A DIFFERENTIAL APPROACH TO THE EULER CHARACTERISTIC

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Abstract. This paper introduces some of the results of Morse theory. These results will be applied to show that every compact, boundaryless, and orientable smooth manifold has the homotopy type of a CW complex. In turn, this will show how one can compute the Euler characteristic, which is a topological invariant, using analysis.

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1. Introduction

The version of Poincare-Hopf Index theorem that the writer is familiar with is as follows:

Theorem 1.1. *Suppose that M is a compact, boundaryless and orientable manifold and that Y is a smooth vector field on M with isolated zeroes. Then,*

$$\sum_{p:Y(p)=0} Ind(Y,p) = \chi(M)$$

In the statement of Theorem 1.1, $\chi(M)$ is defined to be the transversal intersection number between a smooth vector field and the zero section of the manifold. More explicitly, if $\overrightarrow{v}$ is a smooth vector field on M, then

$$\chi(M) = V(M) \# \mathcal{O}(M)$$

where

$$V(x) = (x, \overrightarrow{v}(x)) \in TM \text{ for all } x \in M, \text{ and}$$
$$\mathcal{O}(x) = (x, 0) \in TM \text{ for all } x \in M.$$

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This intersection number is a homotopy invariant. Since any two vector fields on M are homotopic by a linear homotopy, $\chi(M)$ does not depend on the choice of the vector field $\vec{v}$.

In fact, a different definition of $\chi(M)$ tells us that the Euler characteristic is a much tougher invariant; $\chi(M)$ is the alternating sum $\sum_{n\in\mathbb{Z}_{\geq 0}}(-1)^n \text{rank} H_n(M)$ where $H_n(M)$ denotes the n-th singular homology group of M. If two spaces have the same homotopy type, then they have the same homology groups. So, $\chi(M)$ only depends on the homotopy type of the manifold M.

We can see that the first definition of $\chi(M)$ is differential while the second definition of $\chi(M)$ is more algebraic. Using some results from Morse theory, we can establish the equality

$$\sum_{n\in\mathbb{Z}_{\geq 0}} (-1)^n \text{rank} H_n(M) = V(M)\#\mathcal{O}(M)$$

and connect the two seemingly unrelated formulations of $\chi(M)$.

2. Some Terminology

Let $M \subseteq \mathbb{R}^k$ be a smooth n-dimensional manifold embedded in an ambient Euclidean space. For a point $q \in M$, a **chart** at $q \in M$ is a pair (U, ϕ) such that U is an open set in $\mathbb{R}^n$ that contains the point q and

$$\phi : U \to M$$

is a diffeomorphism onto its image. For our convenience, we will further require that $0 \in U$ and $\phi(0) = q$.

Let

$$f : M \to \mathbb{R}$$

be a smooth and real-valued funtion. For a point $p \in M$, we say that p is a **critical point** of f if the derivative of $f : M \to \mathbb{R}$ at p is not a submersion. In our particular case,

$$Df_{|p} : T_p(M) \to \mathbb{R}$$

must be the zero linear transformation. For the purposes of computation, we can pick a chart (U, ϕ) at p and compute

$$D(f \circ \phi)_{|0} : \mathbb{R}^n \to \mathbb{R}$$

Since $D\phi$ maps $\mathbb{R}^n$ to $T_p(M)$ bijectively, p is a critical point of f if and only if

$$D(f \circ \phi) = (0, \dots, 0).$$

The map

$$f \circ \phi : U \to \mathbb{R}$$

is a smooth and real-valued function defined on an open set U in $\mathbb{R}^n$. The **hessian matrix** of $f \circ \phi$ at p is the matrix

$$H = (D_{ij} f \circ \phi_{|0})_{1\leq i,j\leq n} = \begin{pmatrix} D_{11} f \circ \phi_{|0} & \dots & D_{1n} f \circ \phi_{|0} \\ \vdots & \ddots & \vdots \\ D_{n1} f \circ \phi_{|0} & \dots & D_{nn} f \circ \phi_{|0} \end{pmatrix}.$$

Suppose that $p \in M$ is a critical point of f. Then p is a **nondegenerate critical point** of f if the hessian of $(f \circ \phi)$ at 0 is an invertible matrix. Lemma 2.2 will show that this definition of nondegenerate critical points is independent of the choice of the chart (U, ϕ)

We will assume the following result:

Lemma 2.1. *Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}^m$ be a C^1 function. For $i, j \in \mathbb{N}$ such that $1 \leq i, j \leq n$, if $D_{ij} f$ exists on U and is continuous on U, then $D_{ji} f$ exists and $D_{ij} f_{|x_0} = D_{ji} f_{|x_o}$ for any $x_0 \in U$.*

With notation as above, we write the smooth function $f \circ \phi$ as h and the hessian of $f \circ \phi$ at p as H_p. Then, H_p defines a bilinear form on $\mathbb{R}^n$ given by

$$H_p(v, w) = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \begin{pmatrix} D_{11} h_{|p} & \dots & D_{1n} h_{|p} \\ \vdots & \ddots & \vdots \\ D_{n1} h_{|p} & \dots & D_{nn} h_{|p} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_{j=1}^{n} \sum_{i=1}^{n} v_j D_{ji} h_{|p} w_i$$

where $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ are vectors in $\mathbb{R}^n$.

A **homogeneous quadratic form in n-variables**, $x_1, \dots, x_n$, is a polynomial of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i b_{ij} x_j.$$

We see that each term has degree two and that the polynomial can be written in the form

$$XBX^T \text{ where } X = (x_1, \dots, x_n) \text{ and } B = (b_{ij})_{1 \leq i,j \leq n}.$$

In fact, if B is a symmetric matrix then B uniquely determines the quadratic form; there is a bijection between $n \times n$-symmetric matrices and quadratic forms in n-variables. In particular, the Hessian matrix H_p can be viewed as a quadratic form by Lemma 2.2.

We define the **index** of p to be the maximal dimension of a subspace of $\mathbb{R}^n$ such that H_p is negative definite. More explicitly, the index of p is the dimension of the maximal subspace $S \subseteq \mathbb{R}^n$ such that for any non-zero vector s in S, $H_p(s, s) < 0$. The following lemmas show that our definition of the index is independent of the chart (U, ϕ).

Lemma 2.2. *Suppose that (U_1, ϕ_1) and (U_2, ϕ_2) are two charts at p. Then there exists an open neighborhood W of 0 in $\mathbb{R}^n$ such that we can define the function*

$$\psi : W \to \mathbb{R}^n$$

by

$$\psi = \phi_1^{-1} \circ \phi_2.$$

Let H_1 and H_2 be the hessian matrix of $f \circ \phi_1$ and $f \circ \phi_2$, respectively. Then,

$$H_2 = (D\psi_{|0})^T H_1 (D\psi_{|0})$$

Proof. See page 42 of "Differential Topology" by Guillemin and Pollack. □

Lemma 2.3. *A change of basis replaces a quadratic form with matrix A by a quadratic form with matrix $P^T AP$, where P is invertible. On the other hand, if P is an invertible matrix, then a change of a quadratic form represented by the matrix A to a form represented by the matrix $P^T AP$ changes the basis in which we view the form.*

Proof. For the proof of the first statement, see page 268 of "A Survey of Modern Algebra" by Birkhoff and Maclane. □

The two lemmas above tell us that choosing a different chart to compute the Hessian matrix of a smooth and real-valued function on a manifold amounts to choosing a different basis for the quadratic form determined by the Hessian matrix. Hence, what we wish to show is that the dimension of the maximal vector subspace on which the Hessian is negative definite is invariant under a change of basis.

We will assume the following facts:

Lemma 2.4. *By a non-singular linear transformation, any quadratic form that is not identically zero can be reduced to a form with a nonzero leading coefficient.*

Lemma 2.5. *By non-singular linear transformations of the variables, a quadratic form Q can be reduced to a diagonal quadratic form*

$$d_1y_1^2 + d_2y_2^2 + \cdots + d_ry_r^2 \text{ where } d_i \neq 0 \text{ for each } i = 1, \ldots, r.$$

Moreover, the number r, which we call the rank, of nonzero diagonal entries is an invariant of the given form Q.

Theorem 2.6. *Any quadratic form Q can be reduced by non-singular linear transformations to a form*

$$Q(\xi) = z_1^2 + \cdots + z_p^2 - z_{p+1}^2 - \cdots - z_r^2.$$

The proofs of the results above can be found in "A Survey of Modern Algebra" by Birkhoff and Maclane.

Proposition 2.7. *Let $f : M \to \mathbb{R}$ be a smooth and real-valued function and let p be a nondegenerate critical point of f. Then, the index of f at p is well-defined.*

Proof. By Lemmas 2.2, 2.3 and Theorem 2.6, there exists a chart (U, ψ) at p such that the hessian of $f \circ \psi$ at 0 is a diagonal matrix, the diagonal entries of which are either 1 or -1. Set H to be the hessian of $f \circ \psi$ at 0 and λ to be the number of negative diagonal entries of the Hessian. The maximal dimension of the vector subspace $V \subseteq \mathbb{R}^n$ on which H is negative definite is λ. By definition, λ is the index of p.

Let (W, ϕ) be another chart at p. Set H' to be the hessian of $f \circ \phi$ at 0 and set $V' \subseteq \mathbb{R}^n$ to be the vector subspace of maximal dimension on which the hessian of $f \circ \phi$ at 0 is negative definite. Suppose for contradiction that V' has dimension ρ, where $\rho \neq \lambda$. We can assume without loss of generality that $\rho > \lambda$. As ψ and ϕ are diffeomorphisms,

$$f \circ \phi = (f \circ \psi) \circ \gamma$$

where

$$\gamma = \psi^{-1} \circ \phi.$$

γ is a diffeomorphism between two open subsets of $\mathbb{R}^n$. By Lemma 2.2,

$$H' = (D\gamma_{|0})^T (H)(D\gamma_{|0}).$$

Then, there exists a vector $w \in \mathbb{R}^n$ such that $w \in V'$ and $D\gamma_{|0}(w) \notin V$, but

$$H(D\gamma_{|0}(w), D\gamma_{|0}(w)) = H'(w, w) < 0.$$

This contradicts the condition that V is the maximal subspace on which H is negative definite. Therefore, $\rho = \lambda$. □

We will call a smooth and real-valued function $f : M \to \mathbb{R}$ a **Morse function** if all of its critical points are nondegenerate.

3. The Morse lemma

The Morse lemma says that a Morse function f on M can be viewed as a very neat polynomial in a small neighborhood of its nondegenerate critical point. Although it is an important lemma, proving this result in this paper will demand too many pages. Instead, we will state a series of lemmas that will lead to the proof of the Morse lemma.

Lemma 3.1. *Suppose that $f : U \to \mathbb{R}$ is a smooth function defined on an open subset, U, of $\mathbb{R}^n$ that contains the origin and $f(0) = 0$.Then, for an open and convex neighborhood V of the origin that is contained in U,*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n)$$

for some suitable smooth functions $g_1, \ldots, g_n$ defined on V such that $g_i(0) = D_i f_{|0}$ for each $i = 1, \ldots, n$.

Proof. See Milnor's "Morse Theory" for a proof. □

As before, let M be a smooth n-dimensional manifold embedded in some ambient Euclidean space $\mathbb{R}^k$. Suppose that $f : M \to \mathbb{R}$ is a Morse function on M, and let $p \in M$ be a non-degenerate critical point of f.

Lemma 3.2. *Suppose that (U, ϕ) is a chart at p. Then there exists an open subset W contained in U such that W contains the origin and*

$$(f \circ \phi)(x_1, \ldots, x_n) - f(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i h_{ij}(x_1, \ldots, x_n) x_j$$

for any $x = (x_1, \ldots, x_n) \in W$.The function h_{ij} is smooth on W and

$$h_{ij}(0) = D_{ij}(f \circ \phi)_{|0}.$$

Moreover the matrix $(h_{ij}(x_1, \ldots, x_n))_{1 \le i,j \le n}$ is symmetric and invertible when evaluated at any point $x \in W$.

Proof. (Sketch) We can iterate Lemma 3.1 two times so that

$$(f \circ \phi)(x_1, \ldots, x_n) - f(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i h'_{ij}(x_1, \ldots, x_n) x_j.$$

Let $H'(x)$ be the matrix whose coefficients are $h'_{ij}(x)$ and let

$$H(x) = (H'(x) + H'(x)^T)/2.$$

We can guarantee the last condition by observing that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i h'_{ij}(x_1, \ldots, x_n) x_j = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} H(x) \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^T$$

□

Lemma 3.3. *Let $U \subseteq \mathbb{R}^k$ be an open subset. Let n be a natural number such that $0 \leq n < k$. Let*

$$g : U \to \mathbb{R}$$

be a smooth function such that

$$g(x_1, \ldots, x_k) = \sum_{i=n}^{k} \sum_{j=n}^{k} x_i c_{ij}(x_1, \ldots, x_n, \ldots, x_k) x_j$$

where each c_{ij} is a smooth function defined on U. Suppose that $(c_{ij}(0, \ldots, 0))_{n \leq i,j \leq k}$ is not the zero matrix and $(c_{ij}(x_1, \ldots, x_k))_{n \leq i,j \leq k}$ is a symmetric matrix when evaluated at any point $(x_1, \ldots, x_k)$ in U. Then, there exist an open subset W in U that contains the origin and a diffeomorphism $\psi : W \to \psi(W)$ such that $\psi(0) = 0$ and

$$(g \circ \psi)(x_1, \ldots, x_k) = \sum_{i=n}^{k} \sum_{j=n}^{k} x_i d_{ij}(x_1, \ldots, x_n, \ldots, x_k) x_j$$

where d_{ij} is a smooth function defined on W and $d_{nn}(w) \neq 0$ for any $w \in W$.

Lemma 3.4. *There exists a chart (W, ψ) at p such that*

$$(f \circ \psi)(x_1, \ldots, x_n) - f(p) = d_1(x_1, \ldots, x_n) + \cdots + d_n(x_1, \ldots, x_n)$$

where $d_i(w) \neq 0$ for any $w \in W$ and for any i.

Theorem 3.5. *(The Morse lemma)*

Let p be a nondegenerate critical point for f. Then there exists a chart (U, ψ) at p such that

$$f \circ \psi(x_1, ..., x_n) = f(p) - (x_1)^2 - \cdots - (x_\lambda)^2 + (x_{\lambda+1})^2 + \cdots + (x_n)^2$$

for all $(x_1, \ldots, x_n) \in U$, where λ is the index of f at p.

The techniques used for the proofs of Lemmas 3.3, 3.4 and Theorem 3.5 are adaptations of the proofs of Lemmas 2.4, 2.5 and Theorem 2.6 found in "A Survey of Modern Algebra". Although Milnor's "Morse Theory" has a proof of the Morse lemma that does the job in one go, it may be more manageable to prove the result one step at a time.

4. Homotopy Type

For this section, we assume that f is a Morse function on M and we let

$$M^a = f^{-1}(-\infty, a] = \{q \in M : f(q) \leq a\}.$$

We will assume an important theorem:

Theorem 4.1. *Let $a < b$ and suppose that $f^{-1}[a, b]$ is a compact set that does not contain any critical points of f. Then, M^a is diffeomorphic to M^b. Moreover, M^a is a deformation retract of M^b so that the inclusion map $i : M^a \to M^b$ is a homotopy equivalence.*

The proof of this theorem uses the gradient field of f, which we will talk about in Section 6. The zeroes of the gradient field coincide with the critical points of f. As a result, the gradient field of f is a smooth vector field on M such that it does not vanish on the compact set $f^{-1}[a, b]$. Then, we can view the gradient field as a smooth assignment of nonzero vectors that are orthogonal to the level sets of f.

The idea is to "flow" along these orthogonal vectors so that M^b moves to M^a. For a complete proof, see page 12 of Milnor's "Morse Theory".

Now, we prove one of main results of this paper.

Theorem 4.2. *Let $f : M \to \mathbb{R}$ be a Morse function on M and let $c \in \mathbb{R}$ be a critical value of f. Let p be the only critical point in $f^{-1}(c)$ and let λ be the index of p. Suppose that $f^{-1}[c-\delta, c+\delta]$ is a compact set and contains no critical points of f other than p for some $\delta > 0$. Then, we can find a positive number $\epsilon > 0$ small enough such that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell e^λ attached; more explicitly, $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^\lambda$ which is homeomorphic to $M^{c-\epsilon} \cup_{|\partial\psi} D^\lambda$, for some attaching map $\partial\psi$.*

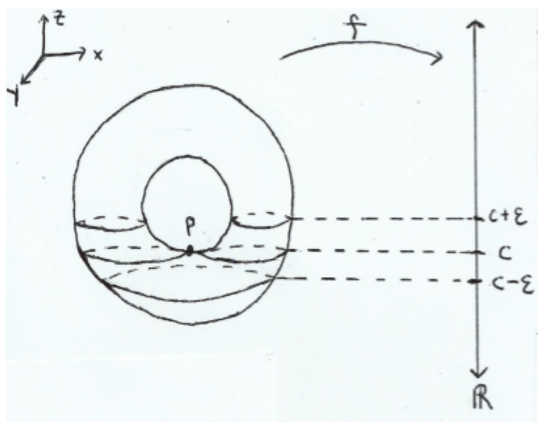
During the course of the proof, it will be useful to keep a concrete example handy. Consider a 2-torus embedded in $\mathbb{R}^3$ and consider the height function

$$f : M \to \mathbb{R}$$

on M given by

$$f(x, y, z) = z.$$

In the figure below, p is a nondegenerate critical point of index 1.



Proof. The first step of this proof is to show that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ and a small region that contains p. To do this, we construct a function F that takes on values that are smaller than f in a small neighborhood of p and show that $F^{-1}(-\infty, c-\epsilon]$ is a deformation retract of $F^{-1}(-\infty, c+\epsilon]$.

Using the Morse lemma, find a chart (U, ϕ) at p so that

$$(f \circ \phi)(x) = c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2 \text{ for each } x = (x_1, \ldots, x_n) \in U.$$

Note that the only critical point of f in $\phi(U)$ is p, because

$$\begin{aligned} D(f \circ \phi)_{|x} &= \begin{pmatrix} -2x_1 & \cdots & -2x_{\lambda_i} & 2x_{\lambda_i+1} & \cdots & 2x_n \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \\ \Leftrightarrow\ & x_i = 0 \text{ for all } i = 1, \ldots, n \end{aligned}$$

for each $x \in U$. Let $\epsilon > 0$ be a positive number such that

(1) $f^{-1}[c-\epsilon, c+\epsilon] \subseteq f^{-1}[c-\delta, c+\delta]$, and
(2) $B = \{x \in \mathbb{R}^n : \|x\| \leq 2\epsilon\} \subseteq U$.

By our assumption on $f^{-1}[c-\delta, c+\delta]$, $f^{-1}[c-\epsilon, c+\epsilon]$ is a compact set that contains no critical points of f other than p.

Find a smooth function

$$\mu : \mathbb{R} \to \mathbb{R}$$

so that

(1) $\mu(0) > \epsilon$,
(2) $\mu^{(k)}(r) = 0$ for all $r \geq 2\epsilon$ and $k \in \mathbb{Z}_{\geq 0}$, and
(3) $-1 < \mu'(r) \leq 0$ for all r.

Define a function

$$F : M \to \mathbb{R}$$

by

$$F = f \text{ on } M - \phi(U), \text{ and}$$

$$(F \circ \phi)(x) = (f \circ \phi)(x) - \mu(x_1^2 + \cdots + x_\lambda^2 + 2x_{\lambda+1}^2 + \cdots + 2x_n^2) \text{ for all } x \in U.$$

One can show that F is a smooth function.

First, we prove that

$$F^{-1}(-\infty, c+\epsilon] = f^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon}.$$

Let W denote the set

$$W = \{x_1^2 + \cdots + x_\lambda^2 + 2x_{\lambda+1}^2 + \cdots + 2x_n^2 < 2\epsilon\}$$

Then, W is an open subset of U that is contained in B. By contstruction, F agrees with f on the set $M - \phi(W)$. On the other hand, for any $x \in W$ we see that

$$\begin{aligned}
&(F \circ \phi)(x) \\
=&(f \circ \phi)(x) - \mu(x_1^2 + \cdots + x_\lambda^2 + 2x_{\lambda+1}^2 + \cdots + 2x_n^2) \\
\leq&(f \circ \phi_i)(x) \\
=&c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2 \\
\leq&c + \frac{1}{2}[x_1^2 + \cdots + x_\lambda^2 + 2x_{\lambda+1}^2 + \cdots + 2x_n^2] \\
\leq&c + \epsilon.
\end{aligned}$$

The inequalities above imply that $\phi(W)$ is contained in both $F^{-1}(-\infty, c+\epsilon]$ and $f^{-1}(-\infty, c+\epsilon]$. Therefore, $F^{-1}(-\infty, c+\epsilon] = f^{-1}(-\infty, c+\epsilon]$.

Next, we prove that the critical points of F are the same as those of f. We only need to concern ourselves with the behavior of F in the open set $\phi(U)$ which contains $\phi(W)$. For convenience, set

$$g : \phi(U) \to \mathbb{R}$$

to be the smooth map given by

$$(g \circ \phi)(x) = \mu(x_1^2 + \cdots + x_\lambda^2 + 2x_{\lambda+1}^2 + \cdots + 2x_n^2)$$

so that $F = f - g$ on $\phi(U)$. Then, for all $x \in U$,

$$\begin{aligned} &D(F \circ \phi)_{|x} \\ =&D(f \circ \phi)_{|x} - D(g \circ \phi)_{|x} \\ =&\begin{pmatrix} -2x_1 & \cdots & -2x_{\lambda_i} & 2x_{\lambda_i+1} & \cdots & 2x_n \end{pmatrix} \\ &- \mu'(g(x)) \circ \begin{pmatrix} 2x_1 & \cdots & 2x_{\lambda_i} & 4x_{\lambda_i+1} & \cdots & 4x_n \end{pmatrix} \\ =&\begin{pmatrix} -2x_1 & \cdots & -2x_{\lambda_i} & 2x_{\lambda_i+1} & \cdots & 2x_n \end{pmatrix} \\ &- \begin{pmatrix} 2\mu'(g(x))x_1 & \cdots & 2\mu'(g(x))x_{\lambda_i} & 4\mu'(g(x))x_{\lambda_i+1} & \cdots & 4\mu'(g(x))x_n \end{pmatrix} \end{aligned}$$

So, for each $i = 1, \dots, n$, the i-th partial of F is

$$D_i(F \circ \phi)_{|x} = \begin{cases} [-2 - 2\mu'(g(x))]x_i & \text{if } i = 1, \dots, \lambda \\ [2 - 4\mu'(g(x))]x_i & \text{if } i = \lambda + 1, \dots, n \end{cases}.$$

Because $-1 < \mu'(r) \leq 0$ for all $r \in \mathbb{R}$, $-2 - 2\mu'(g(x)) < 0$ and $2 - 4\mu'(g(x)) > 0$ for all $x \in U$. Thus,

$$D_i(F \circ \phi)_{|x} = 0 \Leftrightarrow x_i = 0 \text{ for each } i = 1, \dots, n.$$

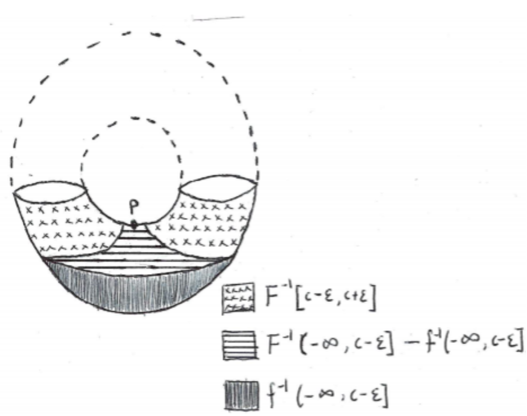
It follows that the only critical point of F in $\phi(U)$ is $\phi(0) = p$.

Lastly, we show that $F^{-1}[c - \epsilon, c + \epsilon]$ is compact and does not contain any critical point of F. We know from above that $F^{-1}(-\infty, c + \epsilon] = f^{-1}(-\infty, c + \epsilon]$. We also know that $F \leq f$ on M because μ is a non-negative function. As a result, $f^{-1}(-\infty, c - \epsilon] \subseteq F^{-1}(-\infty, c - \epsilon]$ and therefore $F^{-1}[c - \epsilon, c + \epsilon] \subseteq f^{-1}[c - \epsilon, c + \epsilon]$. Then, $F^{-1}[c - \epsilon, c + \epsilon]$ is compact, as it is a closed subset of a compact set. Moreover, the subset relation also tells us that the only possible critical point of F that $F^{-1}[c - \epsilon, c + \epsilon]$ can contain is p. However,

$$\begin{aligned} F(p) =&(F \circ \phi)(0) \\ &= (f \circ \phi)(0) - (g \circ \phi)(0) \\ &= c - \mu(0) < c - \epsilon. \end{aligned}$$

Thus, $p \notin F^{-1}[c - \epsilon, c + \epsilon]$ and so $F^{-1}[c - \epsilon, c + \epsilon]$ cannot contain any critical point of F.

In the special case of the height function on the 2-torus, the pre-images of F can be thought of in the following manner:



The three observations above tell us that $F^{-1}[c-\epsilon, c+\epsilon]$ satisfies the hypotheses of Theorem 4.1. Thus, $F^{-1}(-\infty, c-\epsilon]$ is a deformation retract of $M^{c+\epsilon}$. Note that p is contained in the small region $F^{-1}(-\infty, c-\epsilon] - f^{-1}(-\infty, c-\epsilon]$. This completes the first step.

The second step of the proof is to show that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $F^{-1}(-\infty, c-\epsilon]$, where e^λ is homeomorphic to a λ-dimensional unit disk $D^\lambda \subseteq \mathbb{R}^\lambda$.

Let S be the set given by

$$S = \{(u_1, \dots, u_n) \in U : u_1^2 + \dots + u_\lambda^2 \leq \epsilon \text{ and } u_{\lambda+1} = \dots = u_n = 0\}.$$

We define the λ-cell e^λ to be

$$e^\lambda = \phi(S)$$

Define the map

$$\psi : D^\lambda \to e^\lambda$$

by the formula

$$\psi(x_1, \dots, x_\lambda) = \phi(\sqrt{\frac{\epsilon}{\lambda}} x_1, \dots, \sqrt{\frac{\epsilon}{\lambda}} x_\lambda, 0, \dots, 0)$$

Since ϕ is a homeomorphism, ψ is a homeomorphism between D^λ and e^λ.

We need to check that e^λ is contained in $F^{-1}(-\infty, c-\epsilon]$; if e^λ was not contained in $F^{-1}(-\infty, c-\epsilon]$, then it would not make sense to say that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $F^{-1}(-\infty, c-\epsilon]$. Let q be a point in e^λ. Then,

$$q = \phi(x_1, \dots, x_\lambda, 0, \dots, 0), \text{ where } x_1^2 + \dots + x_\lambda^2 \leq \epsilon.$$

For convenience, set $\alpha = x_1^2 + \dots + x_\lambda^2$. We observe that

$$\begin{aligned} F(q) =& (F \circ \phi)(x_1, \dots, x_\lambda, 0, \dots, 0) \\ =& (f \circ \phi)(x_1, \dots, x_\lambda, 0, \dots, 0) - (g \circ \phi)(x_1, \dots, x_\lambda, 0, \dots, 0) \\ =& c - \alpha - \mu(\alpha). \end{aligned}$$

If $\alpha = x_1^2 + \dots + x_\lambda^2 = 0$, then $q = p$ and p is in the set $F^{-1}(-\infty, c-\epsilon]$ by construction. Otherwise, by the Mean Value Theorem, we can write

$$\mu(\alpha) - \mu(0) = \alpha\mu'(\beta) \text{ for some } \beta \in (0, \alpha).$$

Moving the $\mu(0)$ term on the other side and substituting the resulting expression for $\mu(\alpha)$, we get that

$$\begin{aligned} F(q) &= c - [\alpha + \alpha\mu'(\beta) + \mu(0)] \\ &= c - [\alpha(1 + \mu'(\beta)) + \mu(0)] \\ &< c - \epsilon \\ &\text{because } 1 + \mu'(\beta) > 0 \text{ and } \mu(0) > \epsilon \end{aligned}$$

Thus, $q \in F^{-1}(-\infty, c-\epsilon]$. In particular, we note that for any $x \in \phi^{-1}(e^\lambda)$

$$\phi(x_1, \dots, x_n) \in M^{c-\epsilon} \Leftrightarrow x_1^2 + \dots + x_\lambda^2 = \epsilon.$$

Let H be the closure in M of $F^{-1}(-\infty, c-\epsilon] - f^{-1}(\infty, c-\epsilon]$. We know that $F = f$ in the region outside H. This implies that

$$F^{-1}(-\infty, c-\epsilon] - f^{-1}(\infty, c-\epsilon] \subseteq \phi(W).$$

Then, H must be contained in $\phi(B)$.

We wish to construct a homotopy

$$r_t : M^{c-\epsilon} \cup H \to M^{c-\epsilon} \cup H$$

such that r_1 is the identity, r_0 maps $F^{-1}(-\infty, c-\epsilon]$ into $M^{c-\epsilon} \cup e^\lambda$, and r_t is the identity outside H for all time t. Since H is contained in $\phi(U)$, we only need to concern ourselves with $\phi(U)$ to construct the desired homotopy.

For convenience, set

$$\xi(x) = x_1^2 + \cdots + x_\lambda^2 \text{ and}$$
$$\eta(x) = x_{\lambda+1}^2 + \cdots + x_n^2 \text{ for all } x \in U$$

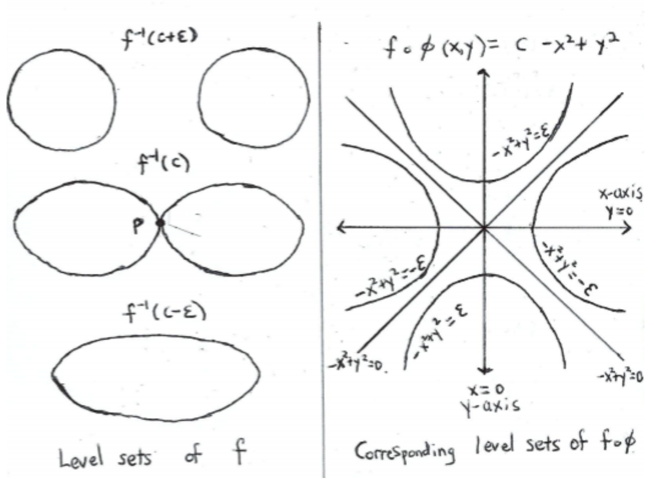
Because $H \subseteq \phi(U)$, we can divide H into the three regions $\phi(R_1)$, $\phi(R_2)$, and $\phi(R_3)$ where R_1, R_2, and R_3 are given by

$$R_1 = \{x \in \phi^{-1}(H) : \xi(x) \leq \epsilon\}$$
$$R_2 = \{x \in \phi^{-1}(H) : \epsilon \leq \xi(x) \leq \eta(x) + \epsilon\}$$
$$R_3 = \{x \in \phi^{-1}(H) : \eta(x) + \epsilon \leq \xi(x)\}$$

To visualize this situation, we return to the case of the 2-torus embedded in $\mathbb{R}^3$. Under the assumption that p is a nondegenerate critical point of index 1,

$$(f \circ \phi)(x, y) = c - x^2 + y^2$$

for all points (x, y) in the open neighborhood U of p.

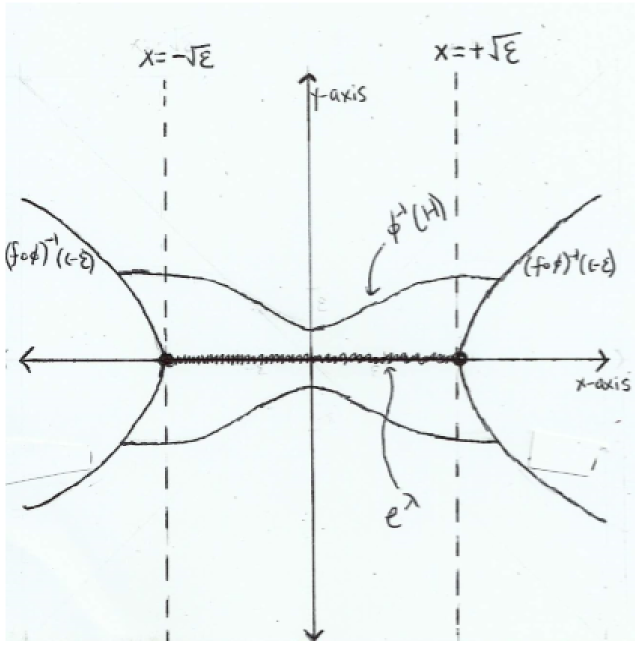


The λ-cell e^λ is given by

$$e^\lambda = \phi(\{(x, 0) \in U : x^2 \leq \epsilon\}),$$

and the regions R_1, R_2 and R_3 are given by

$$R_1 = \{(x, y) \in \phi^{-1}(H) : -\sqrt{\epsilon} \leq x \leq \sqrt{\epsilon}\}$$
$$R_2 = \{(x, y) \in \phi^{-1}(H) : x \leq -\sqrt{\epsilon} \text{ or } x \geq \sqrt{\epsilon}\} \cap \{(x, y) \in \phi^{-1}(H) : -x^2 + y^2 \geq -\epsilon\}$$
$$R_3 = \{(x, y) \in \phi^{-1}(H) : -x^2 + y^2 \leq -\epsilon\}.$$



To obtain the desired deformation retraction in this particular case, we can vertically push the region R_1 into e^λ and the region R_2 into $(f \circ \phi)^{-1}(c - \epsilon)$ which is equal to the set $\{(x, y) \in U : -x^2 + y^2 = -\epsilon\}$. We will adapt this idea to construct the desired homotopy in the general case.

In the region $\phi(R_1)$ let r_t be given by

$$r_t(\phi(x)) = (x_1, \ldots, x_{\lambda_1}, tx_{\lambda_1+1}, \ldots, tx_n) \text{ for all } x \in R_1.$$

We see that r_1 is the identity on R_1 and r_0 maps the region R_1 into e^λ. The image of $\phi(R_1)$ under r_t is contained in $F^{-1}(-\infty, c - \epsilon]$ because, for any $t \in [0, 1]$ and $x \in R_1$,

$$\begin{aligned}
F(r_t(\phi(x))) =& c - \xi(x) + t^2\eta(x) - \mu(\xi(x) + 2t^2\eta(x)) \\
\leq& c - \xi(x) + \eta(x) - \mu(\xi(x) + 2t^2\eta(x)) \\
& \quad \text{because } \eta(x) \text{ is non-negative for all } x \in R_1 \\
\leq& c - \xi(x) + \eta(x) - \mu(\xi(x) + 2\eta(x)) \\
& \quad \text{because } \mu \text{ is a non-increasing function by construction} \\
=& F(\phi(x)) \leq c - \epsilon \\
& \quad \text{because } x \in H.
\end{aligned}$$

In the region $\phi(R_2)$ define r_t to be

$$r_t(\phi(x)) = \begin{cases} (x_1, \ldots, x_{\lambda_1}, 0, \ldots, 0) & \text{if } \eta(x) = 0 \\ (x_1, \ldots, x_{\lambda_1}.s_t(x)x_{\lambda_1+1}, \ldots, s_t(x)x_n) & \text{otherwise} \end{cases}$$

where $s_t : R_2 \to \mathbb{R}$ is given by

$$s_t(x) = t + (1 - t)\sqrt{\frac{\xi(x) - \epsilon}{\eta(x)}}.$$

With a similar argument as above, we can show that the image of r_t lies in $F^{-1}(-\infty, c-\epsilon]$. We also need to check that r_t is continuous when $\eta(x) = 0$. To do this, it suffices to check that

$$\lim_{\eta(x)\to 0} \sqrt{\frac{\xi(x)-\epsilon}{\eta(x)}} x_i = 0 \text{ for each } i = \lambda_1 + 1, \ldots, n.$$

We observe that

$$\begin{aligned} \mid \sqrt{\frac{\xi(x)-\epsilon}{\eta(x)}} x_i \mid &= \mid \frac{x_i}{\sqrt{\eta(x)}} \mid \cdot \mid (\xi(x)-\epsilon) \mid \\ &\leq \mid (\xi(x)-\epsilon) \mid \text{ because } \sqrt{x_i^2} \leq \sqrt{\eta(x)}. \end{aligned}$$

The bounds on the region R_2 forces $\xi(x)$ to approach ϵ as $\eta(x)$ approaches 0. Therefore,

$$\lim_{\eta(x)\to 0} \sqrt{\frac{\xi(x)-\epsilon}{\eta(x)}} x_i \leq \lim_{\eta(x)\to 0} \mid (\xi(x)-\epsilon) \mid = 0.$$

So, r_t is continuous on the region $\phi(R_2)$. By construction, r_1 is the identity on $\phi(R_2)$. r_0 maps $\phi(R_2)$ into $M^{c-\epsilon}$ because

$$\begin{aligned} f(r_0(\phi(x)) &= c - \xi(x) + (s_0(x))^2\eta(x) \\ &= c - \xi(x) + \xi(x) - \epsilon \\ &= c - \epsilon \text{ for any } x \in R_2. \end{aligned}$$

Note that our definition of r_t agree on $\phi(R_1 \cap R_2)$ where

$$R_1 \cap R_2 = \{x \in \phi^{-1}(H) : \xi(x) = \epsilon\}.$$

In the region $\phi(R_3)$, let r_t be the identity. Note that our definitions of r_t agree in the regions $\phi(R_2 \cap R_3)$ where

$$R_2 \cap R_3 = \{x \in \phi^{-1}(H) : \xi(x) = \eta(x) + \epsilon\}$$

and that $R_1 \cap R_3 \subseteq R_2 \cap R_3$.

Finally, we need to check that r_t is the identity in the regions $\phi(R_1) \cap M^{c-\epsilon}$, $\phi(R_2) \cap M^{c-\epsilon}$, and $\phi(R_3) \cap M^{c-\epsilon}$. Note that

$$\phi_1(R_1) \cap M^{c-\epsilon} = \phi(\{x \in \phi^{-1}(H) : \xi(x) = \epsilon \text{ and } \eta(x) = 0\}).$$

So r_t is the identity in $\phi_1(R_1)$. Next,

$$\phi(R_2) \cup M^{c-\epsilon} = \phi(\{x \in \phi^{-1}(H) : \xi(x) = \eta(x) + \epsilon\})$$

and r_t is the identity in this region. Lastly, we note that

$$\phi_1(R_3) \subseteq M^{c-\epsilon},$$

so there is nothing to check.

Note that

$$\begin{aligned} &\phi^{-1}(H) \setminus R_1 = \{x \in \phi^{-1}(H) : \xi(w) > \epsilon\}, \\ &\phi^{-1}(H) \setminus R_2 = \{x \in \phi^{-1}(H) : \xi(w) < \epsilon\} \cup \{w \in \phi^{-1}(H) : (\xi - \eta)(w) > \epsilon\}, \text{ and} \\ &\phi^{-1}(H) \setminus R_3 = \{x \in \phi^{-1}(H) : (\xi - \eta)(w) < \epsilon\} \end{aligned}$$

are all open sets in $\phi^{-1}(H)$ by the continuity of ξ and η. $\phi^{-1}(H)$ is a closed set in $\mathbb{R}^n$ because it is a closed subset of B. Then, R_1, R_2 ,and R_3 are closed sets in

$\phi^{-1}(H)$. Hence, $\phi(R_1)$,$\phi(R_2)$, and $\phi(R_3)$ are closed sets contained in H since ϕ is a homeomorphism. So, r_t is a continuous map on $M^{c-\epsilon} \cup H$ because r_t is continuous when restricted to the closed sets $\phi(R_1)$,$\phi(R_2)$,$\phi(R_3)$, and $M^{c-\epsilon}$. This finishes the second step.

For the third and last step, we prove that

$$M^{c-\epsilon} \cup e^\lambda$$

is homeomorphic to

$$M^{c-\epsilon} \cup_{\partial\psi} D^\lambda$$

for some attaching map $\partial\psi$.

Let the continuous map

$$\partial\psi : \partial D^\lambda \to e^\lambda$$

be given by

$$\partial\psi = \psi_{|\partial D^\lambda}.$$

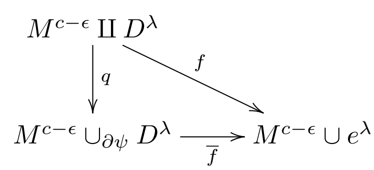
As noted above,

$$e^\lambda \cap M^{c-\epsilon} = \psi(\partial D^n)$$

and

$$M^{c-\epsilon} \cap e^\lambda = im(\partial\psi).$$

Consider the commuting diagram below:



Let the quotient map

$$q : M^{c-\epsilon} \amalg D^\lambda \to M^{c-\epsilon} \cup_{\partial\psi} D^\lambda$$

be defined in the obvious manner. Define f to be

$$f(x) = \begin{cases} x & \text{if } x \in M^{c-\epsilon} \\ \psi(x) & \text{if } x \in D^\lambda. \end{cases}$$

It is true that if A is a compact space, B is a Hausdorff space, and $g : A \to B$ is a contiuous bijection, then g is actually a homeomorphism. We can check that $\overline{f}$ is a continuous bijection. The compactness of M implies that $M^{c-\epsilon} \cup_{\partial\psi} D^\lambda$ is a compact set. Additionally, $M^{c-\epsilon} \cup e^\lambda$ is a Hausdorff space since it is a subspace of $\mathbb{R}^k$. Thus, $\overline{f}$ is a homeomorphism. □

With more bookkeeping, we can use the arguments above to show the following result:

Theorem 4.3. *Let $f : M \to \mathbb{R}$ be a Morse function on M and $c \in \mathbb{R}$ be a critical value of f. Let $\{p_1, \ldots, p_k\}$ be the set of all nondegenerate critical points in $f^{-1}(c)$ and let λ_i be the index of p_i. Suppose that $f^{-1}[c-\delta, c+\delta]$ is a compact set and contains no critical points of f other than $\{p_1, \ldots, p_k\}$ for some $\delta > 0$. Then, we can find a positive number $\epsilon > 0$ small enough such that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with $e_1, \ldots, e_k$ attached, where each e_i is diffeomorphic to the unit ball in $\mathbb{R}^{\lambda_i}$ and $e_i \cap e_j = \emptyset$ whenever $i \neq j$.*

5. CW Complexes

A **CW complex** is a space built in the following manner:

(1) Start with a discrete set $X^0 \subseteq X$ and regard the points in X^0 as 0-cells.

(2) Build the n-skeleton X^n inductively by attaching n-cells e^n_α to X^{n-1}.
More explicitly, let D^n_α be n-dimensional unit disks and $\partial\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$ be continuous maps. Let K be the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ under the identifications $x \sim \partial\varphi_\alpha(x)$ where $x \in \partial D^n_\alpha$.
Then, there is a homeomorphism $\psi : K \to X^n$ such that for any $x \in X^{n-1}$, $\psi([x]) = x$. The cell e^n_α is homeomorphic to $D^n_\alpha - \partial D^n_\alpha$ via $\psi_{|(D^n_\alpha - \partial D^n_\alpha)}$.

(3) $X = \cup_{n\in\mathbb{N}} X^n$. If $X \neq X^m$ for any $m \in \mathbb{N}$, then we require that $A \subseteq X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

From the defintion above, we obtain a map $\varphi_\alpha : D^n_\alpha \to X$ defined to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \to X^n \hookrightarrow X$. $D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha$ is continuous since if a set A in the latter space is open, then $A \cap D^n_\alpha$ is open in D^n_α. $X^{n-1} \amalg_\alpha D^n_\alpha \to X^n$ is continous because it is a composition of a homeomorphism and a quotient map. $X^n \hookrightarrow X$ is continous by condition (3). Therefore φ_α is a continuous map and we call this map the **characteristic map** of e^n_α. By following the composition, we can see that φ_α restricted to the interior of D^n_α is a homeomorphism onto e^n_α.

We will assume some facts about CW complexes.

Proposition 5.1. *CW complexes are Hausdorff.*

Proof. See page 522 of "Algebraic Topology" by Hatcher. □

Lemma 5.2. *If X is a finite CW complex, then X is compact.*

Lemma 5.3. *Suppose that X is a finite CW complex and $\partial\varphi : \partial D^n \to X^{n-1}$ is a continuous map. Then, $X \cup_{\partial\varphi} D^n$ is again a finite CW complex.*

Lemma 5.4. *Suppose that X is a finite CW complex. Let D^n be an n-dimensional unit disk, $K_0 = X \cup_{\partial f_0} D^n$ and $K_1 = X \cup_{\partial f_1} D^n$ where $\partial f_0, \partial f_1 : \partial D^n \to X$ are continuous maps. If ∂f_0 and ∂f_1 are homotopic in X, then $X \cup_{\partial f_0} D^n$ and $X \cup_{\partial f_1} D^n$ have the same homotopy type.*

Proof. The following proof is from "On Simply Connected, 4-dimensional Polyhedra" by Whitehead.

Let f_i be the map given by the composition $D^n \hookrightarrow X \amalg D^n \to X \cup_{\partial f_i} D^n$ for $i = 1, 2$ and let $g_t : \partial D^n \to X$ be the homotopy between $g_0 = \partial f_0$ and $g_1 = \partial f_1$. Define $h_0 : K_0 \to K_1$ such that h_0 restricted to X is the identity and

$$h_0(f_0(ru)) = \begin{cases} f_1(2ru) & \text{for } 0 \le 2r \le 1,\ u \in \partial D^n \\ g_{2-2r}(u) & \text{for } 1 \le 2r \le 2,\ u \in \partial D^n. \end{cases}$$

Similarly, define $h_1 : K_1 \to K_0$ such that h_1 restricted to X is the identity and

$$h_1(f_1(ru)) = \begin{cases} f_0(2ru) & \text{for } 0 \le 2r \le 1,\ u \in \partial D^n \\ g_{2r-1}(u) & \text{for } 1 \le 2r \le 2,\ u \in \partial D^n. \end{cases}$$

One can check that h_0 and h_1 are well-defined and continuous maps.

Since h_1 is the identity on X, $h_1 \circ h_0 : K_0 \to K_0$ is given by

$$h_1(h_0(x)) = x \text{ for } x \in X$$

$$h_1(h_0(f_0(ru))) = \begin{cases} h_1(f_1(2ru)) = f_0(4ru) & \text{for } 0 \le 4r \le 1,\ u \in \partial D^n \\ h_1(f_1(2ru)) = g_{4r-1}(u) & \text{for } 1 \le 4r \le 2,\ u \in \partial D^n \\ h_1(g_{2-2r}(u)) = g_{2-2r}(u) & \text{for } 1 \le 2r \le 2,\ u \in \partial D^n. \end{cases}$$

We define a homotopy ξ_t from $h_1 \circ h_0$ to the identity by

$$\xi_t(x) = x \text{ for } x \in X$$

$$\xi_t(f_0(ru))) = \begin{cases} f_0((4-3t)ru) & \text{for } 0 \le r \le \frac{1}{4-3t},\ u \in \partial D^n \\ g_{(4-3t)r-1}(u) & \text{for } \frac{1}{4-3t} \le r \le \frac{2-t}{4-3t},\ u \in \partial D^n \\ g_{\frac{1}{2}(4-3t)(1-r)}(u) & \text{for } \frac{2-t}{4-3t} \le r \le 1,\ u \in \partial D^n. \end{cases}$$

Note that $\xi_0 = h_1 \circ h_0$ and $\xi_1 = id$ as desired. ξ_t is a well-defined and continuous function, but we will not check this.

In a similar manner, $h_0 \circ h_1 : K_1 \to K_1$ is given by

$$h_0(h_1(x)) = x \text{ for } x \in X$$

$$h_0(h_1(f_1(ru))) = \begin{cases} f_1(4ru) & \text{for } 0 \le 4r \le 1,\ u \in \partial D^n \\ g_{2-4r}(u) & \text{for } 1 \le 4r \le 2,\ u \in \partial D^n \\ g_{2r-1}(u) & \text{for } 1 \le 2r \le 2,\ u \in \partial D^n. \end{cases}$$

If $\frac{1}{4-3t} \le r \le \frac{2-t}{4-3t}$, then

$$g_{1-[(4-3t)r-1]}(u) = \begin{cases} g_{2-4r}(u) & \text{if } t = 0 \\ g_{2r-1}(u) & \text{if } t = 1 \end{cases}$$

and if $\frac{2-t}{4-3t} \le r \le 1$, then

$$g_{1-[\frac{1}{2}(4-3t)(1-r)]}(u) = \begin{cases} g_{2r-1}(u) & \text{if } t = 0 \\ g_1(u) & \text{if } t = 1. \end{cases}$$

Using this, we can contstruct a homotopy η_t from $h_0 \circ h_1$ to id. □

Lemma 5.5. *If a map $F : X \to Y$ has a left homotopy inverse $L : Y \to X$ and a right homotopy inverse $R : Y \to X$, then F is a homotopy equivalence and R and L are 2-sided homotopy inverses.*

Proof. Let h_t^1 be a homotopy from $L \circ F$ to id_X and let h_t^2 be a homotopy from $F \circ R$ to id_Y. Then, $h_t^1 \circ R$ is a homotpy from $(L \circ F) \circ R$ to R, and $L \circ h_t^2$ is a homotopy from $L \circ (F \circ R)$ to L. As a result, we obtain a homotpy h_t^3 from R to L by concatenating the two homotopies $h_{1-t}^1 \circ R$ and $L \circ h_t^2$. Then, we get the relation $RF \cong LF \cong id_X$ by concatenating $h_t^3 \circ F$ and h_t^1. Thus, R is a 2-sided homotopy inverse and F is a homotopy equivalence. We can show that L is a 2-sided homotopy inverse by a similar argument. □

Lemma 5.6. *Let X and Y be finite CW complexes and $\partial\varphi : \partial D^n \to X^{n-1}$ be an attaching map. Then, any homotopy equivalence $f : X \to Y$ extends to a homotopy equivalence $F : X \cup_{\partial\varphi} D^n \to Y \cup_{f \circ \partial\varphi} D^n$.*

Proof. Define $F : X \cup_{\partial\varphi} D^n \to Y \cup_{f\circ\partial\varphi} D^n$ to be

$$F([x]) = \begin{cases} [f(x)] & \text{if } x \in X \\ [x] & \text{if } x \in D^n - \partial D^n. \end{cases}$$

Let $g : Y \to X$ be the homotopy inverse of f and define $G : Y \cup_{f\circ\partial\varphi} D^n \to X \cup_{g\circ f\circ\partial\varphi} D^n$ to be

$$G([y]) = \begin{cases} [g(y)] & \text{if } y \in Y \\ [y] & \text{if } y \in D^n - \partial D^n. \end{cases}$$

F and G are well defined and continuous functions.

Let $\varphi : D^n \to X \cup_{\partial\varphi} D^n$ be the characteristic map associated to D^n in $X \cup_{\partial\varphi} D^n$ so that $\varphi_{|\partial D^n} = \partial\varphi$. Then, the characteristic map associated to D^n in $X \cup_{g\circ f\circ\partial\varphi} D^n$ is the same as $G \circ F \circ \varphi$. Let h_t be the homotopy between $g \circ f$ and id_X. Because $g \circ f \circ \partial\varphi$ is homotopic to $\partial\varphi$ by $h_t \circ \partial\varphi$, there is a homotopy equivalence

$$k : X \cup_{g\circ f\circ\partial\varphi} D^n \to X \cup_{\partial\varphi} D^n$$

by Lemma 5.4. Moreover, the proof of this lemma gives the following formula for k:

$$k(x) = x \text{ if } x \in X$$

$$k(G(F(\varphi(ru))) = \begin{cases} \varphi(2ru) & \text{for } 0 \le 2r \le 1,\ u \in \partial D^n \\ h_{2-2r}(\varphi(u)) & \text{for } 1 \le 2r \le 2,\ u \in \partial D^n. \end{cases}$$

This formula also defines a map

$$k \circ G \circ F : X \cup_{\partial\varphi} D^n \to X \cup_{\partial\varphi} D^n.$$

Additionally, $k \circ G \circ F$ is homotopic to the identity by the map

$$q_t : X \cup_{\partial\varphi} D^n \to X \cup_{\partial\varphi} D^n$$

given by

$$q_t(x) = h_t(x) \text{ for } x \in X$$

$$q_t(\varphi(ru)) = \begin{cases} \varphi(\frac{2}{1+t} ru) & \text{for } 0 \le r \le \frac{1+t}{2} \text{ and } u \in \partial D^n \\ h_{2-2r+t}(\varphi(u)) & \text{for } \frac{1+t}{2} \le r \le 1 \text{ and } u \in \partial D^n \end{cases}$$

q_t is well-defined and continuous. So, F has a left homotopy inverse.

Let $F' : X \cup_{g\circ f\circ\partial\varphi} D^n \to Y \cup_{f\circ g\circ f\circ\partial\varphi} D^n$ be a map given by

$$F'([x]) = \begin{cases} [f(x)] & \text{if } x \in X \\ [x] & \text{if } x \in D^n - \partial D^n. \end{cases}$$

The characteristic map associated to D^n in $Y \cup_{f\circ\partial\varphi} D^n$ and $Y \cup_{f\circ g\circ f\circ\partial\varphi} D^n$ are the same as $F \circ \varphi$ and $F' \circ G \circ F \circ \varphi$, respectively. Let η_t be a homotopy between $f \circ g$ and id_Y. Then, $f \circ g \circ f \circ \partial\varphi$ is homotopic to $f \circ \partial\varphi$ in Y via $\eta_t \circ f \circ \partial\varphi$. Again, by Lemma 5.4, there is a homotopy equivalence

$$k' : Y \cup_{f\circ g\circ f\circ\partial\varphi} D^n \to Y \cup_{f\circ\partial\varphi} D^n$$

given by

$$k'(F'(G(y))) = f(g(y)) \text{ if } y \in Y$$

$$k'(F'(G(F(\varphi(ru)))) = \begin{cases} F(\varphi(2ru)) & \text{for } 0 \le 2r \le 1,\ u \in \partial D^n \\ (h_{2-2r}(F(\varphi(u))) & \text{for } 1 \le 2r \le 2,\ u \in \partial D^n. \end{cases}$$

Also, $k' \circ F' \circ G$ is homotopic to the identity on $Y \cup_{f\circ\partial\varphi} D^n$ via the homotopy p_t given by

$$p_t(x) = \eta_t(y) \text{ for } x \in X$$

$$p_t(\varphi(ru)) = \begin{cases} F(\varphi(\frac{2}{1+t}ru)) & \text{for } 0 \leq r \leq \frac{1+t}{2} \text{ and } u \in \partial D^n \\ h_{2-2r+t}(F(\varphi(u))) & \text{for } \frac{1+t}{2} \leq r \leq 1 \text{ and } u \in \partial D^n \end{cases}$$

One can check that $k' \circ F' \circ G$ and p_t are well-defined and continuous. Hence, G has a left homotopy inverse.

$G \circ F$ is a left homotopy inverse to k. Because k is a homotopy equivalence, it also has a right inverse, and so $G \circ F \circ k$ is also homotopic to the identity by Lemma 5.5. Then, G has a right homotopy inverse $F \circ k$ and also has a left homotopy inverse $k' \circ F'$. So, $F \circ k$ is a 2-sided inverse and $F \circ k \circ G$ is homotopic to the identity by Lemma 5.5. Consequently, F has a right homotopy inverse $k \circ G$. Thus, F is a homotopy equivalence by Lemma 5.5. □

We will assume the following fact:

Theorem 5.7. *Cellular Approximation Theorem*

Every map $f : X \to Y$ of CW complexes is homotopic to a map $f' : X \to Y$ such that $f'(X^n) \subseteq Y^n$ for all $n \in \mathbb{N}$.

Proof. See Hatcher p.349 □

We can now prove a remarkable result.

Theorem 5.8. *If f is a Morse function on a compact manifold M, then M has the homotopy type of a finite CW complex with one cell of dimension λ for each critical point of index λ.*

Proof. We will see in the next section that the compactness of M forces the number of critical points of f to be finite. Then, the number of critical values are also finite and we can write $c_0 < \cdots < c_n$ for the critical values of f. Note that c_0 is the minimum of f on M so that M^a is empty for $a < c_0$. For a real number a, suppose that $c_0 < a < c_1$ and that there are k critical points $p_1, \ldots, p_k$ in the level set $f^{-1}(c_0)$. Then, by Theorem 4.3, M^a has the homotopy type of $\amalg_{i=1}^k D_i^{\lambda_i}$ where λ_i is the index of p_i for each $i = 1, \ldots, k$. So, M^a has the homotopy type of a finite CW complex.

Now, suppose that $j \in \{1, \ldots, n\}$, $c_{j-1} < a < c_j$, and M^a has the homotopy type of a finite CW complex. So, there exists a homotopy equivalence $h' : M^a \to K$ where K is a CW complex. Let $c = c_j$ and choose a positive number δ so that $[c - \delta, c + \delta]$ does not contain any other critical values then c. By Theorem 4.3, there exists a positive number ϵ so that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup_{\partial\phi'_1} D_1^{\lambda_1} \cup_{\partial\phi'_2} \cdots \cup_{\partial\phi'_k} D_k^{\lambda_k}$ where $\partial\phi'_1, \ldots, \partial\phi'_k$ are some attaching maps and k is the number of critical points in $f^{-1}(c)$. Additionally, there is a homotopy equivalence $h : M^{c-\epsilon} \to M^a$ by Theorem 4.1.

Consider the map $h' \circ h \circ \partial\phi'_i : \partial D_i^{\lambda_i} \to K$ for each $i = 1, \ldots, n$. By the Cellular Approximation Theorem, this map is homotopic to a map $\psi_i : \partial D_i^{\lambda_i} \to K^{\lambda_i - 1}$ that maps to the $\lambda_i - 1$ skeleton of K. So, $K \cup_{\psi_1} D_1^{\lambda_1} \cdots \cup_{\psi_k} D_k^{\lambda_k}$ is again a finite CW complex by Lemma 5.3. Then, we see that

$$M^{c-\epsilon} \cup_{\partial\phi'_1} D_1^{\lambda_1} \cup_{\partial\phi'_2} \cdots \cup_{\partial\phi'_k} D_k^{\lambda_k} \cong K \cup_{h'\circ h\circ\partial\phi'_1} D_1^{\lambda_1} \cdots \cup_{h'\circ h\circ\partial\phi'_k} D_k^{\lambda_k}$$

by Lemma 5.6 and that

$$K \cup_{h' \circ h \circ \partial \phi'_1} D_1^{\lambda_1} \cdots \cup_{h' \circ h \circ \partial \phi'_k} D_k^{\lambda_k} \cong K \cup_{\psi_1} D_1^{\lambda_1} \cdots \cup_{\psi_k} D_k^{\lambda_k}$$

by Lemma 5.4. So, $M^{c+\epsilon}$ has the homotopy type of a finite CW complex.

By induction, M has the homotopy type of a finite CW complex. Moreover, it has one cell of dimension λ for each critical point of index λ. □

One immediate application of this theorem is that if M is a compact manifold such that there exists a Morse function on M, then M has the homotopy type of a CW complex. It is true that there exists a Morse function on any compact manifold embedded in an ambient Euclidean space; therefore, every embedded and compact manifold has the homotopy type of a CW complex. Another application of this theorem, the one that we will focus on for the rest of this paper, is to make the connection between the Euler characteristic of M and the transversal intersection number mentioned in the beginning of the paper. Before we do so, however, we need to derive the explicit formula for the gradient field.

6. The Gradient Field

A smooth **vector field** $\vec{v}$ on a manifold M is a smooth map

$$\vec{v} : M \to \mathbb{R}^k$$

such that $\vec{v}(x) \in T_x(M)$ for all $x \in M$. Suppose that f is a smooth and real-valued function defined on M. The **gradient field of** f, denoted $grad(f)$, is a vector field on M that satisfies the property

$$Df_{|p}(w) = grad(f)(p) \cdot w \text{ for } p \in M \text{ and } w \in T_p(M).$$

Just from the definition, it is not clear if $grad(f)$ even exits or if it is smooth. The explicit fomula for $grad(f)$, however, will imply that $grad(f)$ exists for any manifold M and is indeed a smooth vector field.

We will assume the following fact:

Proposition 6.1. *let V be a finite dimensional vector space. Given any basis $\{E_1, \ldots, E_n\}$ for V, let $\epsilon_1, \ldots, \epsilon_n$ be linear maps*

$$\epsilon_i : V \to \mathbb{R}$$

defined by

$$\epsilon_i(E_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Then, $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis for V^*.

For each point x in M, $T_x(M)$ is an n-dimensional vector subspace of $\mathbb{R}^k$. For each vector w in $T_x(M)$, define a linear map

$$\psi_w : T_x(M) \to \mathbb{R}$$

given by the formula

$$\psi_w(u) = w \cdot u \text{ for any } u \in T_x(M).$$

Next, define a $\mathbb{R}$-vector space homomorphism

$$\eta : T_x(M) \to T_x(M)^*$$

by the formula

$$\eta(w) = \psi_w \text{ for each } w \in T_x(M).$$

We claim that η defined above is an isomorphism. Indeed, choose a set of vectors $\{u_1, \ldots, u_n\}$ that forms a basis for $T_x(M)$. By Gram-Schmidt orthogonaliztion, we can find an orthonormal basis $\{E_1, \ldots, E_n\}$ of $T_x(M)$. We set $\eta(E_i) = \epsilon_i$ for each $i = 1, \ldots, n$. Then $\epsilon_1, \ldots, \epsilon_n$ are linear maps from $T_x(M)$ to $\mathbb{R}$ that satisfy the hypothesis of Proposition 6.1. Thus, $\{\epsilon_1, \ldots, \epsilon_n\}$ is a basis for $T_x(M)^*$ and η is an isomorphism.

As above, let f be a smooth and real-valued function defined on M. For each $x \in M$,

$$Df_{|x} : T_x(M) \to \mathbb{R}$$

is a linear functional defined on the vector subspace $T_x(M)$ contained in $\mathbb{R}^k$. With notation as above, let

$$v_x = \eta^{-1}(Df_{|x}) \in T_x(M).$$

By the definition of η,

$$Df_{|x}(w) = v_x \cdot w \text{ for any } w \in T_x(M).$$

Moreover, the vector v_x is uniquely determined for each $x \in M$ since η is an isomorphism. Hence, the gradient field of f exists and is the unique vector field given by

$$grad(f)(x) = v_x.$$

It will be useful for us to derive the formula for $grad(f)$ in some open neighborhood of M. Let (U, ϕ) be a chart at some point in M. Let $G(x)$ be the matrix given by

$$G(x) = (D\phi_{|x})^T \circ D\phi_{|x}$$

for each $x \in U$.

Lemma 6.2. *For each $x \in U$, there exists an invertible matrix $P(x)$ such that*

$$P(x)^T G(x) P(x) = id.$$

Proof. (Sketch)

For each $x \in U$, $G(x)$ defines a quadratic form on $\mathbb{R}^n$ such that

$$(z_1, \ldots, z_n) G(x) (z_1, \ldots, z_n)^T = (D\phi_{|x}(z_1, \ldots, z_n)) \cdot (D\phi_{|x}(z_1, \ldots, z_n))$$

for each $(z_1, \ldots, z_n) \in \mathbb{R}^n$. Then, $G(x)$ is positive definite on $\mathbb{R}^n$. By Theorem 2.6, there exists an invertible matrix $P(x)$ such that $P(x)^T G(x) P(x)$ is a diagonal matrix with either 1 or -1 in its diagonal entries. This matrix must be the identity matrix. □

Lemma 6.2 implies that $G(x)$ is an invertible matrix and that $det(G(x)) = 1$ for each $x \in U$.

Lemma 6.3. *Consider the vector field*

$$\phi^* grad(f) : U \to \mathbb{R}^n$$

on U, defined by the composition

$$\phi^* grad(f)(x) = D\phi^{-1}_{|\phi(x)}(grad(f)(\phi(x))).$$

Then,

$$(\phi^* grad(f)(x)) = D(f \circ \phi)_{|x}(G(x)^{-1}).$$

Proof. (Sketch) By the definition of the gradient field and the Chain Rule,

$$(D\phi_{|x}(\phi^* grad(f)(x))) \cdot D\phi_{|x}(e_i) = D(f \circ \phi)_{|x}(e_i)$$

for each $i = 1, \dots, n$. Set $\phi^* grad(f)(x) = (v_1(x), \dots, v_n(x))$ for each $x \in U$. The first equality implies that

$$D(f \circ \phi)_{|x}(e_i) = v_1(x)(D\phi_{|x}(e_1) \cdot D\phi_{|x}(e_i)) + \cdots + v_n(x)(D\phi_{|x}(e_n) \cdot D\phi_{|x}(e_i)).$$

Therefore,

$$(\phi^* grad(f)(x))G(x) = D(f \circ \phi)_{|x}$$

□

Corollary 6.4. *$grad(f)$ is a smooth vector field on M and the zeroes of $grad(f)$ are precisely the critical points of f.*

Proof. (Sketch)

We write the inverse of $G(x)$ as $(G(x)^{-1}) = (g^{ij}(x))_{1 \le i,j \le n}$. Then, Cramer's rule implies that $g^{ij}(x)$ is a smooth function on U. We can use the formula

$$grad(f)(\phi(x)) = D\phi_{|x} D(f \circ \phi)_{|x}(G(x)^{-1})$$

to show that $grad(f)$ is a smooth vector field on $\phi(U)$ and that the zeroes of $grad(f)$ on $\phi(U)$ are precisely the critical points of f of $\phi(U)$. □

7. The Euler Characteristic

Now we return to the problem posed in the introduction. Suppose that M is a closed and orientable manifold. For a smooth vector field $\vec{v}$ on a manifold M the map

$$V : M \to TM$$

is a smooth map given by

$$V(x) = (x, \vec{v}(x))$$

where $TM = \{(x, v) \in \mathbb{R}^k \times \mathbb{R}^k : x \in M \text{ and } v \in T_x(M)\}$ is the tangent bundle of M.

Lemma 7.1. *The map $V : M \to T(M)$ defined above is an immersion.*

Proof. (Sketch)

We can show that

$$DV_{|p}(w) = (w, D\vec{v}_{|p}(w)) \text{ for each } w \in T_p(M).$$

□

In particular, consider the zero vector field $\vec{o}$ given by

$$\vec{o}(x) = 0 \text{ for all } x \in M.$$

and the associated map $\mathcal{O} : M \to TM$ given by

$$\mathcal{O}(x) = (x, 0).$$

Then, $\mathcal{O}$ is an immersion by Lemma 7.1.

Additionally, let f be a Morse function on M and consider $grad(f)$, the gradient field of f on M. We will denote the associated map to be Y. So, the map

$$Y : M \to TM$$

is defined to be the smooth map given by

$$Y(x) = (x, grad(f)(x)).$$

Y is also an immersion.

We will assume that

Proposition 7.2. *the tangent bundle TM of a manifold M is a $2n$-dimensional manifold where n is the dimension of M.*

Proof. See page 51 of "Differential Topology" by Guillemin and Pollack. □

By Proposition 7.2, we get that

$$dim(M) + dim(M) = dim(TM).$$

Thus the intersection number $Y(M)\#\mathcal{O}(M)$ makes rigorous sense.

Note that the intersections of $Y(M)$ and $\mathcal{O}(M)$ are precisely the zeroes of the vector field $grad(f)$. So, the zeroes of a vector field are the objects that require our attention.

Lemma 7.3. *Suppose that $p \in M$ is a zero of a vector field $\overrightarrow{v}$. Then*

$$D\overrightarrow{v}_{|p} : T_p(M) \to \mathbb{R}^k$$

carries $T_p(M)$ into $T_p(M)$.

Proof. (Sketch)

Let (U, ϕ) be a chart at p. Consider the pulback vector field

$$\phi^*\overrightarrow{v} : U \to \mathbb{R}^n$$

and write

$$\phi^*\overrightarrow{v}(x) = \sum_{i=1}^{n} w_i(x)e_i \text{ for all } x \in U.$$

We can show that

$$\overrightarrow{v}(\phi(x)) = \sum_{i=1}^{n} w_i(x)D_i\phi_{|x} \text{ for all } x \in U.$$

The j-th coordinate function of $\overrightarrow{v}(\phi(x))$ can be written as

$$\sum_{i=1}^{n} w_i(x)D_i(\phi_j)_{|x} \text{ for all } x \in U.$$

The r-th partial derivative of the above j-th coordinate function at 0 is

$$\begin{aligned} D_r(\sum_{i=1}^{n} w_i D_i(\phi_j))_{|0} &= \sum_{i=1}^{n}[D_r(w_i)_{|0}D_i(\phi_j)_{|0} + w_i(0)D_{ri}(\phi_j)_{|0}] \\ &= \sum_{i=1}^{n} D_r(w_i)_{|0}D_i(\phi_j)_{|0} \\ &\text{because } p \text{ is a zero of the vector field } \overrightarrow{w}. \end{aligned}$$

Then,

$$D(\overrightarrow{v} \circ \phi)_{|0}(e_r) = \sum_{i=1}^{n} D_r(w_i)_{|0}D_i(\phi)_{|0}.$$

□

A **nondegenerate zero** of a vector field $\vec{v}$ is a point $p \in M$ such that $\vec{v}(p) = 0$ and $D\vec{v}_{|p} : T_p(M) \to T_p(M)$ is a bijection. Just by comparing the definitions, a statement that seems likely to be true is

Lemma 7.4. *p is a nondegenerate zero of $\vec{v}$ if and only if $V(M) \pitchfork \mathcal{O}(M)$ at $(p, 0)$.*

Proof. Let $p \in M$ be a nondegenerate zero of the vector field $\vec{v}$. Note that

$$DV_{|p}(T_p(M)) = \{(r, s) \in \mathbb{R}^{2k} : r \in T_p(M) \text{ and } s = D\vec{v}_{|x}(r)\} \text{ and}$$
$$D\mathcal{O}_{|p}(T_p(M)) = \{(r, 0) \in \mathbb{R}^{2k} : r \in T_p(M)\} = T_p(M) \times \{0\}.$$

By Lemma 7.1, $DV_{|p}(T_p(M))$ and $D\mathcal{O}_{|p}(T_p(M))$ are n-dimensional vector subspaces of $T_{(p,o)}(TM)$.

We claim that

$$DV_{|p}(T_p(M)) \oplus D\mathcal{O}_{|p}(T_p(M)) = T_p(M) \times T_p(M).$$

Indeed, let $(u, w) \in T_p(M) \times T_p(M)$. By Lemma 7.3 and the fact that $D\vec{v}_{|p}$ is a bijection, there exists $r \in T_p(M)$ such that $D\vec{v}_{|p}(r) = w$. So, $(r, w) \in DV_{|p}(T_p(M))$. Additionally, $(u - r, 0) \in D\mathcal{O}_{|p}(T_p(M))$ because the vector $(u - r)$ is an element of $T_p(M)$. Since $(r, w) + (u - r, 0) = (u, w)$,

$$T_p(M) \times T_p(M) \subseteq DV_{|p}(T_p(M)) + D\mathcal{O}_{|p}(T_p(M)).$$

Since the left-hand side is a vector space that has $2n$-dimensions, and the right-hand side has dimension of at most $2n$, the claim follows.

The claim also implies that

$$DV_{|p}(T_p(M)) \oplus D\mathcal{O}_{|p}(T_p(M)) = T_p(M) \times T_p(M) = T_{(p,0)}(TM).$$

By the definition of transversal intersections,

$$V(M) \pitchfork \mathcal{O}(M) \text{ at } (p, 0) \text{ in } TM.$$

Conversely, suppose that

$$V(M) \pitchfork \mathcal{O}(M) \text{ at } (p, 0) \in TM, \text{ or}$$
$$DV_{|p}(T_p(M)) \oplus D\mathcal{O}_{|p}(T_p(M)) = T_{(p,0)}(TM).$$

Then,

$$DV_{|p}(T_p(M)) \cap D\mathcal{O}_{|p}(T_p(M)) = (0, 0), \text{ or}$$
$$DV_{|p}(T_p(M)) \cap T_p(M) \times \{0\} = (0, 0).$$

The above observation implies that $D\vec{v}_{|p}$ must be an injection. Since $D\vec{v}_{|p}$ is a map from $T_p(M)$ into itself by Lemma 7.3, it must also be a bijection. Thus, p is a nondegenerate zero of $\vec{v}$. □

Lemma 7.5. *Suppose that f is a Morse function on a compact, boundaryless, and orientable manifold, M. For a point $p \in M$, p is a nondegenerate zero of $grad(f)$ if and only if p is a nondegenerate critical point of f.*

Proof. Suppose that $p \in M$ is a critical point of f and let (U, ϕ) be a chart at p. For convenience, set

$$\phi^* grad(f)(x) = (v_1(x), \ldots, v_n(x)) \text{ for each } x \in U.$$

Let $G(x) = D\phi_{|x}^T \circ D\phi_{|x}$ and write $(G(x)^{-1}) = (g^{ij}(x))_{1\leq i,j\leq n}$. By Lemma 6.3,

$$\phi^* grad(f)(x) = D(f \circ \phi)_{|x}(G(x)^{-1}).$$

The j-th coordinate function of $\phi^* grad(f)$ at any $x \in U$ is given by the formula

$$v_j(x) = \sum_{i=1}^{n} D_i(f \circ \phi)_{|x} g^{ij}(x) \text{ for each } x \in U.$$

Then, the r-th partial derivative of the function v_j at 0 is

$$\begin{aligned}
&D_r(v_j)_{|0} \\
=&D_r(\sum_{i=1}^{n} D_i(f \circ \phi) g^{ij})_{|0} \\
=&\sum_{i=1}^{n}[D_{ri}(f \circ \phi)_{|0} g^{ij}(0) + D_i(f \circ \phi)_{|x}(D_r g^{ij})_{|0}] \\
&\text{by the Product Rule.}
\end{aligned}$$

We note that $D_i(f \circ \phi)_{|0} = 0$ for each $i = 1, \ldots, n$ because $\phi(0) = p$ is a critical point of f. Therefore,

$$D_r(v_j)_{|0} = \sum_{i=1}^{n} D_{ri}(f \circ \phi)_{|0} g^{ij}(0).$$

The derivative of $\phi^* grad(f)$ at 0, $D\phi^* grad(f)_{|0}$, is an n-by-n matrix. The entry in the r-th column and the j-th row of this matrix is $D_r(v_j)_{|0}$. As a result,

$$D\phi^* grad(f)_{|0} = ((G(0))^{-1})^T (H(0))^T$$

where $H(0)$ is the Hessian of $f \circ \phi$ at 0.

Additionally, using the identity

$$D\phi_{|x}(\phi^* grad(f)(x)) = grad(f)(\phi(x)),$$

we can write

$$grad(f)(\phi(x)) = (\sum_{i=1}^{n} D_i(\phi_1)_{|x} v_i(x), \ldots, \sum_{j=1}^{n} D_i(\phi_k)_{|x} v_i(x))$$

$$\text{for each } x \in U.$$

The j-th coordinate function of $grad(f) \circ \phi$ is given by

$$\sum_{i=1}^{n} D_i(\phi_j)_{|x} v_i(x)$$

and the r-th partial derivative of this coordinate function at 0 is

$$\begin{aligned} D_r(\sum_{i=1}^{n} D_i(\phi_j)v_i)_{|0} &= \sum_{i=1}^{n}[D_{ri}(\phi_j)_{|0}v_i(0) + D_i(\phi_j)_{|0}D_r(v_i)_{|0}] \\ &\quad \text{by the Product Rule} \\ &= \sum_{i=1}^{n} D_i(\phi_j)_{|0}D_r(v_i)_{|0} \\ &\quad \text{since } v_i(0) = 0 \text{ for each } i = 1, \dots, n \text{ as noted above} \end{aligned}$$

As a result,

$$D(grad(f) \circ \phi)_{|0} = D\phi_{|0} \circ (D(\phi^* grad(f))_{|0}).$$

Therefore, by the Chain Rule and the formula for $D(\phi^* grad(f))_{|0}$ above,

$$D(grad(f))_{|p} = D\phi_{|0} \circ (G(0)^{-1})^T \circ H(0)^T \circ D(\phi^{-1})_{|p}.$$

Since $((G(0)^{-1})^T$ is an invertible matrix, $D(grad(f))_{|p}$ is a bijection onto $T_p(M)$ if and only if $H(0)$ is invertible. □

By Lemma 7.4 and Lemma 7.5, we deduce that

$$Y(M) \pitchfork \mathcal{O}(M).$$

It is true that $Y(M)$ and $\mathcal{O}(M)$ are submanifolds of TM because Y and $\mathcal{O}$ are embeddings. So the intersections between $Y(M)$ and $\mathcal{O}(M)$ form a compact 0 dimensional submanifold of TM. It follows that $Y(M) \cap \mathcal{O}(M)$ is a finite set of points. Then, by Corollary 6.4,

Corollary 7.6. *the set of critical points of f is finite.*

We now compute the oriented intersection number $Y(M)\#\mathcal{O}(M)$.

Theorem 7.7.

$$Y(M)\#\mathcal{O}(M) = \sum_{c \in C}(-1)^{\lambda_c}$$

where C is the set of all critical points of f and λ_c is the index of f at the critical point c.

Proof. Let $(p, 0) \in TM$ be a point of intersection between $V(M)$ and $\mathcal{O}(M)$. Using the Morse lemma, find a chart (U, ϕ) such that

$$f \circ \phi(x_1, \dots, x_n) = f(p) - (x_1)^2 - \cdots - (x_\lambda)^2 + (x_{\lambda+1})^2 + \cdots + (x_n)^2$$

for all $(x_1, \dots, x_n) \in U$, where λ is the index of f at p.

By Lemma 7.5 and the proof of Lemma 7.4,

$$T_{(p,0)}(TM) = T_p(M) \times T_p(M).$$

We choose the bases

$\{D_1\phi_{|0} \times 0, \dots, D_n\phi_{|0} \times 0, 0 \times D_1\phi_{|0}, \dots, 0 \times D_n\phi_{|0}\}$ and
$\{D_1\phi_{|0}, \dots, D_n\phi_{|0}\}$

as positive bases for the tangent spaces $T_{(p,0)}(TM)$ and $T_p(M)$, respectively. We know that

$$DY_{|p}(v) = (v, Dgrad(f)_{|p}(v)) \text{ for each } v \in T_p(M).$$

Then,

$$\begin{aligned}
DY_{|p}(D_j\phi_{|0}) =& DY_{|p}(D\phi_{|0}(e_j)) \\
=& (D\phi_{|0}(e_j), D(grad(f))_{|p}(D\phi_{|0}(e_j))) \\
=& (D\phi_{|0}(e_j), [D\phi_{|0} \circ ((G(0))^{-1})^T \circ H(0)^T \circ D(\phi^{-1})_{|p}](D\phi_{|0}(e_j))) \\
& \text{ from the proof of Lemma 7.5} \\
=& (e_j, ((G(0))^{-1})^T \circ H(0)^T(e_j)) \\
& \text{ in terms of the positively oriented basis} \\
& \text{ of } T_{(p,0)}(T(M)) \text{ that we chose above .}
\end{aligned}$$

The sign of $(p, 0)$ is given by the determinant of the $2n$-by-$2n$ matrix whose columns are the images of the positive basis of $T_p(M)$ under the maps $DV_{|p}$ and $DO_{|p}$ in terms of the positive basis of $T_{(p,0)}(T(M))$. Denote this $2n$-by-$2n$ matrix as S. Our calculations show that the two n-by-n upper blocks of S are the identitry matrices, that the lower-left n-by-n block of S is $((G(0))^{-1})^T \circ H(0)^T$, and that the lower-right n-by-n block of S is the zero matrix.

$G(0)$ has a positive determinant by Lemma 6.2. Also, $H(0)$ is the hessian matrix of $f \circ \phi$ at 0. By the explicit formula for $f \circ \phi$, $H(0)$ is a diagonal matrix of which the first λ diagonal entries are -2 and the rest are 2. Thus, the sign of the determinant of the matrix $((G(0))^{-1})^T \circ H(0)^T$ is $(-1)^\lambda$ and the sign of $(p, 0)$ is $(-1)^{n^2}(-1)^\lambda$ because it takes n^2 transpositions to switch the left n columns of S with the right n columns of S. If n is even, the intersection number is the number that we claimed it would be.

Now, since the intersection number is a homotopy invariant and Y is homotopic to $\mathcal{O}$, we get that

$$\begin{aligned}
& Y(M)\#\mathcal{O}(M) \\
&= (-1)^{n^2}\mathcal{O}(M)\#Y(M) \\
&= (-1)^{n^2}Y(M)\#\mathcal{O}(M).
\end{aligned}$$

So, if n is odd, $Y(M)\#\mathcal{O}(M) = 0$. □

So far, we see that the number above is a differential invariant of a compact, boundaryless, and orientable manifold. To make the connection between $V(M)\#\mathcal{O}(M)$ and $\chi(M)$ as promised in the beginning of this paper, we must figure out how the number $\sum_{c\in C}(-1)^{\lambda_c}$ relates to the topological properties of M. Indeed, the results from Section 5 tell us that M has the homotopy type of a CW-complex K and that K has exactly one cell of dimension λ_c for each critical point $c \in C$. So,

$$\sum_{c\in C}(-1)^{\lambda_c} = \sum_{i=0}^{n}(-1)^i d_i$$

where d_i is the number of i-dimensional cells of K. In fact, for the CW-complex K, it is true that

Theorem 7.8.

$$\chi(K) = \sum_{i=0}^{n}(-1)^n rankH_i(K) = \sum_{i=0}^{n}(-1)^i d_i$$

where $H_i(K)$ is the i-th singular homology group of K and d_i is the number of i-dimensional cells of K.

Proof. See page 146 of Hatcher's "Algebraic Topology" for a proof. □

In particular, $\chi(K)$ is a homotopy type invariant as it is expressed in terms of the homology groups of K. Since M and K homotopically equivalent,

Corollary 7.9.

$$V(M)\#\mathcal{O}(M) = \chi(M)$$

.

Thus, we have essentially shown that we can compute a homotopy type invariant of a manifold using differential methods. Moreover, recall that $V(M)\#\mathcal{O}(M)$ is 0 if M is odd dimensional.

Corollary 7.10. *The Euler characteristic of an odd dimensional, boundaryless, compact and orientable manifold is 0.*

This is a standard consequence of Poincare duality, but the proof presented here does not require any knowledge of that.

8. Further Work

We have talked about how Morse functions imply interesting topological data of the compact, boundaryless, and orientable manifold on which the function is defined.

In fact, we can find a Morse function L_p for an arbitrary manifold X. This function has the special property that $L_p^{-1}(-\infty, a]$ is a bounded subset of $X \subseteq \mathbb{R}^k$ for each $a \in \mathbb{R}$. Using this, we can show that X has the homotopy type of a CW complex as long as X can be embedded as a closed subset of an ambient Euclidean space. With this more general notion of Morse theory, we can study the path space of a Riemannian manifold and make connections between the topology of the path space and the number of geodesics that the manifold can have. Also, Morse theory can be applied to Lie groups to prove the famous Bott periodicity theorem. The proof of these results can be found in the later chapters of Milnor's book on Morse Theory.

The idea of Morse theory also has other applications in symplectic geometry and differential geometry. The subjects that are beyond the scope of this paper, such as Floer homology, must surely be fascinating.

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