MATH 121 Final Exam Solution (version II) Radford 12/03/02

1. Let $y = f(x) = 6 + \sqrt{5x-7}$. The calculation

$$\begin{aligned} y &= 6 + \sqrt{5x-7}, \\ y - 6 &= \sqrt{5x-7}, \\ (y-6)^2 &= 5x - 7, \\ (y-6)^2 + 7 &= 5x, \\ \frac{(y-6)^2+7}{5} &= x \quad (\textbf{15 points}) \end{aligned}$$

shows that the inverse of $f(x)$ is given by $f^{-1}(x) = \dfrac{(x-6)^2+7}{5}$ (**5 points**).

2. Since $f(x) = x^3 + 2x - 3$ has integer coefficients, the rational roots of $f(x)$ are among $\pm 1, \pm 3$. Since $f(1) = 0$ it follows that $x = 1$ is a root of $f(x)$. Therefore $x - 1$ divides $f(x)$. Dividing $x - 1$ into $f(x)$ gives the quotient of $x^2 + x + 3$ and remainder 0. Thus

$$x^3 + 2x - 3 = (x-1)(x^2 + x + 3)$$

By the quadratic formula the roots of $x^2 + x + 3$ are $\dfrac{-1 \pm \sqrt{11}\imath}{2}$. Thus the roots of $f(x)$ are

$$1 \ (\textbf{4 points}), \quad \frac{-1+\sqrt{11}\imath}{2} \ (\textbf{4 points}), \quad \frac{-1-\sqrt{11}\imath}{2} \ (\textbf{4 points}).$$

As a product of linear factors,

$$f(x) = (x-1)\left(x - \frac{-1+\sqrt{11}\imath}{2}\right)\left(x - \frac{-1-\sqrt{11}\imath}{2}\right) \quad (\textbf{8 points}).$$

3. The height function is given by $h(t) = -16t^2 + 10t + 500$ feet.

(a) Rewriting $h(t)$ as

$$-16t^2 + 10t + 500 = -16(t^2 - \frac{10}{16}t) + 500 = -16(t - \frac{10}{32})^2 + (500 + 16\left(\frac{10}{32}\right)^2)$$

we see that the maximal altitude is ≈ 501.563 *feet* (**10 points**).

(b) We want to solve $h(t) = 0$, where $t \geq 0$, or $-16t^2 + 10t + 500 = 0$, where $t \geq 0$. By the quadratic formula $t = \dfrac{-10 \pm \sqrt{10^2 - (4)(-16)(500)}}{2(-16)}$, so $t \approx 5.911$ *seconds.* (**10 points**).

4. (a) The calculation

$$2\sin^2(x) + \cos(x) = 2,$$

$$\begin{aligned} 2(1-\cos^2(x))+\cos(x) &= 2,\\ -2\cos^2(x)+\cos(x) &= 0,\\ \cos(x)\,(-2\cos(x)+1) &= 0, \end{aligned}$$

shows that the set of solutions to the original equation is the set of solutions to $\cos(x) = 0$ together with the set of solutions to $\cos(x) = \frac{1}{2}$ (**3 points**). Thus *all* solutions the original equation are described by

$$x = \frac{\pi}{2} + \pi n \ \ (\textbf{3 points}), \quad \frac{\pi}{3} + 2\pi n \ \ (\textbf{3 points}), \quad \text{or } \frac{5\pi}{3} + 2\pi n \ \ (\textbf{3 points}),$$

where n is any integer.

(b) The calculation

$$\begin{aligned} \log\sqrt{x^2-10} &= 3,\\ \sqrt{x^2-10} &= 10^3,\\ x^2-10 &= (10^3)^2,\\ x^2 &= 10^6+10,\\ x &= \pm\sqrt{10^6+10}, \end{aligned}$$

the steps of which are reversible, shows that $x = \sqrt{10^6+10}$ (**4 points**), $-\sqrt{10^6+10}$ (**4 points**).

5. (a) $600\left(1+\frac{0.08}{4}\right)^{4\cdot 9}$ (**5 points**) ≈ 1044.615 dollars (**5 points**).

(b) Let P be the initial amount invested at $100r\%$ per year compounded continuously. Then $2P = P(15) = Pe^{r15}$ which means $2 = e^{r15}$ (**3 points**) since $P > 0$. Therefore $r = \frac{\ln 2}{15}$ (**3 points**) $\approx .046$ (**4 points**) which can be expressed as 4.6%.

6. a) All real numbers except for $x = 9/17$ (**3 points**). b) $y = 5/17$ (**3 points**). c) $x = -9/17$ (**3 points**). d) At $x = -3/5$ (**3 points**). e) At $y = -1/3$ (**3 points**). f) Labeling (**3 points**) and shape of graph (**2 points**).

7. (a) *no* (**1 points**). Take $x = \frac{\pi}{4}$ for example. In this case $\cot(x+\pi) = \cot(\frac{\pi}{4}+\pi) = 1$ and $-\cot(x) = -\cot(\frac{\pi}{4}) = -1$ (**3 points**).

(b) *no* (**1 points**). Take $x = 0$ for example. In this case $e^{x+5} = e^5$ and $e^x + e^5 = e^0 + e^5 = 1 + e^5$ (**3 points**).

(c) *yes* (**4 points**).

(d) *no* (**1 points**). Take $x = 1$ for example. In this case $\ln(x+10) = \ln(11) \neq 0$ and $(\ln 1)(\ln 10)) = (0)\ln(10) = 0$ (**3 points**).

(e) *no* (**1 points**). Take $x = 2\pi$ for example. In this case $\arccos(\cos(2\pi)) = \arccos(\cos(0)) = 0$, since $0 \leq x \leq \pi$, which is not $x = 2\pi$ (**3 points**).

8. Since $\tan(t) = -\frac{7}{6}$ and $\pi < t < 2\pi$, it follows that the terminal side of t is in the fourth quadrant. Therefore $(6, -7)$ is a point on the terminal side of the angle t. Since $r = \sqrt{6^2 + (-7)^2} = \sqrt{85}$ we conclude that

$$\cos(t) = \frac{6}{\sqrt{85}}, \qquad \sin(t) = -\frac{7}{\sqrt{85}}, \qquad \tan(t) = \frac{\sin(t)}{\cos(t)} = -\frac{7}{6}.$$

The last calculation is not necessary, of course, but does provide a consistency check. Thus

$$(a) \cos(t) = \frac{6}{\sqrt{85}} \ (\mathbf{5\ points}), \quad (b) \csc(t) = -\frac{\sqrt{85}}{7} \ (\mathbf{5\ points}),$$

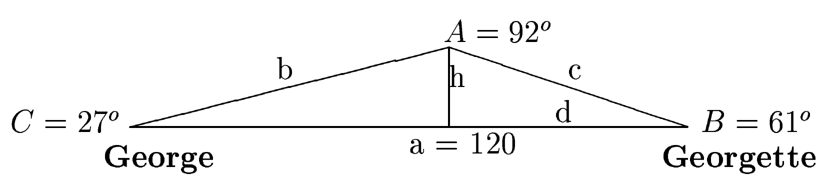
and since $\frac{\pi}{2} < \frac{t}{2} < \pi$,

$$(c) \sin(t/2) = \sqrt{\frac{1 - \cos(t)}{2}} = \sqrt{\frac{1 - \frac{6}{\sqrt{85}}}{2}} = \sqrt{\frac{\sqrt{85} - 6}{2\sqrt{85}}} \ (\mathbf{5\ points}),$$

and finally

$$(d) \cos(2t) = \cos^2(t) - \sin^2(t) = \left(\frac{6}{\sqrt{85}}\right)^2 - \left(\frac{-7}{\sqrt{85}}\right)^2 = \frac{36}{85} - \frac{49}{85} = -\frac{13}{85} \ (\mathbf{5\ points}).$$

9. (a) Let A be the angle formed by the line segments from George to the top of the streetlight to Georgette, let $B = 61^o$, and let $C = 27^o$. Then $A = 180^o - 27^o - 61^o = 92^o$. Below is a picture *which is not meant to be to scale.*



By the Law of Sines

$$\frac{120}{\sin 92^o} = \frac{b}{\sin 61^o}.$$

Therefore

$$b = \left(\frac{120}{\sin 92^o}\right) \sin 61^o \approx 105.018 \ feet \ \ (\mathbf{4\ points}).$$

Consequently the height of the streetlight is $h = b \sin 27^o \approx 47.677 \ feet$ (**8 points**).

(b) The distance from Georgette to the streetlight is

$$d = h \cot 61^o = \frac{h}{\tan 61^o} \approx 26.428 \; feet \;\; (\mathbf{8\ points}).$$

10. (a) The modulus of $1+\sqrt{3}\imath$ is $|1+\sqrt{3}\imath| = \sqrt{1^2+(\sqrt{3})^2} = 2$. Therefore $1+\sqrt{3}\imath = 2\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\imath\right) = 2\left(\cos(\frac{\pi}{3})+\imath\sin(\frac{\pi}{3})\right)$ (**10 points**).

(b) Using the polar form of $1+\imath$ we calculate

$$\begin{aligned}(1+\sqrt{3}\imath)^7 &= \left(2\left(\cos(\frac{\pi}{3})+\imath\sin(\frac{\pi}{3})\right)\right)^7\\ &= 2^7\left(\cos(\frac{\pi}{3})+\imath\sin(\frac{\pi}{3})\right)^7\\ &= 2^7\left(\cos(\frac{7\pi}{3})+\imath\sin(\frac{7\pi}{3})\right)\\ &= 128\left(\cos(\frac{\pi}{3})+\imath\sin(\frac{\pi}{3})\right) \;\; (\mathbf{10\ points}).\end{aligned}$$

Note that $(1+\sqrt{3}\imath)^7 = 64(1+\sqrt{3}\imath)$.