $\mathrm{cf}(\gamma)^V = \mathrm{cf}(\gamma)^{V[G]}$). Let $A \subset \gamma$ with $o.t.(A) = \mathrm{cf}(\gamma)$, $A \in V$.

Let $\xi < \lambda$. Since for all $\delta, \eta \in A$, with $\delta < \eta$, $j_{U_\xi}(F^\xi_\delta)(\kappa) = \delta < \eta = j_{U_\xi}(F^\xi_\eta)(\kappa)$, we have that $Z_{\delta,\eta} = \{x \mid F^\xi_\delta(\kappa_x) < F^\xi_\eta(\kappa_x)\} \in U_\xi$. Using $\lambda < \mathrm{card}(A) < \kappa$, we get $Z = \bigcap_{\delta<\eta;\delta,\eta\in A} Z_{\delta,\eta} \in U_\xi$.

So, $\forall\xi < \lambda$, $\forall_{U_\xi} x$, $\langle F^\xi_\delta(\kappa_x) \mid \delta \in A\rangle$ is increasing.

So for all large ξ, $\langle F^\xi_\gamma(\kappa_\xi) \mid \delta \in A\rangle$ is increasing. I.e. $\langle f_\delta(\xi) \mid \delta \in A\rangle$ is increasing. $\square$

6. Conclusion

To summarize, we have showed that if we start with a supercompact cardinal κ, and a regular λ, then there is a generic extension in which the following holds:

(1) κ has cofinality λ
(2) there is a very good scale at κ
(3) SCH fails at κ
(4) there is a bad scale at κ

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