RESEARCH STATEMENT

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1 Summary

Broadly speaking, my current research interest lies in probability. My research plan consists of two parallel branches. One branch is analytic and probabilistic potential theory for both continuous and discrete Markov processes as well as Dirichlet spaces. The other branch is rough path theory, in particular, path properties for Gaussian processes and stochastic differential equations driven by fractional Brownian motion (fBM in abbreviation).

In the direction of potential theory, I have studied Brownian motion on spaces with varying dimension (BMVD in abbreviation) with an emphasis on their heat kernel estimates which reveals many interesting properties. More recently, using analytic approach, I have studied heat kernel estimates for general strongly local symmetric Dirichlet forms.

Spaces with varying dimension arise in many disciplines including statistics, physics and electrical engineering (e.g. molecular dynamics, plasma dynamics). See, for example, [26, 41] and the references therein. BMVD is constructed via "shorting". The concept of "short diffusion" arises naturally from shorting electric networks: Think of an electric network as a graph with an assignment of resistances to edges. "Shorting" means one or more edges are replaced by edges with zero resistance. More details about this is provided in Section 2.1, Example 1.

In the direction of rough path theory, I am particularly interested in stochastic differential equations of the type

$$X_t = X_0 + \int_{s=0}^{t} V_0(X_s)ds + \sum_{i=1}^{d} \int_{s=0}^{t} V_i(X_s)dB_s^i, \tag{1.1}$$

where $(B)_{t\geq 0}$ is a d-dimensional fBM with Hurst parameter $H > 1/4$ and $(V_i)_{i=0}^d$ is a family of C^∞- bounded vectors on $\mathbb{R}^d$ satisfying the uniform elliptic condition. Processes driven by fBM (instead of BM) have applications in a wide range of disciplines including biotechnology and biophysics, because many complex biological systems usually do have long-range, spatial and temporal correlations. See, for example, [33, 25].

Many basic properties of solutions to (1.1) have been established. For instance, the existence and the uniqueness of the solution X have been established for $H > 1/4$. The existence and smoothness of the density are shown in [21, 32] for $H > 1/2$. When $1/4 < H < 1/2$, the smoothness of the density in the elliptic case has been obtained in [5]. The upper bound estimates for the density functions, the upper bound for the joint density for the random vector (X_s, X_t), and the tail estimate on the increment $X_t - X_s$ are all given in [4]. My interest lies in the studies of such processes including fractal properties, description of local time with respect to X, multiple intersection problems, etc.

The remaining of this statement is structured as follows: Section 2 focuses on potential theory related to Markov processes and Dirichlet spaces; and Section 3 is devoted to rough path theory for SDE driven by fBM. In each section, the first subsection is a description of the background; more detailed description of my problems as well as some highlighted results are given in the following subsection(s); and an overlook of my future research plan in this area is provided in the last subsection.

2 Potential Theory: Markov Processes and Dirichlet Spaces

2.1 Background

The notion of Markov processes plays the center role of modern probability theory. In general, Markov processes are the processes that when the current position (value) is given, the future trajectory only depends on the current position but not the past behavior. In other words, Markov processes forget their path trajectories. An elegant way to characterize Markov processes is via Dirichlet forms. In particular, there is a one-to-one correspondence between the family of regular strongly local symmetric Dirichlet forms and the family of symmetric diffusion processes with no killing. Below we give two examples of Dirichlet forms associated with the best-known processes: electric networks and Brownian motion on Euclidean spaces.

Example 1 (Electric network)**.** A *graph* is a pair $G = (V, E)$ consisting of a finite or countable set V of vertices and a a finite or countable set E of edges. An *electric network* is a connected and locally-finite graph $G = (V, E)$, with an assignment of conductances $C(e) : e \in E$ to edges. Actually $C(e) = R(e)^{-1}$ where $R(e)$ is the resistance on e. Each edge e connecting x and y has two orientations: from x to y or from y to x. Denote the oriented edge e by e_{xy} or e_{yx} for each of the two orientations respectively. Let E^* be the set of all oriented edges of G. Then the Dirichlet form associated with the electric network is the following pair $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F}$ is the domain and $\mathcal{E}$ is a quadratic form on $\mathcal{F}$:

$$\begin{cases} \mathcal{F} = l^2(V), \\ \mathcal{E}(f, g) = \dfrac{1}{2} \displaystyle\sum_{e_{xy} \in E^*} (f(x) - f(y))(g(x) - g(y))C(e). \end{cases} \tag{2.1}$$

Observe that each edge $e \in E$ is represented twice in this sum, once for each orientation, and the two terms of the sum are equal. This explains the factor of $1/2$. The most representative function in $\mathcal{F}$ is the voltage (a.k.a. electrical potential). If we take both f and g to be the voltage, then $\mathcal{E}(f, g)$ gives the energy dissipated through the electric network. This measures how much one needs to be charged for this network by the electric company and explains the reason why Dirichlet forms are sometimes called "energy forms".

In this paragraph we briefly introduce "shorting electric network" which give rise to my research object. In physics, *Shorting Law* states "If two vertices x, y have the same voltage, then the network may be reduced to the network in which x, y are replaced by a single vertex z and the edges with a vertex at x or y are rerouted to the vertex z". In view of (2.1), this is equivalent to restricting the Dirichlet space domain $\mathcal{F}$ to $\{f \in l^2(V) : f(x) = f(y)\}$. In effect one is introducing a new wire of infinite conductivity (zero resistance) between x and y, which explains why such an operation is called "shorting".

As the scaling limit of simple random walks on $\mathbb{Z}^d$, Brownian motion on Euclidean spaces has the following Dirichlet form expression:

Example 2 (Brownian motion on $\mathbb{R}^d$)**.** Standard Brownian motion on $\mathbb{R}^d$ can be characterized by the following Dirichlet form:

$$\begin{cases} \mathcal{F} = W^{1,2}(\mathbb{R}^d), \\ \mathcal{E}(f,g) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) dx. \end{cases} \tag{2.2}$$

Inspired by shorting electric networks, for any non-polar compact set $K \subset \mathbb{R}^d$, one may get a Brownian motion traveling across K at infinite rate(conductivity) by restricting the Dirichlet form domain in (2.2) to $\{f \in W^{1,2}(\mathbb{R}^d) : f|_K \text{ is a constant}\}$. Interested readers may refer to [8, 9] for more examples constructed in this manner. The process BMVD we introduce with more details in Section 2.2 is also such an example.

In the next two sections, with some details, I describe my results for each problem.

2.2 Heat Kernel Estimates for BMVD

In [10], I study Brownian motion on state spaces with varying dimension, which is an example of Markov processes with *darning*. Roughly speaking, the concept of "darning" covers two types of operations: shorting and jointing. As mentioned in last section, such a notion arises naturally from electric networks.

The simplest state space on which Brownian motion with "varying dimension" lives can be thought of as an infinite 1-dimensional pole installed on a 2-dimensional plane. However, a singleton in the plane is "polar" with respect to Brownian motion, meaning that it is never hit by a 2-dimensional Brownian motion in finite time. Consequently, Brownian motion can not be constructed in the usual sense on such a state space because once a Brownian motion particle is on the plane, it never gets the chance to climb up the pole. To circumvent such obstacle, we "short" a small closed disk $B_\epsilon = B(0, \epsilon)$ on $\mathbb{R}^2$ into a single point a^*. See Figure 1. The resulting process hits the "shorted" point a^* when a 2-dimensional Brownian motion hits the closed disk B_ϵ, which happens with probability one in finite time. Then we install a pole at a^*.

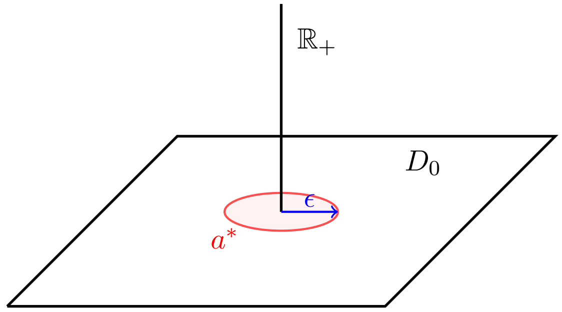

Figure 1: State space of BMVD

As shown in Figure 1, we denote $D_0 := \mathbb{R}^2 \setminus B_\epsilon$. For $p > 0$, let m_p be the underlying measure on the state space whose restriction on $\mathbb{R}_+$ and D_0 is the Lebesgue measure multiplied by p and 1, respectively. In particular, set $m_p(\{a^*\}) = 0$. We denote by $\rho(x, y)$ the shortest path distance between x and y, for any x, y in the state space.

As a standard notation, we let $p(t, x, y)$ denote the heat kernel of BMVD, which describes the distribution of the thermal propagation across the media. To see how the heat kernel relies on the space variable, for instance, we have proved the following estimate in [10]:

$$\frac{C_1}{\sqrt{t}} e^{-\frac{C_2|x-y|^2}{t}} \le p(t, x, y) \le \frac{C_3}{\sqrt{t}} e^{-\frac{C_4|x-y|^2}{t}} \qquad \text{when } t \in (0, 1], x, y \in \mathbb{R}_+. \tag{2.3}$$

(2.3) shows that when t is small and both points are on $\mathbb{R}_+$, the BMVD behaves like 1-dimensional Brownian motion. However, as one point moving towards the plane, the change in the heat kernel behavior is reflected in the following two inequalities proved by us in [10]:

$$\frac{C_1}{\sqrt{t}} e^{-\frac{C_2 \rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_3}{\sqrt{t}} e^{-\frac{C_4 \rho(x,y)^2}{t}} \qquad \text{when } t \in (0, 1], x \in \mathbb{R}_+, y \in D_0, \rho(y, a^*) < 1;$$

whereas

$$\frac{C_1}{t} e^{-\frac{C_2 \rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_3}{t} e^{-\frac{C_4 \rho(x,y)^2}{t}} \qquad \text{when } t \in (0, 1], x \in \mathbb{R}_+, y \in D_0, \rho(y, a^*) \ge 1.$$

From the estimations above we tell that in short time, when $x \in \mathbb{R}_+$, $y \in D_0$, the heat propagates like 1-dimensional Brownian motion when y is close enough to a^*, but like 2-dimensional Brownian motion when y moves away from a^*.

To illustrate how the heat kernel behavior is affected by the time variable, we look at the following estimate established in our work [10]:

$$\frac{C_1}{t} e^{-\frac{C_2 \rho(x,y)^2}{t}} \le p(t, x, y) \le \frac{C_3}{t} e^{-\frac{C_4 \rho(x,y)^2}{t}} \qquad \text{when } t \in [1, \infty), x, y \in D_0. \tag{2.4}$$

Comparing (2.4) with (2.3) shows that in short time, $p(t, a^*, a^*) \asymp t^{-1/2}$, and $p(t, a^*, a^*) \asymp t^{-1}$ in long term. This is consistent with the volume growth rate at a^*: $m_p(B_\rho(a^*, r)) \asymp r$ for $r \le 1$, and $m_p(B_\rho(a^*, r)) \asymp r^2$ for $r \ge 1$, where $B_\rho(a^*, r) = \{x \in E : \rho(a^*, x) < r\}$. Further discussion about this observation will be given in Section 2.3.

Due to the singular nature of the space, the standard Nash inequality and Davies method for obtaining heat kernel upper bound do not give sharp bound for our BMVD. We can not employ either the methods in [19, 35, 3, 37, 38] on obtaining heat kernel estimates through volume doubling and Poincaré inequality or the approach through parabolic Harnack inequality. In fact, the underlying space does not have volume doubling property: $m_p(B_\rho(a^*,r)) \asymp r+r^2$ for $r>0$. Furthermore, it can be shown that the parabolic Harnack inequality fails for BMVD X. The key ingredient of deriving the short time heat kernel estimates is analyzing the "signed radial process" of BMVD. However, for long time estimate, the key is to establish the correct on-diagonal estimate, which is done through some delicate analysis of BMVD and Bessel process on the plane.

More recently, I have been studying BMVD with drift in [11]. Similar to BMVD, Brownian motion with drift on spaces with varying dimension is a diffusion process whose restriction on $\mathbb{R}_+$ or D_0 is 1- or 2-dimensional Brownian motion with drift. It admits no killing or sojourn at the darning point a^*. Such a process can also be conveniently characterized in terms of the Dirichlet form with darning (or "shorting"). However, it turns out that the method of radial process is no longer applicable in this case because the process is no longer rotational-invariant on the plane. The short time heat kernel upper bound is established through Duhamel's formula for BMVD, and the lower bound can be justified by first establishing a near-diagonal estimate and then a chain argument. The main result we get for BMVD with drift is that its transition density has two-sided bounds in the same form as that for BMVD without drift.

2.3 Heat Kernel Estimate for Strongly Local Symmetric Dirichlet Forms

A Markov process is characterized by its transition function. The area studying the estimations of the transition densities of Markov processes is called "heat kernel estimates", owing to the fact that the density of the heat semigroup is not only the fundamental solution to the heat equation, but also the density of Brownian motion. This provides a deep connection among probability, PDE and geometry.

As mentioned in Section 2.1, any strongly local regular symmetric Dirichlet form is uniquely associated with a symmetric diffusion process with no killing. Therefore the heat kernel estimate of a symmetric diffusion process can be reduced to that of its associated Dirichlet form. The most classic result in this direction is the equivalence of the following conditions, which gives characterizations for two-sided Gauusian-type heat kernel estimates on both complete weighted Riemannian manifolds and strongly local regular Dirichlet forms:

- The two-sided Aronson-type Gaussian heat kernel estimate:

$$p(t,x,y) \asymp \frac{C_1}{t^{d/2}} \exp\left(-\frac{C_2|x-y|^2}{t}\right), \quad \forall x,y\in\mathbb{R}^d, t>0,$$

- The parabolic Harnack inequality: A positive solution u of the heat equation in a cylinder of the form $Q=(s,s+r^2)\times B(x,r)$ satisfies

$$\sup_{Q_-}\{u\} \le C \inf_{Q_+}\{u\},$$

where $Q_- = (s + r^2/5, s + 2r^2/5) \times B(x, r/2)$, $Q_+ = (s + 3r^2/5, s + 4r^2/5) \times B(x, r/2)$.

- The conjunction of
  - The volume doubling property: $V(x, 2r) \leq CV(x, r)$, $\forall x \in M, r > 0$.
  - The Poincaré inequality: $\forall x \in M, r > 0, B = B(x, r)$

$$\int_B |f - f_B|^2 \leq Cr^2 \int_B |\nabla f|^2, \quad \forall f \in C_c^\infty(B).$$

The following theorem that I prove in [27] claims that on a strongly local symmetric Dirichlet space, even without volume doubling property, the volume growth rate with respect to the intrinsic metric still "captures" the two-sided on-diagonal heat kernel bounds.

Theorem 3 ('16, L.). *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local symmetric Dirichlet form that is strongly regular and satisfies Nash-type inequality. Fix z outside a properly exceptional set*[1] *. Assume that for all $r > 0$, $\mu(B(z, r)) \leq v(r)$, where $v(r)$ is a continuous monotonically increasing function satisfying doubling property in the following sense: There exists some $A > 0$ such that*

$$v(2r) \leq Av(r), \quad \text{for all } r > 0. \tag{2.5}$$

Suppose also that for some $C_1 > 0$, $T \in (0, \infty]$,

$$p(t, z, z) \leq \frac{C_1}{v(\sqrt{t})}, \quad t \in (0, T). \tag{2.6}$$

Then there exists $C_2 > 0$ such that

$$p(t, z, z) \geq \frac{C_2}{v(\sqrt{t})}, \quad t \in (0, T). \tag{2.7}$$

In the theorem above, Nash-type inequality is a natural assumption to guarantee the existence of heat kernels. See, for example, [2]. Strong regularity ensures that the intrinsic metric induced by the Dirichlet form is non-degenerate and induces the same topology as the original one on the underlying space. We point out that on Dirichlet spaces, for both the classic characterizations of two-sided Gaussain heat kernel estimates and Theorem 3, the metric involved is the intrinsic metric induced by the Dirichlet form[2]. In other words, the underlying space does not need to be equipped with an original metric. As an example, the intrinsic metric induced by (2.2), the Dirichlet form associated with Brownian motion, is d-dimensional Euclidean metric. Another example is that the intrinsic metric induced by the Dirichlet form associated with BMVD is the geodesic metric (the short path distance induced from the Euclidean spaces) on the underlying space.

BMVD can actually be a example of Theorem 3. It has been mentioned that the intrinsic distance induced by its Dirichlet form is the geodesic path distance, and that the underlying

[1] Let E be the underlying space. A set $\mathcal{N} \subset E$ is called properly exceptional if it is Borel, $\mu(\mathcal{N}) = 0$ and $\mathbb{P}_x(X_t \in \mathcal{N} \text{ for some } t \geq 0) = 0$ for all $x \in E \setminus \mathcal{N}$ (see [18, p.134 and Theorem 4.1.1 on p.137]).

[2] The definition for intrinsic metric can be found, for example, in [37, 38]

space does not satisfy volume doubling property at the darning point a^*: $m_p(B_\rho(a^*, r)) \asymp r + r^2$ for all $r > 0$. However, viewing (2.3) and (2.4), we observe that the on-diagonal heat kernel behavior at a^* is still encoded in the volume function: $p(t, a^*, a^*) \asymp m_p(B_\rho(a^*, \sqrt{t}))^{-1}$ for all $t > 0$.

2.4 Future plan on Markov process and Dirichlet form theory

From the discussion in the last two subsections, shorting problems are very interesting in probability because it carries an explicit interpretation in physics. It seems that Brownian motion with darning is the most natural candidate for the scaling limit of simple random walks on "shorted" graphs where vertices falling in a particular region are shorted and replaced by a single vertex. However, although we believe this is true when the "shorted area" has smooth boundary, the further question is what if the "shorted area" is non-smooth? We note that in the Dirichlet form characterization for Brownian motion with shorting, the only condition needed for the "shorted area" is that it is non-polar, which means it will be hit in finite time with positive probability.

To be more precise, Let K be a non-polar compact connected subset in $\mathbb{R}^d$. Let $D_k := K \cap 2^{-k}\mathbb{Z}^d$. We denote the simple random walk (in discrete or continuous time) on $\mathbb{Z}^d$ with D_k being shorted by X^k, which admits a Dirichlet form expression as follows:

$$\begin{cases} \mathcal{F} = \{f \in l^2(\mathbb{Z}^d), f|_{D_k} \text{ is a constant}\}; \\ \mathcal{E}^k(f, f) = \dfrac{1}{2} \displaystyle\sum_{e_{xy} \in E^*} 2^{-(d-2)k}(f(x) - f(y))^2. \end{cases} \tag{2.8}$$

Recall that Brownian motion on $\mathbb{R}^d (d \geq 2)$ with an arbitrary non-polar compact set being shorted in to a single point can be characterized by the following Dirichlet form:

$$\begin{cases} \mathcal{F} = \{f \in W^{1,2}(\mathbb{R}^d),\ f|_K \text{ is a constant}\}; \\ \mathcal{E}(f, g) = \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) dx. \end{cases} \tag{2.9}$$

If we denote the Brownian motion with shorting by X, then when K is a smooth domain with zero-measure boundary, one can show: For every $T > 0$, the simple random walks X^k with shorting converge weakly in the Skorohod space as $k \to \infty$, to the Brownian motion with shorting on K characterized by (2.9). However, when K is non-smooth, even if it can be shown that the distributions of X^k are tight, it may not be true $\mathcal{E}^k(f, f) \to \mathcal{E}(f, f)$ for all $f \in \mathcal{F}$ in (2.9). This suggests that when the boundary is rough, the limiting process, even if it does exist, may not be the one characterized by (2.9).

Another problem along the direction of shorting diffusion processes is to let the shorted area be time-dependent. For instance, we may assume the shorted area is increasing in time. We first wish to give characterization to such type of process via time-dependent Dirichlet forms. Once this can be done, then further questions can be asked about the existence and smoothness of the transition density of the process, the estimations of the density, and other potential properties of the process. Some characterizations for heat kernel estimates of time-dependent Dirichlet forms have been studied by Sturm in [37].

3 Stochastic Differential Equations driven by fBM

3.1 Background

As mentioned in Section 2.1, the concept of Brownian motion was suggested in 1828 to describe the irregular random movements of pollen grains suspended in fluid. The fact that Brownian motion has uncorrelated Gaussian increments gives rise to a wide range of its applications in modeling systems and processes in other disciplines. However, many complex natural systems and processes usually do have long-range, spatial and temporal correlations. This motivates the studies of fractional Brownian motion, which is a generalization of Brownian motion but with positive or negative correlations, depending on its Hurst parameter. Like Brownian motion, fractional Brownian motion is also named after 19th century biologist Robert Brown.

A fractional Brownian motion is a centered Gaussian process parametrized by Hurst parameter $H \in (0,1)$ which determines what kind of process the fBm is:

(i) if $H = 1/2$ then the process is in fact a Brownian motion or Wiener process;

(ii) if $H > 1/2$ then the increments of the process are positively correlated;

(iii) if $H < 1/2$ then the increments of the process are negatively correlated.

Precisely speaking, fractional Brownian motion $(B_t^H)_{t\geq 0}$ with Hurst parameter H is a $d-$dimensional centered Gaussian processes with covariance structure

$$R(t,s) := \mathbb{E}B_s^i B_t^i = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,1] \text{ and } i = 1,\cdots,d.$$

It is well-known that fBM B^H admits a continuous version of its path which is $\alpha-$Hölder continuous for all $\alpha < H$.

The Hurst parameter H also describes the regularity of sample paths of the process. Roughly speaking, the bigger the H is, the more regular the trajectories of the process are. In particular when $H = 1/2$, it is the standard Brownian motion. Currently for $H > 1/4$, SDEs driven by fractional Brownian motions in the form of (1.1) can be defined under the framework of Terry Lyons rough path theory (cf. [31]) which is developed to give meaning of integrals driven by a class of (irregular) paths called geometric rough path. Recent research trends in this area include existence and uniqueness of the solutions to these SDEs, existence and estimates of the (smooth) density functions of the solutions to these SDEs, path behaviors and regularities of the solutions, the most challenging topic among which is to define SDEs driven by fractional Brownian motion with $H < 1/4$.

We now briefly introduce how to make sense of (1.1), since fBM is neither a semimartingale nor a Markov process. To this purpose, we approximate the sample paths of fBM using linear interpolations along dyadic rationals: $t_j^k := j2^{-k}$, $j = 0,\cdots,2^k$ for $k \in \mathbf{N}$. Then for each k and each path, there exists a unique solution to the ODE corresponding to (1.1). Rough path theory guarantees that under some appropriate sense, these approximated paths converge to a limit when the Hurst parameter $H > 1/4$. The limiting path is regarded as the solution to (1.1), the SDE driven by fBM.

3.2 Rough Path Theory and Main Results

To analyze the path properties of the solutions to SDEs in the form of (1.1), the key ingredient is the following upper bound estimate on their finite-dimensional distributions:

Theorem 4 ('15, L., Ouyang)**.** *Fix $\varepsilon \in (0,T)$ and $\gamma < H$. Let $p_{t_1,\dots,t_n}(\xi_1,\dots,\xi_n)$ be the joint density of the random vector $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ for $\varepsilon \le t_1 < \cdots < t_n \le T$. There is a positive constant C such that*

$$\begin{aligned}&\partial^{k_1}_{\xi_1}\dots\partial^{k_n}_{\xi_n} p_{t_1,\dots,t_n}(\xi_1,\dots,\xi_n)\\ &\le C\,\frac{1}{(t_2-t_1)^{(d+k_2)H}}\exp\left(-\frac{|\xi_2|^{2\gamma}}{C|t_2-t_1|^{2\gamma^2}}\right)\cdots\frac{1}{(t_n-t_{n-1})^{(d+k_n)H}}\exp\left(-\frac{|\xi_n|^{2\gamma}}{C|t_n-t_{n-1}|^{2\gamma^2}}\right)\end{aligned}$$

The following primary results on the Hausdorff dimensions of the sample paths and the level sets are proved in [28]. Since the process is non-Gaussian, the explicit transition densities of the process are unknown, and we use the upper bound estimates in Theorem 4 as alternatives.

Theorem 5 ('15, L., Ouyang)**.** *Let X be the solution to (1.1) where the vector fields V_i's satisfy uniform elliptic condition. Then almost surely it holds*

$$\dim_{\mathcal{H}} X([0,1]) = \min\left\{d, \frac{1}{H}\right\},$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension. Moreover, for any $\epsilon \in (0,1)$ and any given $x \in \mathbb{R}^d$,

(i). If $dH > 1$, $L_x := \{t \in [\epsilon, 1] : X_t = x\} = \varnothing$ a.s.;

(ii). If $dH < 1$, $\dim_{\mathcal{H}} L_x = 1 - dH$ with positive probability.

Another natural problem in this area along the line is to establish the smoothness of the local times (in the sense of Meyer-Watanabe) and self-intersection local times for processes driven by fBM. Indeed, the chaos expansion and smoothness of local times and intersection local times of (fractional) Brownian motion have been studied extensively by many authors, for instance, in [22, 23, 24]. In [29], we establish the exact Hölder exponent for the local times of 1-dimensional processes driven by fBM. Given X which is a solution to 1.1, fix a small constant $a > 0$, and let $L^a(t,A)$ and $L^a(t,x)$ be the occupation measure and occupation density respectively on the time interval $[a,t]$. That is

$$L^a(t,A) = L([a,t],A), \quad \text{and} \quad L^a(t,x) = L([a,t],x).$$

The major difficulty is again not knowing the explicit transition density of X. By estalishing the moment estimations on the occupation times, we give in [28] the following exact Hölder exponent of the local times of one dimensional processes driven by fBM.

Theorem 6 ('16, L., Ouyang)**.** *Assume $d = 1$ and $1/4 < H < 1/2$. Let X be the solution to* (1.1) *where the vector fields V_i's satisfy uniform elliptic condition. Let $\alpha(t)$ be the pathwise Hölder exponent of $L^a(t,x)$. It holds*

$$\alpha(t) = 1 - H, \quad a.s.\, for\ all\ t \in [0,T].$$

3.3 Future plan on rough path theory

Many promising questions can be asked regarding rough path properties for solutions to (1.1). First of all, a most natural question following Theorem 6 is, can we establish the continuity of local times to higher dimensional processes driven by fBM?

Based on Theorem 5, a long list of problems can be proposed. Here are a few of them that I am planning to work on in the future: Given two independent processes X^H and X^K driven by fBM with Hurst parameters H and K respectively.

(i) Let E_1, E_2 be Borel sets in $\mathbb{R}_+$. When does it hold that

$$\mathbb{P}(X^H(E_1) \cap X^H(E_2) \neq \emptyset) > 0?$$

(ii) Let F be a Borel set in $\mathbb{R}^d$ and let I_1, I_2 be two compact intervals in $\mathbb{R}_+$. Can we find the necessary and sufficient condition for F to contain the intersection points of $X^H(E_1)$ and $X^K(E_2)$, namely,

$$\mathbb{P}(X^H(E_1) \cap X^K(E_2) \cap F \neq \emptyset) > 0?$$

(iii) What is the Hausdorff dimension of the intersection points of X^H and X^K, namely,

$$\dim_{\mathcal{H}}\{x \in \mathbb{R}^d : x = X^H(t) = X^K(s)\}?$$

Note that the set in the display above can be written as $\dim_{\mathcal{H}}\{x \in \mathbb{R}^d : x \in X^H(\mathbb{R}_+) \cap X^K(\mathbb{R}_+)\}$.

Similar multiple intersection questions have been answered for Brownian motions by Evans in [16], Tongring in [39], Fitzsimmons and Salisbury in [17], and Peres in [34], for independent Brownian sheets by Dalang, et al. in [15], and for anisotropic Gaussian random fields by Chen and Xiao in [7], and by Wu and Xiao in [40].

Along the line of studying the existence and smoothness of the local times of SDE driven by fBM, further natural questions can be asked about the self-intersection local times of two such processes. To be more precise, let X^H and X^K be two independent processes driven by fBM with Hurst parameters H and K respectively, and let $Y(s,t) := X^H(s) - X^K(t)$ be a process from $\mathbb{R}^2_+$ to $\mathbb{R}^d$. Let L be the occupation density of $Y(s,t)$ at 0 over an closed rectangle in $\mathbb{R}^2_+$, which is called the intersection local time of X^H and x^K. We hope to find the necessary and sufficient condition for the existence of L, and naturally, the joint continuity of L.

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